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Interaction of Codazzi Pairs with Almost Para Norden Manifolds

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Abstract. In this paper, we research some properties of Codazzi pairs on almost para Norden manifolds. Let $(M_{2n},\ \varphi,\ g, G)$ be an almost para Norden manifold. Firstly, g-conjugate connection, G-conjugate connection and φ-conjugate connection of a linear connection ∇ on M_{2n} denoted by ∇^* , $\nabla^\dagger$ and ∇^φ are defined and it is demonstrated that on the spaces of linear connections, $(id,\ *, \dagger, \varphi)$ acts as the four-element Klein group. We also searched some properties of these three types conjugate connections. Then, Codazzi pairs (∇, φ) , (∇, g) and (∇, G) are introduced and some properties of them are given. Let R , R^* and $R^\dagger$ are $(0, 4)$-curvature tensors of conjugate connections ∇ , ∇^* and $\nabla^\dagger$, respectively. The relationship among the curvature tensors is investigated. The condition of $N_\varphi = 0$ is obtained, where N_φ is Nijenhuis tensor field on M_{2n} and it is known that the condition of integrability of almost para complex structure φ is $N_\varphi = 0$. In addition, Tachibana operator is applied to the pure metric g and a necessary and sufficient condition $(M,\ \varphi,\ g, G)$ being a para Kahler Norden manifold is found. Finally, we examine φ-invariant linear connections and statistical manifolds.

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1. Introduction

Codazzi tensors are an important issue encountered in various fields of geometry. The conditions of being Codazzi tensors for various types of tensors and the properties of Codazzi tensors have been studied by many authors in [4–7, 10, 12, 13, 18, 23–29].

When we study this subject in the paper, we are inspired by two papers [10] and [13]. In the first paper [10], they research Codazzi couplings of an affine connection ∇ with a tangent bundle isomorphism L on smooth manifolds, a pseudo-Riemannian metric g and a nondegenerate 2-form ω. They consider Codazzi (para) Kahler structures which are a generalization of special (para) Kahler structure, without requiring ∇ to be flat. Moreover, they give some results about ω-conjugate, g-conjugate and L-gauge tarnsformations of ∇, along with identity, form an involute Abelian group. Their findings indicate that any statistical manifold might admit a (para) Kahler structure in the condition that L which is compatible to g and Codazzi coupled with ∇ can be found. In the second paper [13], the authors first define conjugate connections of linear connections regarding Norden metric g, twin Norden metric G and almost complex structure J. They give relationship between curvature tensors of conjugate connections. Moreover, they prove conjugations along with an identity operation together act as a Klein group. In addition, some properties of Codazzi pairs (∇,J) and (∇,G) are given. They give a necessary and sufficient condition the an almost anti-Hermitian manifold $(M,\ J,\ g, G)$ is an

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anti-Kähler relative to a torsion-free linear connection ∇, assuming (∇, J) being a Codazzi pair. Finally, they give some results about statistical structures on M.

Based on all these studies, in this paper, we consider some properties of Codazzi pairs on almost para Norden manifolds. In section 2, we give some basic definitions which we use later. In section 3, we first introduce three types of conjugate connections of linear connections relative to φ, g and G. These connections are called φ-conjugate connection, g-conjugate connection and G-conjugate connection and signed by ∇^{φ}, ∇^{*} and $\nabla^{\dagger}$ respectively. We obtain, $(id, *, \dagger, \varphi)$ acts as the 4-element Klein group on the space of linear connections. In section 4, we introduce Codazzi pair of ∇ with φ, Codazzi pair of ∇ with g , Codazzi pair of ∇ with G and searched properties of these pairs. In section 5, we consider the relationship among the $(0,4)$-curvature tensor fields of conjugate connections ∇^{φ}, ∇^{*}, $\nabla^{\dagger}$ and we get $R(X,Y,\varphi Z,T) = -R^{*}(X,Y,T,\varphi Z) = R^{\varphi}(X,Y,Z,\varphi T)$. In section 6, we compute Nijenhuis tensor N_{φ} and obtained $N_{\varphi}(X,Y) = 0$ if (∇, φ) is a Codazzi pair. Later, Φ-operator (or Tachibana operatör) applied to pure tensors is applied to the pure tensor g and found $\left(\Phi_{\varphi}G\right)(X,Y,Z) = \left(\Phi_{\varphi}g\right)(X,\varphi Y,Z) = \left(\nabla_{\varphi X}G\right)(Y,Z) - (\nabla_{X}g)(\varphi Y,\varphi Z)$ if (∇,φ) is a Codazzi pair. Moreover, the necessary and sufficient condition is obtained for the (M_{2n},φ,g,G) to be para Kahler Norden. In section 7, we study φ-invariant linear connections. Under what conditions do we find that ∇^{*} and $\nabla^{\dagger}$ are φ-invariant. In section 8, finally we investigate statistical manifolds.

2. Preliminaries

Let M_{2n} be a Riemannian manifold with neutral metric, i.e., with pseudo-Riemannian metric g of signature (n,n). In this paper we assume manifolds, connections and tensor fields to be differentiable and of class C^{∞}. The set of all tensor fields of type (p,q) on M_{2n} is denoted by $\Im_{q}^{p}(M_{2n})$.

Let M_{2n} be a differentiable manifold. φ is called an almost paracomplex structure if $\varphi^2 = id$ for the an affinor field $\varphi \in \Im_{1}^{1}(M_{2n})$. The pair (M_{2n},φ) is called an almost paracomplex manifold [14]. The metric g is an almost para Norden metric relative to φ if

$$g(\varphi X,\varphi Y) = g(X,Y)$$

or equivalently

$$g(\varphi X,Y) = g(X,\varphi Y)$$

for any $X,Y \in \Im_{0}^{1}(M_{2n})$ [21]. These metrics have also been called pure metrics, B-metrics and anti-Hermitian metrics in ([9,11,15,19,30,31]). In this case, (M_{2n}, φ, g) is called an almost para Norden manifold. (M_{2n}, φ, g) is called a para Norden manifold, if φ is integrable. It is known that the condition of integrability of almost para complex structure φ is $N_{\varphi} = 0$, where N_{φ} is Nijenhuis tensor field on M_{2n} defined by

$$N_{\varphi}(X,Y) = [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y] + [X,Y].$$

On the other hand, the paracomplex structure φ is intagrable if and only if $\nabla\varphi = 0$, where ∇ is a torsion-free linear connection.

Let (M_{2n}, φ, g) be an almost para Norden manifold. G is defined as

$$G(X,Y) = (g\circ\varphi)(X,Y) = g(\varphi X,Y)$$

by means of the para Norden metric g, is called almost twin para Norden metric for all vector fields $X,Y \in \Im_{0}^{1}(M_{2n})$ [21]. From now on, we assign the quadruple (M_{2n}, φ, g, G) as an almost para Norden manifold. We can easily see that:

$$\begin{aligned} G(X,Y) &= (g\circ\varphi)(X,Y) = g(\varphi X,Y) \\ &= g(X,\varphi Y) \\ &= G(Y,X), \end{aligned}$$

and

$$\begin{aligned} G(\varphi X,Y) &= (g\circ\varphi)(\varphi X,Y) = g\left(\varphi^2 X,Y\right) \\ &= g(\varphi X,\varphi Y) = (g\circ\varphi)(X,\varphi Y) \\ &= G(X,\varphi Y). \end{aligned}$$

The covariant differentiation of the Levi-Civita connection of g denoted by ∇_g. In this case, we have

$$\nabla_g G = \left(\nabla_g g\right) \circ \varphi + g \circ \left(\nabla_g \varphi\right) = g \circ \left(\nabla_g \varphi\right)$$

and from Theorem 2 in [22], $\nabla_g G = 0$ is obtained.

3. Conjugate Connections

Let ∇ be a linear connection on $(M_{2n},\ \varphi,\ g, G)$. The conjugate connections of ∇ with respect to $\varphi,\ g$ and G are defined as the linear connections given with the following equations:

$$\nabla^{\varphi}(X, Y) = \nabla^{\varphi}_X Y = \varphi^{-1}\left(\nabla_X \varphi Y\right),$$

$$Zg\,(X, Y) = g\,(\nabla_Z X, Y) + g(X, \nabla^*_Z Y), \tag{3.1}$$

$$ZG\,(X, Y) = G\,(\nabla_Z X, Y) + G(X, \nabla^{\dagger}_Z Y), \tag{3.2}$$

for all vector fields X, Y, Z on M_{2n}. These connections ∇^{φ}, ∇^* and $\nabla^{\dagger}$ are called φ-conjugate connection, g-conjugate connection and G-conjugate connection, respectively. The conjugate connections have been studied by many authors in [1–3, 8, 10, 13, 17]. From the relationship among these connections of ∇, we can write the theorem below.

Theorem 3.1. *Let $(M_{2n},\ \varphi,\ g, G)$ be an almost para Norden manifold. φ-conjugate connection, g-conjugate connection and G-conjugate connection of a linear connection ∇ are denoted by respectively ∇^{φ}, ∇^* and $\nabla^{\dagger}$. Hence, $(id,\ *, \dagger, \varphi)$ realizes a 4-element Klein group action on the space of linear connections:*

1. $(\nabla^*)^* = (\nabla^{\dagger})^{\dagger} = (\nabla^{\varphi})^{\varphi} = \nabla$,
2. $(\nabla^{\dagger})^{\varphi} = (\nabla^{\varphi})^{\dagger} = \nabla^*$,
3. $(\nabla^*)^{\varphi} = (\nabla^{\varphi})^* = \nabla^{\dagger}$,
4. $(\nabla^*)^{\dagger} = (\nabla^{\dagger})^* = \nabla^{\varphi}$.

