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Group divisible designs of four groups and block size five with configuration $(1, 1, 1, 2)$

Research Article

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Abstract: We present constructions and results about GDDs with four groups and block size five in which each block has Configuration $(1, 1, 1, 2)$, that is, each block has exactly one point from three of the four groups and two points from the fourth group. We provide the necessary conditions of the existence of a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 1, 2)$, and show that the necessary conditions are sufficient for a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 1, 2)$ if $n \not\equiv 0 \pmod 6$, respectively. We also show that a GDD$(n, 4, 5; 2n, 6(n-1))$ with Configuration $(1, 1, 1, 2)$ exists, and provide constructions for a GDD$(n = 2t, 4, 5; n, 3(n-1))$ with Configuration $(1, 1, 1, 2)$ where $n \neq 12$, and a GDD$(n = 6t, 4, 5; 4t, 2(6t-1))$ with Configuration $(1, 1, 1, 2)$ where $n \neq 6$ and 18, respectively.

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1. Introduction

Group divisible designs (GDDs) have been studied for their usefulness in statistics and for their universal application to constructions of new designs [13, 17, 18]. Certain difficulties are present especially when the number of groups is smaller than the block size. In [3, 4], the question of existence of GDDs

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for block size three was settled. There is a more technical proof given in the book "Triple System" [2]. Similar results were established for GDDs with block size four in [6, 8, 9, 14, 19]. In [7, 16], results about GDDs with two groups and block size five with fixed block configuration were presented. In [10], results about GDDs with block size six with fixed block configuration were established.

A group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k-element subsets of a v-set V called *blocks* which satisfies the following properties: each point of V appears in r (called *replication number*) of the b blocks; the $v = nm$ elements of V are partitioned into m subsets (called *groups*) of size n each; points within the same group are called *first associates* of each other and appear together in λ_1 blocks; any two points not in the same group are called *second associates* of each other and appear together in λ_2 blocks. We note that in [13], the term GDD always refer to the case where $\lambda_1 = 0$. When λ_1 is not zero, the designs here are called group divisible PBIBDs [18].

In [6, 19], the necessary conditions are proved to be sufficient for the existence of a GDD$(n, 3, 4; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 2)$ where each block has exactly one point from two of the three groups and two points from the third group. The purpose of this paper is to establish results for GDDs with block size five and four groups (i.e. GDD$(n, 4, 5; \lambda_1, \lambda_2)$) in which each block has Configuration $(1, 1, 1, 2)$, that is, each block has exactly one point from three of the four groups and two points from the fourth group. Unless otherwise stated, GDDs addressed in this paper all have the Configuration $(1, 1, 1, 2)$. First we find the relationship between λ_2 and λ_1.

Theorem 1.1. *The necessary conditions for the existence of a* GDD$(n, 4, 5;\ \lambda_1, \lambda_2)$ *are* $n \geq 2$ *and* $\lambda_2 = \frac{3(n-1)\lambda_1}{n}$.

Proof. Suppose a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ exists, then the replication number r for an arbitrary point is $\frac{\lambda_1(n-1)+\lambda_2(3n)}{4}$. Also, since $vr = bk$, we have $b = \frac{n\times[\lambda_1(n-1)+\lambda_2(3n)]}{5}$. On the other hand, since every block must contain exactly one first associate pair (with Configuration $(1, 1, 1, 2)$), the group size n should be greater than or equal to 2, and the number of the first associates pairs $\frac{4n(n-1)}{2}$ times λ_1 must be equal to the number of blocks b. We have $2n(n-1)\lambda_1 = \frac{n\times[\lambda_1(n-1)+\lambda_2(3n)]}{5}$, that is, $\lambda_2 = \frac{3(n-1)\lambda_1}{n}$. □

Corollary 1.2. *A necessary condition for the existence of a* GDD$(3, 4, 5;\ \lambda_1, \lambda_2)$ *is* $\lambda_2 = 2\lambda_1$ *and a necessary condition for the existence of a* GDD$(n, 4, 5; \lambda_1, \lambda_2)$ *reduces to* $\lambda_2 = (n-1)t$ *(for* $t \geq 1$*) if* $n \neq 3$.