Proof. **i.** The definition of conjugate connections leads to the statement.

ii. We calculate

$$\begin{aligned} G\left((\nabla^{\dagger})^{\varphi}_Z X, Y\right) &= G\left(\varphi^{-1}{\nabla_Z}^{\dagger}\varphi X, Y\right) \\ &= G\left({\nabla_Z}^{\dagger}\varphi X, \varphi Y\right) = G\left(\varphi Y, {\nabla_Z}^{\dagger}\varphi X\right) \\ &= ZG\,(\varphi Y, \varphi X) - G\,(\nabla_Z \varphi Y, \varphi X) \\ &= Zg\,(Y, \varphi X) - g\,(\varphi \nabla_Z \varphi Y, \varphi X) \\ &= Zg\,(\varphi Y, X) - g\,(\nabla_Z \varphi Y, X) \\ &= g\,(\varphi Y, \nabla^*_Z X) \\ &= G(\nabla^*_Z X, Y) \end{aligned}$$

which gives $(\nabla^{\dagger})^{\varphi} = \nabla^*$. From the definition of the G-conjugate connection given in (3.2), we compute

$$\begin{aligned} ZG\,(X, Y) &= G\left(\nabla^{\varphi}_Z X, Y\right) + G(X, (\nabla^{\varphi})^{\dagger}_Z Y), \\ Zg\,(\varphi X, Y) &= g\left(\varphi \nabla^{\varphi}_Z X, Y\right) + g(\varphi X, (\nabla^{\varphi})^{\dagger}_Z Y), \\ Zg\,(\varphi X, Y) &= g\left(\varphi \varphi^{-1} \nabla_Z \varphi X, Y\right) + g(\varphi X, (\nabla^{\varphi})^{\dagger}_Z Y, \\ Zg\,(\varphi X, Y) - g\,(\nabla_Z \varphi X, Y) &= g(\varphi X, (\nabla^{\varphi})^{\dagger}_Z Y), \\ g\,(\varphi X, \nabla^*_Z Y) &= g(\varphi X, (\nabla^{\varphi})^{\dagger}_Z Y), \end{aligned}$$

which gives $(\nabla^{\varphi})^{\dagger} {=} \nabla^*$. Thus, we obtain $(\nabla^{\dagger})^{\varphi} = (\nabla^{\varphi})^{\dagger} = \nabla^*$.

iii. We calculate

$$\begin{aligned} g(\varphi X, (\nabla^{\varphi})^{*}_{Z} Y) &= Zg(\varphi X, Y) - g\left(\nabla^{\varphi}_{Z}\varphi X, Y\right) \\ &= ZG(X, Y) - G(\varphi^{-1}\nabla^{\varphi}_{Z}\varphi X, Y) \\ &= ZG(X, Y) - G\left(\varphi\left(\varphi^{-1}\left(\nabla_{Z}\varphi^{2}X\right)\right), Y\right) \\ &= ZG(X, Y) - G(\nabla_{Z}X, Y) \\ &= G\left(X, \nabla^{\dagger}_{Z}Y\right) = g(\varphi X,\ \nabla^{\dagger}_{Z}Y) \end{aligned}$$

which gives $(\nabla^{\varphi})^{*}=\nabla^{\dagger}$. Similarly,

$$\begin{aligned} g\left((\nabla^{*})^{\varphi}_{Z}X, Y\right) &= g\left(\varphi^{-1}\nabla_{Z}{}^{*}\varphi X, Y\right) = g(\varphi Y, \nabla_{Z}{}^{*}\varphi X) \\ &= Zg(\varphi Y, \varphi X) - g(\nabla_{Z}\varphi Y, \varphi X) \\ &= Zg(\varphi Y, \varphi X) - g(\varphi\nabla_{Z}\varphi Y, X) \\ &= ZG(Y, \varphi X) - G(\nabla_{Z}\varphi Y, X) \\ &= ZG(\varphi Y, X) - G(\nabla_{Z}\varphi Y, X) \\ &= G\left(\varphi Y, \nabla^{\dagger}_{Z}X\right) = g\left(Y, \nabla^{\dagger}_{Z}X\right) \\ &= g(\nabla^{\dagger}_{Z}X, Y) \end{aligned}$$

which gives $(\nabla^{*})^{\varphi} = \nabla^{\dagger}$. Thus, we obtain $(\nabla^{\varphi})^{*}=(\nabla^{*})^{\varphi}=\nabla^{\dagger}$.

iv. From the definition of the g-conjugate connection given in (3.1), we compute

$$\begin{aligned} g\left(\varphi X, (\nabla^{\dagger})^{*}_{Z}Y\right) &= Zg(\varphi X, Y) - g\left(\nabla^{\dagger}_{Z}\varphi X, Y\right) \\ &= Zg(Y, \varphi X) - g(Y, \nabla^{\dagger}_{Z}\varphi X) \\ &= ZG\left(\varphi^{-1}Y, \varphi X\right) - G(\varphi^{-1}Y, \nabla^{\dagger}_{Z}\varphi X) \\ &= ZG(\varphi Y, \varphi X) - G(\varphi Y, \nabla^{\dagger}_{Z}\varphi X) \\ &= G(\nabla_{Z}\varphi Y, \varphi X) = G(\varphi\nabla_{Z}\varphi Y, X) = G\left(\varphi^{-1}\nabla_{Z}\varphi Y, X\right) \\ &= G\left(\nabla^{\varphi}_{Z}Y, X\right) \\ &= g(\varphi X, \nabla^{\varphi}_{Z}Y) \end{aligned}$$

which gives $(\nabla^{\dagger})^{*} = \nabla^{\varphi}$. Similarly, from the definition of the G-conjugate connection given in (3.2), we compute

$$\begin{aligned} ZG(X, Y) &= G(\nabla^{*}_{Z}X, Y) + G(X, (\nabla^{*})^{\dagger}_{Z}Y), \\ Zg(\varphi X, Y) &= g(\varphi\nabla^{*}_{Z}X, Y) + g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ Zg(\varphi Y, X) - g(\varphi Y, \nabla^{*}_{Z}X) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ g(\nabla_{Z}\varphi Y, X) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ g(\varphi\nabla_{Z}\varphi Y, \varphi X) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ g\left(\varphi^{-1}\nabla_{Z}\varphi Y, \varphi X\right) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ g\left(\nabla^{\varphi}_{Z}Y, \varphi X\right) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \\ g\left(\varphi X, \nabla^{\varphi}_{Z}Y\right) &= g(\varphi X, (\nabla^{*})^{\dagger}_{Z}Y), \end{aligned}$$

which gives $(\nabla^{*})^{\dagger} =\nabla^{\varphi}$. Thus, $(\nabla^{\dagger})^{*} = (\nabla^{*})^{\dagger} = \nabla^{\varphi}$.

The proof is completed. □

4. Codazzi Pairs

4.1. **Codazzi Pair of ∇ with φ.** Let ∇ be a linear connection and φ be an almost paracomplex structure on M_{2n}. We demonstrate that the pair (∇, φ) is a Codazzi pair, if the identity is given below holds

$$(\nabla_X \varphi) Y = (\nabla_Y \varphi) X,$$

for all vector fields X and Y on M_{2n}.

Theorem 4.1. *Let ∇ be a linear connection and φ be an almost para complex structure on M_{2n}. If (∇, φ) is a Codazzi pair, the followings are equivalent:*
1. ∇ and ∇^{φ} have equal torsions.
2. $\left(\nabla^{\varphi}, \varphi^{-1}\right)$ is a Codazzi pair.

Proof. **i.** Let (∇, φ) is a Codazzi pair. The torsion of connection ∇^{φ} is given as follows;

$$S^{\nabla^{\varphi}}(X, Y) = \nabla^{\varphi}_X Y - \nabla^{\varphi}_Y X - [X, Y].$$

Based on this, we obtain

$$\begin{aligned} S^{\nabla^{\varphi}}(X, Y) - S^{\nabla}(X, Y) &= \nabla^{\varphi}_X Y - \nabla_X Y - \nabla^{\varphi}_Y X + \nabla_Y X \\ &= \varphi^{-1}(\nabla_X \varphi Y) - \nabla_X Y - \varphi^{-1}(\nabla_Y \varphi X) + \nabla_Y X \\ &= \varphi\left[(\nabla_X \varphi) Y + \varphi(\nabla_X Y)\right] - \nabla_X Y - \varphi\left[(\nabla_Y \varphi) X + \varphi(\nabla_Y X)\right] + \nabla_Y X \\ &= \varphi\left[(\nabla_X \varphi) Y - (\nabla_Y \varphi) X\right] \\ &= 0, \end{aligned}$$

which gives $S^{\nabla} = S^{\nabla^{\varphi}}$.

ii. Let (∇, φ) be a Codazzi pair. To show that $\left(\nabla^{\varphi}, \varphi^{-1}\right)$ is a Codazzi pair, it is necessary to show that $\left(\nabla^{\varphi}_X \varphi^{-1}\right) Y - \left(\nabla^{\varphi}_Y \varphi^{-1}\right) X = 0$.

$$\begin{aligned} \left(\nabla^{\varphi}_X \varphi^{-1}\right) Y - \left(\nabla^{\varphi}_Y \varphi^{-1}\right) X &= \left(\nabla^{\varphi}_X \varphi\right) Y - \left(\nabla^{\varphi}_Y \varphi\right) X \\ &= \nabla^{\varphi}_X(\varphi Y) - \varphi\left(\nabla^{\varphi}_X Y\right) - \nabla^{\varphi}_Y(\varphi X) + \varphi\left(\nabla^{\varphi}_Y X\right) \\ &= \varphi^{-1}(\nabla_X Y) - \varphi\left(\varphi^{-1}\nabla_X \varphi Y\right) - \varphi^{-1}(\nabla_Y X) + \varphi\left(\varphi^{-1}\nabla_Y \varphi X\right) \\ &= \varphi(\nabla_X Y) - \left[(\nabla_X \varphi) Y + \varphi(\nabla_X Y)\right] - \varphi(\nabla_Y X) + \left[(\nabla_Y \varphi) X + \varphi(\nabla_Y X)\right] \\ &= -(\nabla_X \varphi) Y + (\nabla_Y \varphi) X \\ &= 0. \end{aligned}$$

On the contrary, if $(\nabla^{\varphi}, \varphi^{-1})$ is a Codazzi pair, (∇, φ) is so.
The proof is completed. □

Theorem 4.2. *Let ∇ be a linear connection and φ be an almost para complex structure on M_{2n}. $(\nabla^{\varphi}, \varphi)$ is a Codazzi pair if and only if (∇, φ) is so.*

Proof. Let (∇, φ) be a Codazzi pair.

$$\begin{aligned} \left(\nabla^{\varphi}_X \varphi\right) Y - \left(\nabla^{\varphi}_Y \varphi\right) X &= \left(\nabla^{\varphi}_X \varphi Y\right) - \varphi\left(\nabla^{\varphi}_X Y\right) - \left(\nabla^{\varphi}_Y \varphi X\right) + \varphi\left(\nabla^{\varphi}_Y X\right) \\ &= \varphi^{-1}(\nabla_X Y) - \varphi\left(\varphi^{-1}\nabla_X \varphi Y\right) - \varphi^{-1}(\nabla_Y X) + \varphi\left(\varphi^{-1}\nabla_Y \varphi X\right) \\ &= \varphi(\nabla_X Y) - \left[(\nabla_X \varphi) Y + \varphi\nabla_X Y\right] - \varphi(\nabla_Y X) + \left[(\nabla_Y \varphi) X + \varphi\nabla_Y X\right] \\ &= -(\nabla_X \varphi) Y + (\nabla_Y \varphi) X \\ &= 0, \end{aligned}$$

which gives $(\nabla^{\varphi}, \varphi)$ is a Codazzi pair. On the contrary, if $(\nabla^{\varphi}, \varphi)$ is a Codazzi pair, (∇, φ) is so.
The proof is completed. □

4.2. **Codazzi Pair of ∇ with g.** Let ∇ be a linear connection and g be an almost para Norden metric on M_{2n}. We demonstrate that the pair (∇, g) is a Codazzi pair, if

$$(\nabla_Z g)(X,Y) = (\nabla_X g)(Z,Y) \tag{4.1}$$

holds for all vector fields X, Y and Z on M_{2n}.
Let (∇, g) be a Codazzi pair on M_{2n}. The $(0,3)$-tensor field C defined as

$$C(X,Y,Z) = (\nabla_Z g)(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \tag{4.2}$$

and called cubic form of Codazzi pair (∇, g). Since g is symetric, the tensor C is also symetric, i.e., $C(X,Y,Z) = C(Y,X,Z)$.
In addition, the necessary and sufficient condition for g to be parallel with respect to ∇ is that $C \equiv 0$.
With the substitution of (3.1) into (4.2), we have

$$C(X,Y,Z) = (\nabla_Z g)(X,Y) = Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = g(X, (\nabla^* - \nabla)_Z Y).$$

Besides, $(0,3)$-tensor field C^* is defined as

$$C^*(X,Y,Z) = (\nabla^*_Z g)(X,Y)$$

and is obtained

$$\begin{aligned} C^*(X,Y,Z) = (\nabla^*_Z g)(X,Y) &= Zg(X,Y) - g(\nabla^*_Z X, Y) - g(X, \nabla^*_Z Y) \\ &= -g(X, (\nabla^* - \nabla)_Z Y) = -C(X,Y,Z). \end{aligned}$$

If (∇, g) is a Codazzi pair, from equation (4.1), we have $C(X,Y,Z) = C(Z,Y,X)$.

Theorem 4.3. *Let (M_{2n}, g) be an almost para Norden manifold, ∇ be a linear connection and ∇^* be g-conjugate connection. In this case, the followings are equivalent:*
1. *(∇, g) is a Codazzi pair.*
2. *(∇^*, g) is a Codazzi pair.*
3. *C is totally symmetric.*
4. *$S^{\nabla} = S^{\nabla^*}$.*

Proof. **i. ⇒ ii.** If (∇, g) is a Codazzi pair, we have

$$\begin{aligned} (\nabla^*{}_Z g)(X,Y) - (\nabla^*{}_X g)(Z,Y) &= Zg(X,Y) - g(\nabla^*_Z X, Y) - g(X, \nabla^*_Z Y) \\ &\quad - Xg(Z,Y) + g(\nabla^*_X Z, Y) + g(Z, \nabla^*_X Y) \\ &= -g(X, (\nabla^* - \nabla)_Z Y) + g(Z, (\nabla^* - \nabla)_X Y) \\ &= -C(X,Y,Z) + C(Z,Y,X) \\ &= 0. \end{aligned}$$

So, (∇^*, g) is a Codazzi pair.
ii. ⇒ iii. Due to symmetry of g,

$$C(X,Y,Z) = (\nabla_Z g)(X,Y) = (\nabla_Z g)(Y,X) = C(Y,X,Z).$$

For (∇, g) being a Codazzi pair,

$$C(X,Y,Z) = (\nabla_Z g)(X,Y) = (\nabla_X g)(Z,Y) = C(Z,Y,X).$$

Also due to symmetry of g and for (∇, g) being a Codazzi pair,

$$C(X,Y,Z) = (\nabla_Z g)(X,Y) = (\nabla_Z g)(Y,X) = (\nabla_Y g)(Z,X) = (\nabla_Y g)(X,Z) = C(X,Z,Y).$$

That is, C is totally symmetric in all of its indices.

iii. ⇒ iv. If C is totally symmetric, we have

$$\begin{aligned} C(X,Y,Z) - C(X,Z,Y) &= (\nabla_Z g)(X,Y) - (\nabla_Y g)(X,Z) \\ &= Zg(X,Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) - Yg(X,Z) - g(\nabla_Y X, Z) - g(X, \nabla_Y Z) \\ &\Rightarrow g(X,\ (\nabla^* - \nabla)_Z Y - (\nabla^* - \nabla)_Y Z) = 0 \\ &\Rightarrow g\left(X, S^{\nabla^*}(Z,Y) - S^{\nabla}(Z,Y)\right) = 0 \\ &\Rightarrow S^{\nabla} = S^{\nabla^*} \end{aligned}$$

where, $S^{\nabla^*}(Z,Y) = \nabla^*_Z Y - \nabla^*_Y Z - [Z,Y]$.

The proof is completed. □

Corollary 4.4. *Let (M_{2n}, g) be an almost para Norden manifold, ∇ and ∇^* be connections respectively linear and g-conjugate. For the tensors C^* and C, we have $C^*(X,Y,Z) = -C(X,Y,Z)$. So, C^* is totally symmetric if and only if C is so.*

4.3. **Codazzi Pair of ∇ with G.** Let ∇ be a linear connection and G be a twin metric on M_{2n}. We demonstrate that the pair (∇, G) is a Codazzi pair, if

$$(\nabla_Z G)(X,Y) = (\nabla_X G)(Z,Y) \tag{4.3}$$

holds for all vector fields X, Y and Z on M_{2n}.

Let (∇, G) be a Codazzi pair on M_{2n}. $\mathcal{F}$ is the $(0,3)$-tensor field and defined as

$$\mathcal{F}(X,Y,Z) = (\nabla_Z G)(X,Y) = ZG(X,Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y). \tag{4.4}$$

Since G is symetric, the tensor $\mathcal{F}$ is also symetric, i.e., $\mathcal{F}(X,Y,Z) = \mathcal{F}(Y,X,Z)$.

With the substitution of (3.2) into (4.4), we have

$$\mathcal{F}(X,Y,Z) = (\nabla_Z G)(X,Y) = ZG(X,Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y) = G\left(X, \left(\nabla^\dagger - \nabla\right)_Z Y\right).$$

Besides, $(0,3)$-tensor field $\mathcal{F}^\dagger$ is defined as

$$\mathcal{F}^\dagger(X,Y,Z) = \left(\nabla^\dagger_Z G\right)(X,Y)$$

and is obtained

$$\begin{aligned} \mathcal{F}^\dagger(X,Y,Z) &= \left(\nabla^\dagger_Z G\right)(X,Y) = ZG(X,Y) - G\left(\nabla^\dagger_Z X, Y\right) - G\left(X, \nabla^\dagger_Z Y\right) \\ &= -G\left(X, \left(\nabla^\dagger - \nabla\right)_Z Y\right) = -\mathcal{F}(X,Y,Z). \end{aligned}$$

If (∇, G) is a Codazzi pair, from equation (4.3), we have $\mathcal{F}(X,Y,Z) = \mathcal{F}(Z,Y,X)$.