Proof. By Theorem 1.1, $\lambda_2 = 2\lambda_1$ if $n = 3$. If $n \neq 3$, then $3\lambda_1 \equiv 0 (\text{mod } n) = nt$ $(t \geq 1)$, thus $\lambda_1 = \frac{nt}{3}$, and $\lambda_2 = (n-1)t$ for $t \geq 1$. □

Corollary 1.3. *For* $n \not\equiv 0(mod\ 3)$*, the minimum* λ_1 *for the existence of a* GDD$(n, 4, 5; \lambda_1, \lambda_2)$ *is* n*. For* $n \equiv 0(mod\ 3)$*, the minimum* λ_1 *is* $\frac{n}{3}$*.*

Proof. By Theorem 1.1, if $n \not\equiv 0(\text{mod } 3)$, then $\lambda_1 \equiv 0(\text{mod } n)$, thus the minimum λ_1 for the existence of a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ is n. If $n \equiv 0(\text{mod } 3)$, then $\lambda_1 \equiv 0(\text{mod } \frac{n}{3})$, thus the minimum λ_1 is $\frac{n}{3}$. □

Notice that if a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ exists, then a GDD$(n, 4, 5; t\lambda_1, t\lambda_2)$ exists by taking t multiples of GDD$(n, 4, 5; \lambda_1, \lambda_2)$. Therefore, we can reduce the problem to find a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ for the minimum value of λ_1 (which are given in Corollary 1.3).

Remark 1.4. *If a* GDD$(n, 4, 5; \lambda_1, \lambda_2)$ *for the minimum value of* λ_1 *exists (it's* n *for* $n \not\equiv 0(mod\ 3)$ *and* $\frac{n}{3}$ *for* $n \equiv 0(mod\ 3)$*), then a* GDD$(n, 4, 5; t\lambda_1, t\lambda_2)$ *exists for* $t \geq 1$*.*

2. GDD$(n, 4, 5; \lambda_1, \lambda_2)$ for $n = 2, 3, 4$ and $n \equiv 1, 5(\text{mod } 6)$

Theorem 2.1. *Necessary conditions given in Theorem 1.1 are sufficient for the* GDD*s with* $n = 2, 3$ *and* 4.

Proof. By Theorem 1.1, the necessary condition for the existence of a GDD$(2, 4, 5; \lambda_1, \lambda_2)$ is $2\lambda_2 = 3\lambda_1$, that is, $\lambda_1 \equiv 0(\text{mod } 2)$ and $\lambda_2 \equiv 0(\text{mod } 3)$. The minimum values of λ_1 and λ_2 are 2 and 3, respectively. A GDD$(2, 4, 5; 2, 3)$ on the four groups $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ and $\{7, 8\}$ is as follows: $\{1, 3, 5, 7, 8\}$, $\{2, 4, 6, 7, 8\}$, $\{3, 6, 7, 1, 2\}$, $\{4, 5, 8, 1, 2\}$, $\{5, 7, 3, 2, 4\}$, $\{6, 8, 1, 3, 4\}$, $\{7, 1, 4, 5, 6\}$, and $\{8, 2, 3, 5, 6\}$. By Remark 1.4, we have a GDD$(2, 4, 5; \lambda_1, \lambda_2)$.

By Corollary 1.2, the necessary condition for the existence of a GDD$(3, 4, 5; \lambda_1, \lambda_2)$ is $\lambda_2 = 2\lambda_1$. The minimum values of λ_1 and λ_2 are 1 and 2, respectively. A construction for a GDD$(3, 4, 5; 1, 2)$ on the four groups $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$ and $\{a, b, c\}$ is as follows: $\{1, 2, 6, 7, b\}$, $\{1, 3, 4, 9, a\}$, $\{2, 3, 5, 8, c\}$, $\{4, 5, 7, 3, b\}$, $\{5, 6, 9, 2, a\}$, $\{6, 4, 8, 1, c\}$, $\{7, 8, a, 1, 5\}$, $\{8, 9, b, 2, 4\}$, $\{9, 7, c, 3, 6\}$, $\{c, b, 1, 5, 9\}$, $\{b, a, 3, 6, 8\}$, $\{c, a, 2, 4, 7\}$. Note that this construction is also listed in Clatworthy's table (number 513 on page 902 in [1]). By Remark 1.4, we have a GDD$(3, 4, 5; \lambda_1, \lambda_2 = 2\lambda_1)$.