Theorem 4.5. *Let ∇ be a linear connection on (M_{2n}, φ, g, G). If (∇, G) is a Codazzi pair, the followings are provided:*

1. $\left(\nabla^*_{\varphi Z} G\right)(X,Y) = \left(\nabla^*_{\varphi X} G\right)(Z,Y)$.
2. $S^{\nabla} = S^{\nabla^*}$ *if and only if (∇^*, φ) is a Codazzi pair.*
3. $S^{(\nabla^*)^\varphi} = S^{\nabla}$.

Proof. **i.** If (∇, G) is a Codazzi pair, $(\nabla_Z G)(X,Y) = (\nabla_X G)(Z,Y)$. From here

$$\begin{aligned} ZG(X,Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y) &= XG(Z,Y) - G(\nabla_X Z, Y) - G(Z, \nabla_X Y), \\ Zg(\varphi X, Y) - g(\varphi \nabla_Z X, Y) - g(\varphi X,\ \nabla_Z Y) &= Xg(\varphi Z, Y) - g(\varphi \nabla_X Z, Y) - g(\varphi Z, \nabla_X Y), \\ Zg(Y, \varphi X) - g(\varphi \nabla_Z X, Y) - g(\ \nabla_Z Y, \varphi X) &= Xg(Y, \varphi Z) - g(\varphi \nabla_X Z, Y) - g(\nabla_X Y, \varphi Z), \\ g(Y, \nabla^*_Z(\varphi X)) - g(\nabla_Z X, \varphi Y) &= g(Y, \nabla^*_X(\varphi Z)) - g(\nabla_X Z, \varphi Y) \\ &\quad - Zg(X, \varphi Y) + g(X, \nabla^*_Z(\varphi Y)) + g(Y, \nabla^*_Z(\varphi X)) \\ &= -Xg(Z, \varphi Y) + g(Z, \nabla^*_X(\varphi Y)) + g(Y, \nabla^*_X(\varphi Z)) \end{aligned}$$

is obtained. Where φX, φY and φZ are written instead of X, Y and Z,

$$-\varphi Zg(\varphi X,Y)+g\left(\varphi X,\nabla^*_{\varphi Z}Y\right)+g\left(\varphi Y,\nabla^*_{\varphi Z}X\right)=-\varphi Xg(\varphi Z,Y)+g\left(\varphi Z,\nabla^*_{\varphi X}Y\right)+g\left(\varphi Y,\nabla^*_{\varphi X}Z\right),$$
$$-\varphi ZG(X,Y)+G\left(X,\nabla^*_{\varphi Z}Y\right)+G\left(Y,\nabla^*_{\varphi Z}X\right)=-\varphi XG(Z,Y)+G\left(Z,\nabla^*_{\varphi X}Y\right)+G\left(Y,\nabla^*_{\varphi X}Z\right),$$
$$-\varphi ZG(X,Y)+G(\nabla^*_{\varphi Z}X,Y)+G\left(X,\nabla^*_{\varphi Z}Y\right)=-\varphi XG(Z,Y)+G\left(\nabla^*_{\varphi X}Z,Y\right)+G\left(Z,\nabla^*_{\varphi X}Y\right),$$

so that

$$\left(\nabla^*_{\varphi Z}G\right)(X,Y)=\left(\nabla^*_{\varphi X}G\right)(Z,Y)$$

is obtained.

ii. If (∇,G) is a Codazzi pair, $(\nabla_Z G)(X,Y)=(\nabla_X G)(Z,Y)$. From here

$$\begin{aligned}
ZG(X,Y)-G(\nabla_Z X,Y)-G(X,\nabla_Z Y)&=XG(Z,Y)-G(\nabla_X Z,Y)-G(Z,\nabla_X Y),\\
Zg(\varphi X,Y)-g(\varphi\nabla_Z X,Y)-g(\varphi X,\ \nabla_Z Y)&=Xg(\varphi Z,Y)-g(\varphi\nabla_X Z,Y)-g(\varphi Z,\nabla_X Y),\\
Zg(Y,\varphi X)-g(\nabla_Z Y,\varphi X,)-g(\varphi\nabla_Z X,Y)&=Xg(Y,\varphi Z)-g(\nabla_X Y,\varphi Z)-g(\varphi\nabla_X Z,Y),\\
g(Y,\nabla^*_Z(\varphi X))-g(\varphi\nabla_Z X,Y)&=g(Y,\nabla^*_X(\varphi Z))-g(\varphi\nabla_X Z,Y),\\
g(\nabla^*_Z(\varphi X),Y)-g(\varphi\nabla_Z X,Y)&=g(\nabla^*_X(\varphi Z),Y)-g(\varphi\nabla_X Z,Y),\\
G(\varphi\nabla^*_Z(\varphi X),Y)-G(\nabla_Z X,Y)&=G(\varphi\nabla^*_X(\varphi Z),Y)-G(\nabla_X Z,Y),\\
G(\varphi\{\nabla^*_Z(\varphi X)-\nabla^*_X(\varphi Z)\},Y)&=G(\nabla_Z X-\nabla_X Z,Y).
\end{aligned}\tag{4.5}$$

Thus,

$$\begin{aligned}
\varphi\{\nabla^*_Z(\varphi X)-\nabla^*_X(\varphi Z)\}&=\nabla_Z X-\nabla_X Z,\\
\varphi\{(\nabla^*_Z\varphi)X+\varphi\nabla^*_Z X-(\nabla^*_X\varphi)Z-\varphi\nabla^*_X Z\}&=\nabla_Z X-\nabla_X Z,\\
\varphi\{(\nabla^*_Z\varphi)X-(\nabla^*_X\varphi)Z\}+(\nabla^*_Z X-\nabla^*_X Z-[Z,X])&=\nabla_Z X-\nabla_X Z-[Z,X],\\
\varphi\{(\nabla^*_Z\varphi)X-(\nabla^*_X\varphi)Z\}+S^{\nabla^*}(Z,X)&=S^{\nabla}(Z,X),
\end{aligned}$$

is obtained. From the last equation, it is seen that the necessary and sufficient condition for $S^{\nabla^*}=S^{\nabla}$ is that (∇^*,φ) must be a Codazzi pair.

iii. From equation (4.5), we have

$$\begin{aligned}
G\left((\nabla^*)^{\varphi}_Z X-(\nabla^*)^{\varphi}_X Z,Y\right)&=G(\nabla_Z X-\nabla_X Z,Y),\\
G\left((\nabla^*)^{\varphi}_Z X-(\nabla^*)^{\varphi}_X Z-[Z,X],Y\right)&=G(\nabla_Z X-\nabla_X Z,Y-[Z,X],Y),\\
G\left(S^{(\nabla^*)^{\varphi}}(Z,X),Y\right)&=G(S^{\nabla}(Z,X),Y),
\end{aligned}$$

and can be seen that $S^{(\nabla^*)^{\varphi}}=S^{\nabla}$.

The proof is completed. □

Theorem 4.6. *Let ∇ be a linear connection on (M_{2n},φ,g,G). The followings are equivalent:*

1. (∇,G) is a Codazzi pair.
2. $(\nabla^\dagger,G)$ is a Codazzi pair.
3. $\mathcal{F}$ is totally symmetric.
4. $S^{\nabla}=S^{\nabla^\dagger}$.

Proof. **i. ⇒ ii.** Let (∇,G) is a Codazzi pair, then we have

$$\begin{aligned}
\left(\nabla^\dagger_Z G\right)(X,Y)-\left(\nabla^\dagger_X G\right)(Z,Y)&=ZG(X,Y)-G\left(\nabla^\dagger_Z X,Y\right)-G\left(X,\nabla^\dagger_Z Y\right)\\
&\quad-XG(Z,Y)+G\left(\nabla^\dagger_X Z,Y\right)+G(Z,\nabla^\dagger_X Y)\\
&=-G\left(X,\left(\nabla^\dagger-\nabla\right)_Z Y\right)+G\left(Z,\left(\nabla^\dagger-\nabla\right)_X Y\right)\\
&=-\mathcal{F}(X,Y,Z)+\mathcal{F}(Z,Y,X)\\
&=0.
\end{aligned}$$

So, $(\nabla^\dagger,G)$ is a Codazzi pair.

ii. ⇒ iii. Due to symmetry of G,

$$\mathcal{F}(X,Y,Z) = (\nabla_Z G)(X,Y) = (\nabla_Z G)(Y,X) = \mathcal{F}(Y,X,Z).$$

For (∇, G) being a Codazzi pair,

$$\mathcal{F}(X,Y,Z) = (\nabla_Z G)(X,Y) = (\nabla_X G)(Z,Y) = \mathcal{F}(Z,Y,X).$$

Also due to symmetry of G and for (∇, G) being a Codazzi pair,

$$\mathcal{F}(X,Y,Z) = (\nabla_Z G)(X,Y) = (\nabla_Z G)(Y,X) = (\nabla_Y G)(Z,X) = (\nabla_Y G)(X,Z) = \mathcal{F}(X,Z,Y).$$

That is, $\mathcal{F}$ is totally symmetric in all of its indices.

iii. ⇒ iv. If $\mathcal{F}$ is totally symmetric, we have

$$\begin{aligned}
\mathcal{F}(X,Y,Z) - \mathcal{F}(X,Z,Y) &= (\nabla_Z G)(X,Y) - (\nabla_Y G)(X,Z) \\
&= ZG(X,Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y) - YG(X,Z) - G(\nabla_Y X, Z) - G(X, \nabla_Y Z) \\
&\Rightarrow G\left(X, \left(\nabla^\dagger - \nabla\right)_Z Y - \left(\nabla^\dagger - \nabla\right)_Y Z\right) = 0 \\
&\Rightarrow G\left(X, S^{\nabla^\dagger}(Z,Y) - S^{\nabla}(Z,Y)\right) = 0 \\
&\Rightarrow S^{\nabla} = S^{\nabla^\dagger},
\end{aligned}$$

where $S^{\nabla^\dagger}(Z,Y) = \nabla^\dagger_Z Y - \nabla^\dagger_Y Z - [Z,Y]$.

The proof is completed. □

Corollary 4.7. *Let (M_{2n}, g) be an almost para Norden manifold, ∇ and $\nabla^\dagger$ be connections linear and G-conjugate, respectively. For the tensors $\mathcal{F}^\dagger$ and $\mathcal{F}$, we have $\mathcal{F}^\dagger(X,Y,Z) = -\mathcal{F}(X,Y,Z)$. So, $\mathcal{F}^\dagger$ is totally symmetric if and only if $\mathcal{F}$ is so.*

Theorem 4.8. *Let ∇ be a linear connection on (M_{2n}, φ, g, G), $\nabla^\dagger$ be a G-conjugate connection and (∇, G) be a Codazzi pair. In this case, the necessary and sufficient condition for $\left(\nabla^\dagger, \varphi\right)$ to be a Codazzi pair is that (∇, g) is so.*

Proof.

$$\begin{aligned}
G\left(\left(\nabla^\dagger_Z \varphi\right)X - \left(\nabla^\dagger_X \varphi\right)Z, Y\right) &= G\left(\nabla^\dagger_Z(\varphi X) - \varphi\nabla^\dagger_Z X, Y\right) - G\left(\nabla^\dagger_X(\varphi Z) - \varphi\nabla^\dagger_X Z, Y\right) \\
&= G\left(Y, \nabla^\dagger_Z(\varphi X)\right) - G\left(\varphi\nabla^\dagger_Z X, Y\right) - G\left(Y, \nabla^\dagger_X(\varphi Z)\right) + G\left(\varphi\nabla^\dagger_X Z, Y\right) \\
&= ZG(Y,\varphi X) - G(\nabla_Z Y, \varphi X) - G\left(\varphi\nabla^\dagger_Z X, Y\right) - XG(Y,\varphi Z) + G(\nabla_X Y, \varphi Z) + G\left(\varphi\nabla^\dagger_X Z, Y\right) \\
&= ZG(\varphi X, Y) - G(\varphi X, \nabla_Z Y) - XG(\varphi Z, Y) + G(\varphi Z, \nabla_X Y) \\
&\quad + G\left(\varphi\left(\nabla^\dagger_X Z - \nabla^\dagger_Z X - [X,Z]\right) + \varphi[X,Z], Y\right).
\end{aligned}$$

Because of (∇, G) is a Codazzi pair, $S^{\nabla^\dagger} = S^{\nabla}$. Then, from the last equation, we obtain

$$\begin{aligned}
&ZG(\varphi X, Y) - G(\varphi X, \nabla_Z Y) - XG(\varphi Z, Y) + G(\varphi Z, \nabla_X Y) \\
&\quad + G(\varphi(\nabla_X Z - \nabla_Z X - [X,Z]) + \varphi[X,Z], Y) \\
&\quad = Zg(X,Y) - g(X, \nabla_Z Y) - Xg(Z,Y) + g(Z, \nabla_X Y) + g(\nabla_X Z, Y) - g(\nabla_Z X, Y) \\
&\quad = (\nabla_Z g)(X,Y) - (\nabla_X g)(Z,Y).
\end{aligned}$$

The proof is completed. □

5. Curvature Properties

Let $(M_{2n},\ g)$ a pseudo-Riemannian manifold. The curvature tensor field of a linear connection denoted by R is defined as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Moreover, R is called the $(0,4)$-curvature tensor field and is defined as

$$R(X,Y,Z,T) = g(R(X,Y)Z,T)$$

for all vector fields X, Y and Z on M_{2n}.

The curvature tensors of the ∇ , ∇^* and ∇^φ connections are R, R^* and R^φ ,respectively, the relationship between these curvatures is given by the theorem below.

Theorem 5.1. *Let (M_{2n}, φ, g, G) be an almost para Norden manifold, ∇^* ve ∇^φ be g-conjugate and φ-conjugate connections, R , R^* and R^φ be curvature tensors of ∇ , ∇^* and ∇^φ connections, respectively. There is a relation between these curvatures as follows;*

$$R(X,Y,\varphi Z,T) = -R^*(X,Y,T,\varphi Z) = R^\varphi(X,Y,Z,\varphi T)$$

for all vector fields X , Y and Z on M_{2n}.

Proof. Using Lie brackets $[X,Y] = [Y,T] = [T,Z] = 0$ is written, where X , Y, Z $\in \left\{\frac{\partial}{\partial x^1}, \ \ldots, \ \frac{\partial}{\partial x^{2n}}\right\}$. From here, we obtain

$$\begin{aligned} XYG(Z,T) &= X(YG(Z,T)) = X(Yg(\varphi Z,T)) \\ &= X(g(\nabla_Y\varphi Z,T)) + X(g(\varphi Z,\nabla^*_Y T)) \\ &= g(\nabla_X\nabla_Y\varphi Z,T) + g(\nabla_Y\varphi Z,\nabla^*_X T) + g(\nabla_X\varphi Z,\nabla^*_Y T) + g\left(\varphi Z,\nabla^*_X\nabla^*_Y T\right), \end{aligned} \tag{5.1}$$

$$\begin{aligned} YXG(Z,T) &= (XG(Z,T)) = Y(Xg(\varphi Z,T)) \\ &= Y(g(\nabla_X\varphi Z,T)) + Y(g(\varphi Z,\nabla^*_X T)) \\ &= g(\nabla_Y\nabla_X\varphi Z,T) + g(\nabla_X\varphi Z,\nabla^*_Y T) + g(\nabla_Y\varphi Z,\nabla^*_X T) + g\left(\varphi Z,\nabla^*_Y\nabla^*_X T\right). \end{aligned} \tag{5.2}$$

From the equations (5.1) and (5.2), we obtain

$$\begin{aligned} &[X,Y]G(Z,T) = XYG(Z,T) - YXG(Z,T) = 0 \\ &\Rightarrow g(\nabla_X\nabla_Y\varphi Z - \nabla_Y\nabla_X\varphi Z,T) + g\left(\varphi Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow g(R(X,Y)\varphi Z,T) + g(R^*(X,Y)T,\varphi Z) = 0 \\ &\Rightarrow R(X,Y,\varphi Z,T) + R^*(X,Y,T,\varphi Z) = 0. \end{aligned}$$

That is,

$$R(X,Y,\varphi Z,T) = -R^*(X,Y,T,\varphi Z). \tag{5.3}$$

Similarly,

$$\begin{aligned} &[X,Y]G(Z,T) = XYG(Z,T) - YXG(Z,T) = 0 \\ &\Rightarrow g(\nabla_X\nabla_Y\varphi Z - \nabla_Y\nabla_X\varphi Z,T) + g\left(\varphi Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow G\left(\varphi^{-1}\nabla_X\nabla_Y\varphi Z - \varphi^{-1}\nabla_Y\nabla_X\varphi Z,T\right) + G\left(Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow G\left(\varphi^{-1}\nabla_X\varphi\left(\varphi^{-1}\nabla_Y\varphi Z\right) - \varphi^{-1}\nabla_Y\varphi(\varphi^{-1}\nabla_X\varphi Z),T\right) + G\left(Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow G\left(\varphi^{-1}\nabla_X\varphi\left(\nabla^\varphi_Y Z\right) - \varphi^{-1}\nabla_Y\varphi\left(\nabla^\varphi_X Z\right),T\right) + G\left(Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow G\left(\nabla^\varphi_X\nabla^\varphi_Y Z - \nabla^\varphi_Y\nabla^\varphi_X Z,T\right) + G\left(Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow g\left(\varphi\left(\nabla^\varphi_X\nabla^\varphi_Y Z - \nabla^\varphi_Y\nabla^\varphi_X Z\right),T\right) + g\left(\varphi Z,\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T\right) = 0 \\ &\Rightarrow g\left(\left(\nabla^\varphi_X\nabla^\varphi_Y Z - \nabla^\varphi_Y\nabla^\varphi_X Z\right),\varphi T\right) + g\left(\nabla^*_X\nabla^*_Y T - \nabla^*_Y\nabla^*_X T,\varphi Z\right) = 0 \\ &\Rightarrow g(R^\varphi(X,Y)Z,\varphi T) + g(R^*(X,Y)T,\varphi Z) = 0 \\ &\Rightarrow R^\varphi(X,Y,Z,\varphi T) + R^*(X,Y,T,\varphi Z) = 0. \end{aligned}$$

So that,

$$R^\varphi(X,Y,Z,\varphi T) = -R^*(X,Y,T,\varphi Z) \tag{5.4}$$

is obtained.

From equations (5.3) and (5.4),

$$R(X,Y,\varphi Z,T) = -R^*(X,Y,T,\varphi Z) = R^\varphi(X,Y,Z,\varphi T)$$

is obtained.