The necessary condition for the existence of a GDD$(4, 4, 5; \lambda_1, \lambda_2)$ is $\lambda_1 \equiv 0(\text{mod } 4)$. The minimum values of λ_1 and λ_2 are 4 and 9, respectively. A construction for a GDD$(4, 4, 5; 4, 9)$ on the four groups $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$ and $\{13, 14, 15, 16\}$ is as follows in Figure 1 (where each column represents a block). By Remark 1.4, we have a GDD$(4, 4, 5; \lambda_1, \lambda_2)$. □

2	1	1	1	5	5	5	6	2	1	1	1	5	5	6	5	14	13	13	13	9	9	10	9	13	13	14	13	9	10	9	9
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	4	3	1	2	3	4	1	2	3	4
6	5	7	8	8	7	6	5	5	6	8	7	7	8	5	6	5	6	7	8	8	7	5	6	6	5	8	7	7	8	5	6
9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11	9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11
13	14	15	16	14	13	16	15	15	16	13	14	16	15	14	13	13	14	15	16	13	14	16	15	15	16	13	14	15	16	14	13

3	3	2	2	6	6	7	7	3	3	2	2	6	6	7	7	15	15	14	14	10	10	11	11	14	14	15	15	11	11	10	10
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	7	8	8	7	6	5	6	5	8	7	7	8	5	6	5	6	7	8	8	7	6	5	6	5	8	7	7	8	5	6
9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11	9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11
13	14	15	16	14	13	16	15	15	16	13	14	16	15	14	13	13	14	15	16	14	13	16	15	15	16	13	14	16	15	14	13

4	4	4	3	7	8	8	8	4	4	4	3	8	7	8	8	16	16	16	15	12	11	12	12	16	15	16	16	12	12	11	12
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
5	6	8	7	8	7	6	5	6	5	7	8	7	8	5	6	5	6	7	8	8	7	6	5	6	5	8	7	7	8	5	6
9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11	9	10	11	12	11	12	9	10	12	11	10	9	10	9	12	11
13	14	15	16	14	13	16	15	15	16	13	14	16	15	13	14	13	14	15	16	14	13	16	15	15	16	13	14	16	15	14	13

Figure 1. A GDD$(4, 4, 5; 4, 9)$

We use a different construction below for a GDD$(n, 4, 5; \lambda_1 = 2n, \lambda_2 = 6(n-1))$ where λ_1 is not of its minimum value (but twice of its minimum value if $n \not\equiv 0(\text{mod } 3)$ or six times of its minimum value if $n \equiv 0(\text{mod } 3)$). It's an interesting construction as it uses a special kind of group divisible design GDD$(n, k, k; 0, 1)$. Such a GDD is called a transversal design, TD(k, n). The construction also uses a *resolvable* GDD (RGDD). A design is *resolvable* if the blocks of the design can be partitioned into *parallel classes* $P_1, \ldots, P_s$, where every point of V occurs exactly once in each P_i. Similarly, one can define a resolvable transversal design, RTD(k, n). The following several theorems from the Handbook of Combinatorial Designs (2nd edition) [1], also see Rees [15] and Ge and Ling [5], are well-known theorems of RGDDs that we will use in our proof.

Theorem 2.2. [1] (Theorem 5.35 on page 264) *The necessary condition for the existence of a* RGDD$(n, m, k; 0, \lambda))$ *are (1)* $m \geq k$*, (2)* $nm \equiv 0(\text{mod } k)$*, and (3)* $\lambda n(m-1) \equiv 0(\text{mod } (k-1))$*.*

Theorem 2.3. [1] (Theorem 5.43 on page 265) *A* RGDD$(n, m, 3; 0, \lambda)$ *exists if and only if* $m \geq 3$*,* $\lambda n(m-1)$ *is even,* $nm \equiv 0(\text{mod } 3)$*, and* $(\lambda, n, m) \notin \{(1, 2, 6), (1, 6, 3)\} \cup \{(2j+1, 2, 3), (4j+2, 1, 6) : j \geq 0\}$*.*

Theorem 2.4. [1] (Theorem 5.44 on page 265) *The necessary conditions for the existence of a* RGDD$(n, m, 4; 0, 1)$*, namely,* $m \geq 4$*,* $nm \equiv 0(\text{mod } 4)$ *and* $n(m-1) \equiv 0(\text{mod } 3)$*, are also sufficient except for* $(n, m) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:* $n = 2$ *and* $m \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$*;* $n = 10$ *and* $m \in \{4, 34, 52, 94\}$*;* $n \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $m \in \{10, 70, 82\}$*;* $n = 6$ *and* $m \in \{6, 54, 68\}$*;* $n = 18$ *and* $m \in \{18, 38, 62\}$*;* $n = 9$ *and* $m = 44$*;* $n = 12$ *and* $m = 27$*;* $n = 24$ *and* $m = 23$*; and* $n = 36$ *and* $m \in \{11, 14, 15, 18, 23\}$*.*

A *latin square* L of side (or order) n is an $n \times n$ array in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and exactly once in each column. Two latin squares L_1 and L_2 of the same order are *orthogonal* if $L_1(a,b) = L_1(c,d)$ and $L_2(a,b) = L_2(c,d)$, implies $a = c$ and $b = d$. A set of latin squares $L_1, \ldots, L_m$ is *mutually orthogonal*, or a set of *MOLS*, if for every $1 \le i \le j \le m$, L_i and L_j are orthogonal.

Theorem 2.5. [1] (Theorem 3.18 on page 161) *The existence of* k *MOLS* (n), *the existence of a* $\mathrm{TD}(k+2, n)$ *and the existence of a* $\mathrm{RTD}(k+1, n)$ *are equivalent where* $k \ge 1$.

Theorem 2.4 implies the existence of a $\mathrm{RGDD}(n,4,4;0,1) = \mathrm{RTD}(4,n)$ except for $n = 2,3,6$ and 10. A construction of a $\mathrm{TD}(4,3)$ (it is also a $\mathrm{RTD}(4,3)$) and a set of 2 MOLS(10) (which implies the existence of a $\mathrm{TD}(4,10)$ by Theorem 2.5) can be found in examples 6.5.1 and 6.5.10 in [11], respectively. Therefore, we have the following Lemma 2.6.

Lemma 2.6. *A* $\mathrm{TD}(4,n)$ *and a* $\mathrm{RTD}(3,n)$ *exist except for* $n = 2$ *and* 6.

Theorem 2.7. *If a* $\mathrm{RGDD}(n,3,3;0,n-1)$ *exists (i.e.* $n \neq 2$ *by Theorem 2.3), then a* $\mathrm{GDD}(n,4,5;2n, 6(n-1))$ *also exists. Hence a* $\mathrm{GDD}(n,4,5;2n,6(n-1))$ *exists for all* $n > 2$.

Proof. A $\mathrm{RGDD}(n,3,3;0,n-1)$ has $n^2(n-1)$ blocks and these are partitioned into $n(n-1)$ parallel classes. First, we construct a $\mathrm{RGDD}(n,3,3;0,\, n-1)$ on three groups G_1, G_2, and G_3. There are $n(n-1)$ parallel classes. Attach each pair of distinct points from G_4 with blocks of two parallel classes to make blocks of size 5. In the same way, we construct RGDDs on G_1, G_2, and G_4 and attach a pair from G_3, and then construct RGDDs on G_1, G_3, and G_4 and attach a pair from G_2, and then construct RGDDs on G_2, G_3, and G_4 and attach a pair from G_1, on two parallel classes each. Now a parallel class has n triples and each pair from a G_i is attached to these triples of two parallel classes, λ_1 is $2n$. Now we show $\lambda_2 = 6(n-1)$. Let $i \in G_i$ and $j \in G_j$. When we attach a pair from G_i containing i to two parallel classes from the RGDD that misses G_i, the pair $\{i,j\}$ occurs in $2(n-1)$ blocks. Likewise, when a pair from G_j containing j is attached to two parallel classes from the RGDD that misses G_j, the pair $\{i,j\}$ occurs $2(n-1)$ times. The other two RGDDs intersect both G_i and G_j. Thus the pair $\{i,j\}$ occurs $n-1$ times in each of these RGDDs. Hence the pair $\{i,j\}$ occurs a total of $6(n-1)$ times. □

By using the same proof as in Theorem 2.7, we have the following corollary.

Corollary 2.8. *If a* $\mathrm{RGDD}(n,3,3;0,\frac{n-1}{2})$ *exists, then a* $\mathrm{GDD}(n,4,5;\, n,3(n-1))$ *also exists. Hence a* $\mathrm{GDD}(n,4,5;n,3(n-1))$ *exists for all* $n \equiv 1 (mod\ 2)$, *i.e. odd numbers.*

Combine Corollary 1.3, Remark 1.4 and Corollary 2.8, we have the following result.