The proof is completed. □

6. Φ-Operator and Nijenhuis Tensor

Let $(M_{2n},\ g)$ be an almost para Norden manifold. Φ is called a Tachibana operator or Φ-operator and applied to pure tensors [20, 30]. If the Φ- operator is applied to the pure tensor g,

$$\begin{aligned}\left(\Phi_{\varphi}g\right)(X,Y,Z) &= (\varphi X)(g(Y,Z)) - X(g(\varphi Y,Z)) + g((L_Y\varphi)X,Z) + g(Y,(L_Z\varphi)X)\\ &= \left(L_{\varphi X}g - L_X(g\circ\varphi)\right)(Y,Z) \end{aligned} \tag{6.1}$$

is obtained, where L is Lie derivative and X , Y and Z are vector fields on M_{2n} [20].
Twin metric G defined on almost para Norden manifold is also a Norden metric, so the Φ- operator can be applied to this metric and

$$\begin{aligned}\left(\Phi_{\varphi}G\right)(X,Y,Z) &= \left(L_{\varphi X}G - L_X(G\circ\varphi)\right)(Y,Z)\\ &= \left(\Phi_{\varphi}g\right)(X,\varphi Y,Z) + g\left(N_{\varphi}(X,Y),Z\right), \end{aligned} \tag{6.2}$$

(see [20]).

Proposition 6.1. *Let ∇ be a linear connection on almost para Norden manifold (M_{2n},φ,g). If (∇,φ) is a Codazzi pair, $N_{\varphi}(X,Y)=0$.*

Proof. The Nijenhuis tensor is as follows;

$$\begin{aligned}N_{\varphi}(X,Y) &= [\varphi X,\varphi Y] - \varphi[X,\varphi Y] - \varphi[\varphi X,Y] + \varphi^2[X,Y]\\
&= \nabla_{\varphi X}\varphi Y - \nabla_{\varphi Y}\varphi X - \varphi\left(\nabla_X\varphi Y - \nabla_{\varphi Y}X\right) - \varphi\left(\nabla_{\varphi X}Y - \nabla_Y\varphi X\right) + (\nabla_X Y - \nabla_Y X)\\
&= \left(\nabla_{\varphi X}\varphi\right)Y + \varphi\nabla_{\varphi X}Y - \left(\nabla_{\varphi Y}\varphi\right)X - \varphi\nabla_{\varphi Y}X - \varphi\left((\nabla_X\varphi)Y + \varphi\nabla_X Y - \nabla_{\varphi Y}X\right)\\
&\quad - \varphi\left(\nabla_{\varphi X}Y - (\nabla_Y\varphi)X - \varphi\nabla_Y X\right) + (\nabla_X Y - \nabla_Y X)\\
&= \left(\nabla_{\varphi X}\varphi\right)Y + \varphi\nabla_{\varphi X}Y - \left(\nabla_{\varphi Y}\varphi\right)X - \varphi\nabla_{\varphi Y}X - \varphi(\nabla_X\varphi)Y - \nabla_X Y + \varphi\nabla_{\varphi Y}X\\
&\quad - \varphi\nabla_{\varphi X}Y + \varphi(\nabla_Y\varphi)X + \nabla_Y X + \nabla_X Y - \nabla_Y X\\
&= \left(\nabla_{\varphi X}\varphi\right)Y - \left(\nabla_{\varphi Y}\varphi\right)X - \varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X)\\
&= -\varphi\left(\nabla_{\varphi X}\varphi\right)\varphi Y + \varphi\left(\nabla_{\varphi Y}\varphi\right)\varphi X - \varphi((\nabla_X\varphi)Y - (\nabla_Y\varphi)X)\\
&= -\varphi\left\{\left(\nabla_{\varphi X}\varphi\right)\varphi Y - \left(\nabla_{\varphi Y}\varphi\right)\varphi X\right\} - \varphi\{(\nabla_X\varphi)Y - (\nabla_Y\varphi)X\}.\end{aligned}$$

That is, if (∇,φ) is a Codazzi pair, then $N_{\varphi}(X,Y)=0$. □

Theorem 6.2. *Let ∇ be a linear connection on (M_{2n},φ,g,G). If (∇,φ) is a Codazzi pair, then*

$$\left(\Phi_{\varphi}G\right)(X,Y,Z) = \left(\Phi_{\varphi}g\right)(X,\varphi Y,Z) = \left(\nabla_{\varphi X}G\right)(Y,Z) - (\nabla_X g)(\varphi Y,\varphi Z).$$

Proof. Since ∇ is a torsion-free linear connection, $\nabla_X Z - \nabla_Z X = [X,Z]$ is written. Using equation (6.1), we have

$$\begin{aligned}\left(\Phi_{\varphi}g\right)(X,\varphi Y,Z) &= \left(L_{\varphi X}g - L_X(g\circ\varphi)\right)(\varphi Y,Z) = \left(L_{\varphi X}g\right)(\varphi Y,Z) - (L_X g\circ\varphi)(\varphi Y,Z)\\
&= \varphi Xg(\varphi Y,Z) - g\left(L_{\varphi X}\varphi Y,Z\right) - g\left(\varphi Y,L_{\varphi X}Z\right) - Xg\circ\varphi(\varphi Y,Z) + g\circ\varphi(L_X\varphi Y,Z) + g\circ\varphi(\varphi Y,L_X Z)\\
&= \varphi Xg(\varphi Y,Z) - g([\varphi X,\varphi Y],Z) - g(\varphi Y,[\varphi X,Z] - Xg\circ\varphi(\varphi Y,Z)\\
&\quad + g\circ\varphi([X,\varphi Y],Z) + g\circ\varphi(\varphi Y,[X,Z])\\
&= \varphi Xg(\varphi Y,Z) - g\left(\nabla_{\varphi X}\varphi Y - \nabla_{\varphi Y}\varphi X,Z\right) - g\left(\varphi Y,\nabla_{\varphi X}Z - \nabla_Z\varphi X\right)\\
&\quad - Xg\circ\varphi(\varphi Y,Z) + g\circ\varphi\left(\nabla_X\varphi Y - \nabla_{\varphi Y}X,Z\right) + g\circ\varphi(\varphi Y,\nabla_X Z - \nabla_Z X)\\
&= \varphi Xg(\varphi Y,Z) - g\left(\left(\nabla_{\varphi X}\varphi\right)Y + \varphi\nabla_{\varphi X}Y - \left(\nabla_{\varphi Y}\varphi\right)X - \varphi\nabla_{\varphi Y}X,Z\right) - g\left(\varphi Y,\nabla_{\varphi X}Z - (\nabla_Z\varphi)X - \varphi\nabla_Z X\right)\\
&\quad - Xg\circ\varphi(\varphi Y,Z) + g\circ\varphi\left((\nabla_X\varphi)Y + \varphi\nabla_X Y - \nabla_{\varphi Y}X,Z\right) + g\circ\varphi(\varphi Y,\nabla_X Z - \nabla_Z X)\\
&= \varphi Xg(\varphi Y,Z) - g\left(\left(\nabla_{\varphi X}\varphi\right)Y + \varphi\nabla_{\varphi X}Y - \left(\nabla_{\varphi Y}\varphi\right)X - \varphi\nabla_{\varphi Y}X,Z\right) - g\left(\varphi Y,\nabla_{\varphi X}Z - (\nabla_Z\varphi)X - \varphi\nabla_Z X\right)\\
&\quad - Xg(\varphi Y,\varphi Z) + g\left((\nabla_X\varphi)Y + \varphi\nabla_X Y - \nabla_{\varphi Y}X,\varphi Z\right) + g(\varphi Y,\varphi(\nabla_X Z - \nabla_Z X))\end{aligned}$$

$$
\begin{aligned}
&= \varphi Xg(\varphi Y, Z) - g\left(\left(\nabla_{\varphi X}\varphi\right)Y, Z\right) - g\left(\varphi\nabla_{\varphi X}Y, Z\right) + g\left(\left(\nabla_{\varphi Y}\varphi\right)X, Z\right) + g\left(\varphi\nabla_{\varphi Y}X, Z\right)\\
&\quad - g\left(\varphi Y, \nabla_{\varphi X}Z\right) - Xg(\varphi Y, \varphi Z) + g(\varphi Y, \ (\nabla_Z\varphi)X) + g(\varphi Y, \varphi\nabla_Z X) + g((\nabla_X\varphi)Y, \varphi Z)\\
&\quad + g(\varphi\nabla_X Y, \varphi Z) - g\left(\nabla_{\varphi Y}X, \varphi Z\right) + g(\varphi Y, \varphi\nabla_X Z) - g(\varphi Y, \varphi\nabla_Z X)\\
&= \varphi Xg(\varphi Y, Z) - g\left(\left(\nabla_{\varphi X}\varphi\right)Y, Z\right) - g\left(\varphi\nabla_{\varphi X}Y, Z\right) + g\left(\left(\nabla_{\varphi Y}\varphi\right)X, Z\right) + g\left(\varphi\nabla_{\varphi Y}X, Z\right)\\
&\quad - g\left(\varphi Y, \nabla_{\varphi X}Z\right) + g(\varphi Y, \ (\nabla_Z\varphi)X) + g(\varphi Y, \varphi\nabla_Z X) - Xg(\varphi Y, \varphi Z) - g\left(\varphi\nabla_{\varphi Y}X, Z\right)\\
&\quad + g((\nabla_X\varphi)Y, \varphi Z) + g(\varphi\nabla_X Y, \varphi Z) + g(\varphi Y, \varphi\nabla_X Z) - g(\varphi Y, \varphi\nabla_Z X).
\end{aligned}
$$