Corollary 2.9. *Necessary conditions are sufficient for a* $\mathrm{GDD}(n,4,5;\lambda_1,\ \lambda_2)$ *if* $n \equiv 1,5 (mod\ 6)$, *i.e.* $n \equiv 1 (mod\ 2)$ *and* $n \not\equiv 0 (mod\ 3)$.

The following is a different proof of Theorem 2.7.

Theorem 2.10. *A* $\mathrm{GDD}(n,4,5;\lambda_1 = 2n, \lambda_2 = 6(n-1))$ *exists for all* $n \ge 2$.

Proof. In Theorem 2.1, we have proved that the necessary conditions are sufficient for $n = 2$. A $\mathrm{TD}(4,n)$ exists for every n except for $n = 2$ and 6 by Lemma 2.6. For any $n > 2$ and $n \neq 6$, a $\mathrm{TD}(4,n)$ has n^2 blocks each of size 4 and replication number $r = n$. Any two elements from group G_i occur together 0 times and pairs from different G_i's occur once. Take any block $\{a_1,a_2,a_3,a_4\}$, where $a_j \in G_j$. and replace it by $\{x,a_1,a_2,a_3,a_4\}$, where $x \in G_i - \{a_i\}$. We have $4n^2(n-1)$ blocks. It is easy to check the parameters $\lambda_1 = 2n$, (because replication for $\mathrm{TD}(4,n)$ is n) and $\lambda_2 = 6(n-1)$. Therefore, a $\mathrm{GDD}(n,4,5;\lambda_1 = 2n, \lambda_2 = 6(n-1))$ always exists. The reason for $\lambda_2 = 6(n-1)$ is that suppose we have two elements $a \in G_i$, $b \in G_j$, and $i \neq j$. There are only three types of blocks which will involve a and/or b, that is, when both appear in one block, when a appears and b does not appear, and when a does not appear while b appears, in a block. In case 1, the number of pairs are $4(n-1)$, while in cases 2 and 3, there are $n-1$ pairs in each, and this gives a total of $6(n-1)$.

For $n = 6$, an RGDD$(6, 3, 3; 0, 5)$ exists by Theorem 2.3 since $m \geq 3$ and $(\lambda, n, m) = (5, 6, 3)$ does not belong to that set of exceptions in Theorem 2.3. Hence we get a GDD$(6, 4, 5; 12, 30)$ by Theorem 2.7, that is, a GDD$(n, 4, 5; 2n, 6(n-1))$ with $n = 6$ exists. Thus, a GDD$(n, 4, 5; \lambda_1 = 2n, \lambda_2 = 6(n-1))$ exists for all $n \geq 2$. □

3. GDD$(n, 4, 5; \lambda_1, \lambda_2)$ for $n \equiv 2, 4 (\text{mod } 6)$

A *balanced incomplete block design* BIBD(v, k, λ) $(\lambda \geq 1)$ is a pair (V, B) where B is a collection of binary blocks of V such that every block contains exactly $k < v$ points and every pair of distinct elements is contained in exactly λ blocks. A resolvable BIBD(v, k, λ) is denoted as RBIBD(v, k, λ).

A *1-factor* of a graph G is a set of pairwise disjoint edges which partition the vertex set. A *1-factorization* of a graph G is the set of 1-factors which partition the edge set of the graph. A 1-factorization of a K_{2n} (also a RBIBD$(2n, 2, 1)$) exists, and for all $n \geq 1$ contains $2n - 1$ 1-factors [12].

Theorem 3.1. *Necessary conditions are sufficient for a* GDD$(n, 4, 5; \lambda_1, \lambda_2)$ *if* $n \equiv 2, 4(mod\ 6)$*, i.e. a* GDD$(6t + 2, 4, 5; 6t + 2, 3(6t + 1))$ *and a* GDD$(6t + 4, 4, 5; 6t + 4, 3(6t + 3))$ *exist.*

Proof. The construction provided in this proof uses a TD$(4, t)$, and it works for all $n = 2t$ except for $t = 2$ and $t = 6$ (since a TD$(4, t)$ does not exist for $t = 2$ or 6 from Lemma 2.6). Let $n = 2t$ where $t \neq 2$ and 6. A 1-factorization of a K^i_{2t} on $2t$ elements of G_i has $(2t - 1)$ 1-factors. Each 1-factor has t edges. Let F^i_j be the jth 1-factor. We partition $2t$ elements of G_i according to the edges of F^i_j, that is, $F^i_{j1}, F^i_{j2}, \ldots, F^i_{jt}$. Construct a TD$(4, t)$ on four groups $H_i = \{F^i_{j1}, F^i_{j2}, \ldots, F^i_{jt}\}$. From each block of the TD$(4, t)$, which gives naturally four groups each of size 2, we construct a GDD$(2, 4, 5; 2, 3)$ with 8 blocks. We repeat this for each 1-factor of the 1-factorization. A detailed counting gives the required values for λ_1 and λ_2 (see Example 3.2 below for an illustration of the construction).

For $t = 2$ (i.e., $n = 4$), a GDD$(4, 4, 5; \lambda_1, \lambda_2)$ exists from Theorem 2.1. For $t = 6$ (i.e., $n = 12$), it is considered in the case of a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ for $n \equiv 0(\text{mod } 6)$, a GDD$(6t + 2, 4, 5; 6t + 2, 3(6t + 1))$ and a GDD$(6t + 4, 4, 5; 6t + 4, 3(6t + 3))$ exist. □

Example 3.2. *A* GDD$(6, 4, 5; 6, 15)$ *based on the construction procedure in Theorem 3.1 is as follows.*

Here $t = 3$ and we want to construct a GDD$(6, 4, 5; 6, 15)$. The number of blocks for the GDD is 360. We start with a TD$(4, 3)$. If we use the groups $\{A_1, A_2, A_3\}$, $\{B_1, B_2, B_3\}$, $\{C_1, C_2, C_3\}$, and $\{D_1, D_2, D_3\}$ then the blocks of the TD are $\{\{A_1, B_1, C_1, D_1\}$, $\{A_1, B_2, C_2, D_2\}$, $\{A_1, B_3, C_3, D_3\}$, $\{A_2, B_1, C_2, D_3\}$, $\{A_2, B_2, C_3, D_1\}$, $\{A_2, B_3, C_1, D_2\}$, $\{A_3, B_1, C_3, D_2\}$, $\{A_3, B_2, C_1, D_3\}$, $\{A_3, B_3, C_2, D_1\}\}$.

Take a RBIBD$(6, 2, 1)$ on G_i and call it β_i. Essentially this is a 1-factorization on K^i_6, a complete graph on six vertices where the vertices are the elements of group G_i and we have $6 - 1 = 5$ 1-factors. The sets G_i are given as, say, $G_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $G_2 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, $G_3 = \{z_1, z_2, z_3, z_4, z_5, z_6\}$, and $G_4 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. Taking, for example, the set G_i, with $i = 1$, the five 1-factors will appear as $F^i_1 = \{(x_1, x_2), (x_3, x_4), (x_5, x_6)\}$, $F^i_2 = \{(x_1, x_3), (x_2, x_5), (x_4, x_6)\}$, $F^i_3 = \{(x_1, x_4), (x_2, x_6), (x_3, x_5)\}$, $F^i_4 = \{(x_1, x_5), (x_2, x_4), (x_3, x_6)\}$, and $F^i_5 = \{(x_1, x_6), (x_2, x_3), (x_4, x_5)\}$.

In general, we write F^i_j, where i, j are the group number and one-factor position, respectively. Take for example, $j = 3$ and $i = 1, 2, 3, 4$. This gives $F^1_3 = \{(x_1, x_4), (x_2, x_6), (x_3, x_5)\}$, $F^2_3 = \{(y_1, y_4), (y_2, y_6), (y_3, y_5)\}$, $F^3_3 = \{(z_1, z_4), (z_2, z_6), (z_3, z_5)\}$, and $F^4_3 = \{(w_1, w_4), (w_2, w_6), (w_3, w_5)\}$.