If the para Norden metric g and the Codazzi pair (∇, φ) are used in the last equation, we have

$$
\begin{aligned}
&= \varphi Xg(\varphi Y, Z) - g\left(\varphi\nabla_{\varphi X}Y, Z\right) - g\left(\varphi Y, \nabla_{\varphi X}Z\right) + g(\varphi Y, \ (\nabla_Z\varphi)X)\\
&\quad - Xg(\varphi Y, \varphi Z) + g((\nabla_X\varphi)Y, \varphi Z) + g(\varphi\nabla_X Y, \varphi Z) + g(\varphi Y, \varphi\nabla_X Z)\\
&= \varphi Xg(\varphi Y, Z) - g\left(\varphi\nabla_{\varphi X}Y, Z\right) - g\left(\varphi Y, \nabla_{\varphi X}Z\right) + g(\varphi Y, \ (\nabla_X\varphi)Z)\\
&\quad - Xg(\varphi Y, \varphi Z) + g((\nabla_X\varphi)Y, \varphi Z) + g(\varphi\nabla_X Y, \varphi Z) + g(\varphi Y, \varphi\nabla_X Z)\\
&= \varphi XG(Y, Z) - G\left(\nabla_{\varphi X}Y, Z\right) - G\left(Y, \nabla_{\varphi X}Z\right) - Xg(\varphi Y, \varphi Z) + g(\nabla_X\varphi Y, \varphi Z) + g(\varphi Y, \nabla_X\varphi Z)\\
&= \left(\nabla_{\varphi X}G\right)(Y, Z) - (\nabla_X g)(\varphi Y, \varphi Z).
\end{aligned}
\tag{6.3}
$$

From Proposition 6.1, equation (6.2) and (6.3), we have

$$\left(\Phi_\varphi G\right)(X, Y, Z) = \left(\Phi_\varphi g\right)(X, \varphi Y, Z) = \left(\nabla_{\varphi X}G\right)(Y, Z) - (\nabla_X g)(\varphi Y, \varphi Z).$$

The proof is completed. □

Let g be a para Norden metric, φ be a paracomplex structure such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g. In that case the triple $(M_{2n}, \varphi, \ g)$ is called a para Kahler Norden manifold. In addition to that, on an almost paracomplex manifold the condition $\nabla\varphi = 0$ is equivalent to $\Phi_\varphi g = 0$, where g is a para Norden metric [22]. So, if a para Norden metric g is a paraholomorphic then $(M_{2n}, \varphi, \ g)$ is called a para Kahler Norden manifold. Based on this information, the following theorem is written [21].

Theorem 6.3. *Let ∇ be a torsion-free linear connection on para Norden manifold (M_{2n}, φ, g, G) and (∇, φ) be a Codazzi pair. Then, $(M_{2n}, \varphi, \ g)$ is a para Kahler Norden manifold if and only if*

$$\left(\nabla_{\varphi X}G\right)(Y, Z) = (\nabla_X g)(\varphi Y, \varphi Z).$$

Proof. Within the conditions given from the Theorem 6.2 we have

$$\left(\Phi_\varphi G\right)(X, Y, Z) = \left(\Phi_\varphi g\right)(X, \varphi Y, Z) = \left(\nabla_{\varphi X}G\right)(Y, Z) - (\nabla_X g)(\varphi Y, \varphi Z).$$

If the triple $(M_{2n}, \varphi, \ g)$ is a para Kahler Norden manifold then $\Phi_\varphi g = 0$. So, we have

$$\left(\nabla_{\varphi X}G\right)(Y, Z) = (\nabla_X g)(\varphi Y, \varphi Z).$$

The proof is completed. □

7. φ- Invariant Linear Connections

Let $(M_{2n}, \ g)$ be a paracomplex manifold and ∇ be a linear connection. If ∇ satisfies the following condition

$$\nabla_X\varphi Y = \varphi\nabla_X Y$$

∇ is named a φ-invariant linear connection, where X , Y and Z are vector fields on M_{2n}.

Theorem 7.1. *Let ∇ be a linear connection on (M_{2n}, φ, g, G) and ∇^*, $\nabla^\dagger$ are g-conjugate and G-conjugate connections, respectively. In that case,*

1. *∇^* is φ-invariant if and only if ∇ is φ-invariant.*
2. *$\nabla^\dagger$ is φ-invariant if and only if ∇ is φ-invariant.*

Proof. **i.**
$$\begin{aligned} G\left(\nabla^*_X\varphi Y-\varphi\nabla^*_X Y,Z\right) &= G\left(\nabla^*_X\varphi Y,Z\right)-G\left(\varphi\nabla^*_X Y,Z\right)\\ &= g\left(\nabla^*_X\varphi Y,\varphi Z\right)-g\left(\varphi\nabla^*_X Y,\varphi Z\right)\\ &= g\left(\varphi Z,\nabla^*_X\varphi Y\right)-g\left(Z,\nabla^*_X Y\right)\\ &= Xg\left(\varphi Z,\varphi Y\right)-g\left(\nabla_X\varphi Z,\varphi Y\right)-Xg\left(Z,Y\right)+g\left(\nabla_X Z,Y\right)\\ &= Xg\left(Z,Y\right)-g\left(\varphi Y,\nabla_X\varphi Z\right)-Xg\left(Z,Y\right)+g\left(Y,\nabla_X Z\right)\\ &= -g\left(\varphi Y,\nabla_X\varphi Z\right)+g\left(\varphi Y,\varphi\nabla_X Z\right)\\ &= -G\left(Y,\nabla_X\varphi Z\right)+G\left(Y,\varphi\nabla_X Z\right)\\ &= G\left(\varphi\nabla_X Z-\nabla_X\varphi Z,Y\right). \end{aligned}$$

From here, we have $\nabla^*_X\varphi Y=\varphi\nabla^*_X Y$ if and only if $\varphi\nabla_X Z=\nabla_X\varphi Z$.

ii.
$$\begin{aligned} G\left(\nabla^\dagger_X\varphi Y-\varphi\nabla^\dagger_X Y,Z\right) &= G\left(Z,\nabla^\dagger_X\varphi Y\right)-G\left(\varphi Z,\nabla^\dagger_X Y\right)\\ &= XG\left(Z,\varphi Y\right)-G\left(\nabla_X Z,\varphi Y\right)-XG\left(\varphi Z,Y\right)+G\left(\nabla_X\varphi Z,Y\right)\\ &= -G\left(\nabla_X Z,\varphi Y\right)+G\left(\nabla_X\varphi Z,Y\right)\\ &= -G\left(\varphi\nabla_X Z,Y\right)+G\left(\nabla_X\varphi Z,Y\right)\\ &= G\left(\nabla_X\varphi Z-\varphi\nabla_X Z,Y\right). \end{aligned}$$

From here, we have $\nabla^\dagger_X\varphi Y=\varphi\nabla^\dagger_X Y$ if and only if $\nabla_X\varphi Z=\varphi\nabla_X Z$. □

Theorem 7.2. *Let ∇ be a φ-invariant connection on (M_{2n},φ,g,G) and ∇^*, $\nabla^\dagger$ are g-conjugate and G-conjugate connections, respectively. In this case, the followings are provided:*

1. $\nabla^*=\nabla^\dagger$,

2. (∇,G) *is a Codazzi pair if and only if* (∇,g) *is so.*

Proof. **i.**
$$\begin{aligned} ZG\left(X,Y\right)-G\left(\nabla_Z X,Y\right) &= G\left(X,\nabla^\dagger_Z Y\right),\\ Zg\left(\varphi X,Y\right)-g\left(\varphi\nabla_Z X,Y\right) &= g\left(\varphi X,\nabla^\dagger_Z Y\right),\\ Zg\left(\varphi X,Y\right)-g\left(\nabla_Z\varphi X,Y\right) &= g\left(\varphi X,\nabla^\dagger_Z Y\right),\\ g\left(\varphi X,\nabla^*_Z Y\right) &= g\left(\varphi X,\nabla^\dagger_Z Y\right). \end{aligned}$$

So that $\nabla^*=\nabla^\dagger$.

ii. We have
$$\begin{aligned} &\left(\nabla_Z G\right)\left(X,Y\right)=\left(\nabla_X G\right)\left(Z,Y\right)\\ \Longleftrightarrow\ & ZG\left(X,Y\right)-G\left(\nabla_Z X,Y\right)-G\left(X,\nabla_Z Y\right)=XG\left(Z,Y\right)-G\left(\nabla_X Z,Y\right)-G\left(Z,\nabla_X Y\right)\\ \Longleftrightarrow\ & Zg\left(\varphi X,Y\right)-g\left(\varphi\nabla_Z X,Y\right)-g\left(\varphi X,\nabla_Z Y\right)=Xg\left(\varphi Z,Y\right)-g\left(\varphi\nabla_X Z,Y\right)-g\left(\varphi Z,\nabla_X Y\right)\\ \Longleftrightarrow\ & Zg\left(X,\varphi Y\right)-g\left(\nabla_Z X,\varphi Y\right)-g\left(X,\nabla_Z\varphi Y\right)=Xg\left(Z,\varphi Y\right)-g\left(\nabla_X Z,\varphi Y\right)-g\left(Z,\nabla_X\varphi Y\right)\\ \Longleftrightarrow\ & \left(\nabla_Z g\right)\left(X,\varphi Y\right)=\left(\nabla_X g\right)\left(Z,\varphi Y\right). \end{aligned}$$

The proof is completed. □

8. Statistical Manifolds

Let ∇ be a torsion-free linear connection and g a pseudo-Riemannian metric on M_{2n}. If (∇,g) is a Codazzi pair, (M_{2n},g) is called a statistical manifold [16].