Construct a TD$(4, 3)$ where the groups are $\{F^1_{j1}, F^1_{j2}, F^1_{j3}\}$, $\{F^2_{j1}, F^2_{j2}, F^2_{j3}\}$, $\{F^3_{j1}, F^3_{j2}, F^3_{j3}\}$, and $\{F^4_{j1}, F^4_{j2}, F^4_{j3}\}$. Take each block of the transversal design, for example, the first block, $\{\{x_1, x_4\}, \{y_1, y_4\}, \{z_1, z_4\}, \{w_1, w_4\}\}$. The elements of this block give the groups for a GDD$(2, 4, 5; 2, 3)$. The second block will have the groups $\{\{x_1, x_4\}, \{y_2, y_6\}, \{z_2, z_6\}, \{w_2, w_6\}\}$, and so on, up to the ninth block with groups $\{\{x_3, x_5\}, \{y_3, y_5\}, \{z_2, z_6\}, \{w_1, w_4\}\}$. From each of these nine blocks from a 1-factor we get $9 \times 8 = 72$ blocks. From 5 1-factors, we have constructed 360 required blocks for a GDD$(6, 4, 5; 6, 15)$. Now we show that $\lambda_1 = 6$ and $\lambda_2 = 15$. For example, observe that the pair $\{x_1, x_4\}$ appears in one 1-factor. Through that 1-factor, the element $\{x_1, x_4\}$ appears in three blocks of the TD$(4,3)$. In each block of the TD$(4,3)$, the pair $\{x_1, x_4\}$ appears two times. Thus, $\lambda_1 = 6$. On the other hand, the pair $\{x_1, y_1\}$ appears in 3 blocks of the GDD$(2,4,5;2,3)$. Since there are five 1-factors, we get $\lambda_2 = 15$.

Remark 3.3. *Theorem 3.1 provides constructions for a* GDD$(n,4,5;n,\ 3(n-1))$ *for* $n = 2t$ ($n \neq 4$ *and* 12).

4. GDD$(n,4,5;\lambda_1,\lambda_2)$ for $n \equiv 0,3 (\text{mod } 6)$

If $n \equiv 0,3(\text{mod } 6)$, then $n \equiv 0(\text{mod } 3) = 3s$. The minimum value of λ_1 is $\frac{n}{3} = s$ by Corollary 1.3. Thus, if a GDD$(n,4,5;s,3s-1)$ exists, then a GDD$(n,4,5;\lambda_1,\lambda_2)$ for $n \equiv 0(\text{mod } 3)$ exists by Theorem 1.1 and Remark 1.4.

Theorem 4.1. *Necessary conditions are sufficient for a* GDD$(n,4,5;\lambda_1,\ \lambda_2)$ *if* $n \equiv 3(\text{mod } 6)$, *i.e., a* GDD$(6t+3,4,5;2t+1,6t+2)$ *exists for* $t \geq 0$.

Proof. We know that a TD$(4,2t+1)$ exists (Lemma 2.6) and has a replication number $2t+1$. We also know that a RBIBD$(6t+3,3,1)$ exists and has $3t+1$ parallel classes [1]. We also have a GDD$(3,4,5;1,2)$. We wish to construct a GDD$(6t+3,4,5;2t+1,6t+2)$. Let the groups be $G_1 = \{a_1, a_2, \ldots, a_{6t+3}\}$, $G_2 = \{b_1, b_2, \ldots, b_{6t+3}\}$, $G_3 = \{c_1, c_2, \ldots, c_{6t+3}\}$, and $G_4 = \{d_1, d_2, \ldots, d_{6t+3}\}$. Let $\pi_1,\ \pi_2, \ldots,\ \pi_{3t+1}$ be parallel classes of a RBIBD$(6t+3,3,1)$ on $\{1,2,\ldots,6t+3\}$. Use each π_i to partition each of the four groups by relabelling the elements, i.e., if $\{j_1, j_2, j_3\}$ is the jth block of π_i, then the jth partition set G_{1j} of G_1 is $\{a_{j_1}, a_{j_2}, a_{j_3}\}$. Similarly for other G_i, for $i = 2,3,4$. Use a TD$(4,2t+1)$ on groups $\{G_{i1}, G_{i2}, \ldots, G_{i,2t+1}\}$, $i = 1,2,3,4$. If a block of the TD$(4,2t+1)$ is $\{G_{1r}, G_{2s}, G_{3t}, G_{4u}\}$, construct a GDD$(3,4,5;1,2)$ on groups G_{1r}, G_{2s}, G_{3t}, and G_{4u}. The union of all the blocks of the GDDs thus constructed using all the π_i's is a GDD$(6t+3,4,5;2t+1,6t+2)$. Clearly $\lambda_1 = 2t+1$ because in a TD$(4,2t+1)$ each element occurs $2t+1$ times. It means that G_{ij} will be in $2t+1$ blocks of the TD and hence, when GDD$(3,4,5;1,2)$ is formed with G_{ij} as one of the groups, pairs of elements within G_{ij} will occur $2t+1$ times. Also, λ_2 is $6t+2$ because pairs of elements between G_i and G_j $(i \neq j)$ occur twice for each parallel class π_i and there are $3t+1$ parallel classes. □

For $n \equiv 0(\text{mod } 6) = 6t$, we provide constructions for a GDD$(6t,4,5;4t,\ 2(6t-1))$ where $\lambda_1 = 4t$ is not of its minimum value (but twice of its minimum value which is $2t$).

Example 4.2. *A construction of a* GDD$(12,4,5;8,22)$ *is as follows.*

First note that a TD$(4,4)$ exists (by Lemma 2.6), and it has 16 blocks of size 4. Let G_1, G_2, G_3, and G_4 be the groups, each of size four for a GDD$(12,4,5;8,22)$ which we wish to construct. Let $H_i = \{G_{i1}, G_{i2}, G_{i3},\ G_{i4}\}$, $i = 1,2,3,4$, where each G_{ij} has size 3. We construct TD$(4,4)$s on groups $H = \{H_1, H_2, H_3, H_4\}$. Any G_{ij} is in four blocks. But a block of H gives four subsets each of size 3. So these groups can be used to get a GDD$(3,4,5;1,2)$. Do this for each block of H. The pair of elements within G_{ij} occur four times and the pairs from $G_{ij}, G_{st}, i \neq s$ occur two times.

Now, construct a RBIBD$(12,3,2)$ with 11 parallel classes. Use each of the parallel classes and apply the construction. As pairs in BIBD appear twice we have any two elements (a,b) from G_i in eight blocks and (c,d) where $c \in G_i$ and $d \in G_j$, $i \neq j$ occur 22 times.

Remark 4.3. *A* GDD$(6t,4,5;4t,2(6t-1))$ *where* $t \neq 1$ *and* 3 *can be constructed using a* TD$(4,2t)$ *and a* RBIBD$(6t,3,2)$.

By Lemma 2.6, a TD$(4,n)$ exist except for $n = 2$ and 6. Use similar ideas as in Example 4.2, we construct a GDD$(6t,4,5;4t,2(6t-1))$ except for $t = 1$ and 3 using a TD$(4,n = 2t)$ and a GDD$(3,4,5;1,2)$. We use a partition of $6t$ elements according to the parallel classes π_{ij} of a RBIBD$(6t,3,2)$ on G_i, $i = 1,2,3,4$ and $j = 1,2,\cdots,6t-1$. Note for a RBIBD$(6t,3,2)$, there are $6t-1$ parallel classes and the number of blocks is $b = 2t(6t-1)$. We use a partition π_s, to get groups, say G_{ij}, $i = 1, 2, 3, 4$ and $j = 1, 2, \cdots, 2t$. The pair of elements within G_{ij} occur $2t$ times and pairs from G_{ij}, G_{st}, $i \neq s$ occur two times. This construction is repeated $6t-1$ times once for each of the $6t-1$ parallel classes. Therefore elements from the same group will occur $4t$ times and pairs of elements from two different groups will occur $2(6t-1)$ times and we have GDD$(6t, 4, 5; 4t, 2(6t-1))$ where $t \neq 1$ and 3.

To completely solve the case for $n \equiv 0 (\text{mod } 6) = 6t$, one should construct a GDD$(6t, 4, 5; 2t, 6t-1)$.

5. Summary

In this paper we studied constructions and results about GDD$(n, 4, 5; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 1, 2)$. We provide the necessary conditions of the existence of a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 1, 2)$, and show that the necessary conditions are sufficient for a GDD$(n, 4, 5; \lambda_1, \lambda_2)$ with Configuration $(1, 1, 1, 2)$ if $n \not\equiv 0 (\text{mod } 6)$, respectively. We also show that a GDD$(n, 4, 5; 2n, 6(n-1))$ with Configuration $(1, 1, 1, 2)$ exists, and provide constructions for a GDD$(n = 2t, 4, 5; n, 3(n-1))$ with Configuration $(1, 1, 1, 2)$ where $n \neq 12$, and a GDD$(n = 6t, 4, 5; 4t, 2(6t-1))$ with Configuration $(1, 1, 1, 2)$ where $n \neq 6$ and 18, respectively. The remaining case of the problem is to show that the necessary conditions are sufficient for $n \equiv 0 (\text{mod } 6)$, i.e., to show the existence of a GDD$(6t, 4, 5; 2t, 6t-1)$ with Configuration $(1, 1, 1, 2)$.

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