Theorem 8.1. *Let ∇ be a φ-invariant torsion-free linear connection on (M_{2n},φ,g,G). ∇^*, $\nabla^\dagger$ are g-conjugate and G-conjugate connections, respectively. In that case, if (∇,G) is a statistical structure, the followings are provided:*

1. (∇,g) *is a statistical structure,*

2. $\left(\nabla^\dagger,g\right)$ *is a statistical structure,*

3. (∇^*,g) *is a statistical structure,*

On the contrary, if the above statements are provided, (∇,G) is a statistical structure.

Proof. **i.** From the Theorem 7.2 it is known that, (∇, G) is a Codazzi pair if and only if (∇, g) is so. Therefore, (∇, g) is a statistical structure.

ii. To show that $\left(\nabla^{\dagger}, g\right)$ is a statistical structure, it is necessary to show $\left(\nabla^{\dagger}, g\right)$ is a Codazzi pair. So,

$$\left(\nabla^{\dagger}_Z g\right)(X,Y) - \left(\nabla^{\dagger}_X g\right)(Z,Y) = Zg(X,Y) - g\left(\nabla^{\dagger}_Z X, Y\right) - g\left(X, \nabla^{\dagger}_Z Y\right) - Xg(Z,Y) + g\left(\nabla^{\dagger}_X Z, Y\right) + g(Z, \nabla^{\dagger}_X Y).$$

From the Theorem 7.2, it is known that $\nabla^{\dagger} = \nabla^{*}$. Using this, we have

$$= Zg(X,Y) - g(\nabla^{*}_Z X, Y) - g(X, \nabla^{*}_Z Y) - Xg(Z,Y) + g(\nabla^{*}_X Z, Y) + g(Z, \nabla^{*}_X Y).$$

Also, using the definition g-conjugate connection $Zg(X,Y) = g(\nabla_Z X, Y) + g\left(X, \nabla^{*}_Z Y\right)$ and $Xg(Z,Y) = g(\nabla_X Z, Y) + g\left(Z, \nabla^{*}_X Y\right)$, we get from the last equation above,

$$\begin{aligned}
&= g(\nabla_Z X, Y) + g(X, \nabla^{*}_Z Y) - g(\nabla^{*}_Z X, Y) - g(X, \nabla^{*}_Z Y) - g(\nabla_X Z, Y) - g(Z, \nabla^{*}_X Y) + g(\nabla^{*}_X Z, Y) + g(Z, \nabla^{*}_X Y)\\
&= g(\nabla_Z X, Y) - g(\nabla^{*}_Z X, Y) - g(\nabla_X Z, Y) + g(\nabla^{*}_X Z, Y)\\
&= -g(Y, (\nabla^{*} - \nabla)_Z X) + g(Y, (\nabla^{*} - \nabla)_X Z)\\
&= -C(Y,X,Z) + C(Y,Z,X)\\
&= 0
\end{aligned}$$

is obtained. That is, $\left(\nabla^{\dagger}_Z g\right)(X,Y) - \left(\nabla^{\dagger}_X g\right)(Z,Y) = 0$ and $\left(\nabla^{\dagger}, g\right)$ is a statistical structure.

iii. To show that (∇^{*}, g) is a statistical structure, it is necessary to show (∇^{*}, g) is a Codazzi pair. So, similar to the proof above, we have

$$\begin{aligned}
(\nabla^{*}{}_Z g)(X,Y) - (\nabla^{*}{}_X g)(Z,Y) &= Zg(X,Y) - g(\nabla^{*}_Z X, Y) - g(X, \nabla^{*}_Z Y) - Xg(Z,Y) + g(\nabla^{*}_X Z, Y) + g(Z, \nabla^{*}_X Y)\\
&= g(\nabla_Z X, Y) + g(X, \nabla^{*}_Z Y) - g(\nabla^{*}_Z X, Y) - g(X, \nabla^{*}_Z Y)\\
&\quad - g(\nabla_X Z, Y) - g(Z, \nabla^{*}_X Y) + g(\nabla^{*}_X Z, Y) + g(Z, \nabla^{*}_X Y)\\
&= g(\nabla_Z X, Y) - g(\nabla^{*}_Z X, Y) - g(\nabla_X Z, Y) + g(\nabla^{*}_X Z, Y)\\
&= -g(Y, (\nabla^{*} - \nabla)_Z X) + g(Y, (\nabla^{*} - \nabla)_X Z)\\
&= -C(Y,X,Z) + C(Y,Z,X)\\
&= 0,
\end{aligned}$$

which gives (∇^{*}, g) is a statistical structure.

The proof is completed. □

Theorem 8.2. *Let ∇ be a φ-invariant torsion-free linear connection and ∇^{*} be g-conjugate connection on (M_{2n}, φ, g, G). (∇, G) is a statistical structure if and only if (∇^{*}, G) is so.*

Proof. For (∇^{*}, G) to be a statistical structure, $\left(\nabla^{*}_Z G\right)(X,Y) = \left(\nabla^{*}_X G\right)(Z,Y)$. So,

$$\begin{aligned}
(\nabla^{*}_Z G)(X,Y) - (\nabla^{*}_X G)(Z,Y) &= ZG(X,Y) - G(\nabla^{*}_Z X, Y) - G(X, \nabla^{*}_Z Y) - XG(Z,Y) + G(\nabla^{*}_X Z, Y) + G(Z, \nabla^{*}_X Y)\\
&= Zg(\varphi X, Y) - g(\varphi Y, \nabla^{*}_Z X) - g(\varphi X, \nabla^{*}_Z Y) - Xg(\varphi Z, Y) + g(\varphi Y, \nabla^{*}_X Z) + g(\varphi Z, \nabla^{*}_X Y)\\
&= g(\nabla^{*}_Z \varphi X, Y) + g(\varphi X, \nabla_Z Y) - g(\varphi Y, \nabla^{*}_Z X) - g(\varphi X, \nabla^{*}_Z Y) - g(\nabla^{*}_X \varphi Z, Y)\\
&\quad - g(\varphi Z, \nabla_X Y) + g(\varphi Y, \nabla^{*}_X Z) + g(\varphi Z, \nabla^{*}_X Y).
\end{aligned}$$

From Theorem 7.1, we know ∇ is φ-invariant if and only if ∇^{*} is so. Then, we have

$$\begin{aligned}
&= g(\varphi \nabla^{*}_Z X, Y) + g(\varphi X, \nabla_Z Y) - g(\varphi Y, \nabla^{*}_Z X) - g(\varphi X, \nabla^{*}_Z Y) - g(\varphi \nabla^{*}_X Z, Y) - g(\varphi Z, \nabla_X Y) + g(\varphi Y, \nabla^{*}_X Z) + g(\varphi Z, \nabla^{*}_X Y)\\
&= g(\varphi Y, \nabla^{*}_Z X) + g(\varphi X, \nabla_Z Y) - g(\varphi Y, \nabla^{*}_Z X) - g(\varphi X, \nabla^{*}_Z Y) - g(\varphi Y, \nabla^{*}_X Z) - g(\varphi Z, \nabla_X Y) + g(\varphi Y, \nabla^{*}_X Z) + g(\varphi Z, \nabla^{*}_X Y)\\
&= -g(\varphi X, (\nabla^{*} - \nabla)_Z Y) + g(\varphi Z, (\nabla^{*} - \nabla)_X Y)\\
&= -G(X, (\nabla^{*} - \nabla)_Z Y) + G(Z, (\nabla^{*} - \nabla)_X Y).
\end{aligned}$$

From Theorem 7.2, we know if ∇ is a φ-invariant connection, then $\nabla^* = \nabla^\dagger$. So, we have from the last equation above

$$\begin{aligned} -G\left(X, (\nabla^* - \nabla)_Z Y\right) + G\left(Z, (\nabla^* - \nabla)_X Y\right) &= -G\left(X, \left(\nabla^\dagger - \nabla\right)_Z Y\right) + G\left(Z, \left(\nabla^\dagger - \nabla\right)_X Y\right) \\ &= -\mathcal{F}(X, Y, Z) + \mathcal{F}(Z, Y, X) \\ &= 0. \end{aligned}$$

The proof is completed. □

9. Results and Suggestions

In this section, the results obtained in the study will be presented.

The main aim of the current study is to research some properties of Codazzi pairs on an almost para Norden manifold. Firstly, ∇^φ , $\varphi-$ conjugate, ∇^* , $g-$ conjugate, $\nabla^\dagger$, $G-$conjugate connections are defined on (M_{2n}, φ, g, G) and properties of these connections are investigated. In addition, it is demonstrated that on the spaces of linear connections, $(id, *, \dagger, \varphi)$ acts as the four-element Klein group. (∇, φ) , (∇, g) and (∇, G) Codazzi pairs are given on (M_{2n}, φ, g, G), some properties and relationship between them are investigated. The curvature tensors of ∇ , ∇^*, ∇^φ connections denoted as R , R^*, R^φ respectively, the relation between these curvatures is given on (M_{2n}, φ, g, G). Necessary and sufficient conditions are obtained for the (M_{2n}, φ, g, G) to be para Kahler Norden. φ-invariant connection on (M_{2n}, φ, g, G) is studied. Finally, statistical structures and some properties of them are given.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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