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Araştırma Makalesi/Research Article

On Eigenvalue And Eigenvector Perceptions of Undergraduate Pre-service Mathematics Teachers

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Abstract – Analysis of systems that can be expressed in matrices is very important in the field of application. Whether such a system works properly is determined by eigenvalues of the matrix representing the system. Eigenvalue and eigenvector concepts are taught within the scope of linear algebra course at undergraduate level. In this study, perceptions of undergraduate students who took the linear algebra course about the eigenvalue-eigenvector concepts are investigated. The research is conducted with the participation of 95 students from the Faculty of Education, Department of Mathematics Education. A scale measuring the students' approached about eigen theory was developed. For the reliability of the scale, Kuder-Richardson 20 (KR-20) reliability analysis was done and 0,72 was obtained. To see the relationship between learning outcomes and academic achievement is used the chi-square test and descriptive analysis are made in the study. Problems arising in the perception of eigenvalue and solutions are presented.

Key words: linear algebra, matrices, eigenvalue, eigenvector.

Introduction

Undergraduate level students first encounter the eigenvalue and eigenvector concepts in linear algebra classes. Linear algebra is one of the first undergraduate courses that students have to understand theoretically systematically, so it is cognitively a frustrating experience for both students and supervisors (Stewart & Thomas, 2011; Thomas & Stewart, 2011). The students taking the course face two basic difficulties, cognitive and conceptual difficulties related to the nature of linear algebra (Dorier & Sierpinska, 2001). Stewart and Thomas (2011) stated that this course was very intense for students and they found it difficult to relate the definitions and theoretical results to their previous knowledge. According to these authors, students' difficulties are due to the fact that many teachers and texts do not introduce the geometry of eigenvalues and eigenvectors. When geometric representation is left aside and the relation of algebraic and geometric representation is not discussed, many students learn these concepts mechanically, while when both representations are considered in the teaching of these concepts, students understand these concepts better (Salgado, 2015). International studies show that linear algebra students often develop an analytical-arithmetic thinking style that leads to the development of procedural knowledge (Gol Tabaghi & Sinclair, 2013). The fact that mathematical equations are not considered geometrically in two and three-dimensional spaces prevents students from developing their synthetic-geometric thinking skills. The relationship between mathematical operations and their geometrical equivalents should be given in the field of algebra in the most intense and clearest way. The resulted difficulties in the linear algebra course are described in Dorier et al. (2000): "The main criticisms made by the students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics". Although students can perform operations that require calculation in linear algebra, they have difficulty understanding concepts and establishing relationships between concepts (Dorier, 1998; Harel, 1989). Thomas and Stewart (2011) show that students generally do not understand the meaning of eigenvalue and eigenvector definitions and try to algebraically change symbols without understanding these concepts. Moreover, they observed that many students did not know the geometric image of the eigenvalue and eigenvector. In linear algebra courses, students have difficulties in topics such as linear dependence and independence, row and column spaces, and vector spaces of matrices, geometric interpretation of the action of linear transformations (Carlson, 1993). Larson et al. (2007); Larson, Zandieh and Rasmussen (2008) studied the way students approached a basic equation about eigen theory. The results of these few studies have inspired us to identify the problems.

The learning outcomes of the linear algebra course on eigenvalue and eigenvector can be expressed in the following 7 items:

i. Defines the concepts of eigenvalue and eigenvector

ii. Expresses the explicit state of the system of equations given in implicit form

iii. Uses the theoretical knowledge about the existence of the solution of the system

iv. Graphically interprets a system of linear equations *v.* Algebraically calculates the eigenvalues of the two-dimensional square matrix

vi. Knows that if the eigenvalues of a square matrix of dimension 2 are complex, the eigenvectors corresponding to these eigenvalues cannot be represented in the Euclidean plane

vii. Knows that a square matrix has at least one eigenvalue

In order for the eigenvalues and eigenvectors corresponding to a square matrix to be well understood and correctly determined, these outcomes must be realized. In this study, the perceptions of the undergraduate students are investigated by asking in line with the outcomes in these 7 items. For this purpose, the students are asked questions about the evaluation of the above items. The answers given to these questions are evaluated.

Let us express some remarkable points about the mathematical definitions of eigenvalue and eigenvector concepts. In linear algebra and engineering mathematics books, the concepts of eigenvalue and eigenvector are usually started like the question "When does the image of an x vector become a scalar multiple of itself under linear transformation given by a square matrix?". In basic linear algebra and engineering mathematics books, the term scalar is used in the introduction to the subject, and it is clarified in the next chapters that the term scalar corresponds to what kind of numbers (e.g. Meyer 2000; Burden, 1993; Bretscher, 2013; Szabo, 2000; Kreyszig, 2006). Since the subject is not usually mentioned in courses, or because the students' attention is not drawn, our study shows that the students think that this scalar is a real number. However, with the scalar concept, a number associated with the field of the linear vector space in which the problem is addressed should be understood, which is a real or complex number. In practice, vector spaces that are usually built on a complex number field are studied. Let the n-dimensional real Euclidean space be $\mathbb{R}^n$ and the complex Euclidean space $\mathbb{C}^n$.

In the linear algebra book of Szabo (2000), the definition of eigenvalue and eigenvector is defined as follows:

> A real number λ is an eigenvalue of a real $n \times n$ matrix A if there exists a nonzero column vector $x \in \mathbb{R}^n$ for which $Ax = \lambda x$. A nonzero column vector $x \in \mathbb{R}^n$ is an eigenvector of a real $n \times n$ matrix if there exists a real number λ for which $Ax = \lambda x$ (p. 375).

According to this definition, for example consider the matrix $A = \begin{bmatrix} 3 & -5 \\ 1 & -1 \end{bmatrix}$ For all nonzero $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, the equation $Ax = \lambda x$ does not satisfy. However, this matrix A can also be the coefficient matrix of a linear differential equation with a constant coefficient, and the λ scalar and the vector x must be obtained for the solution of this differential equation. It is not possible to find eigenvalues and eigenvectors of the system without working with complex numbers ($\mathbb{C}$) to obtain the solution of that system. In fact, each real number is also a complex number, and the basics of the complex number system are taught to students in both high school mathematics and general mathematics (calculus). When the eigenvalue and eigenvector definitions are given with mathematical notation as follows, it is easily understood which structure of the scalar and vector in $Ax = \lambda x$ equation:

Definition: Let A be a given $n \times n$ dimensional matrix. If a $\lambda \in \mathbb{C}$ and nonzero vector $x \in \mathbb{C}^n$ satisfy the equation $Ax = \lambda x$, then λ is called an eigenvalue of A and x is called an eigenvector of A corresponding to λ. (Horn & Johnson, 2013).

Using the definition, the eigenvalues and corresponding eigenvectors of matrix A given above are obtained as follows: $\lambda_1 = 1 + i,\ x_1 = (5,2 - i)^T$ ve $\lambda_2 = 1 - i,\ x_2 = (5,2 + i)^T$. (Here, "$i$" and "$T$" stand for the complex unit element and the transpose of the matrix respectively).

Linear transformations can be expressed in matrices, and in low dimensional examples, the image of the vector x can be displayed on the graph under this linear transformation. If the eigenvalues of the matrix are real, the transformed image of the eigenvector corresponding to that eigenvalue can be illustrated. For this, by using virtual manipulatives (e.g. GeoGebra application), the student can see any vector of this vector under transformation by selecting any vector that the student wants on the plane with the help of dynamic vectors. In this way, the student sees the definition of eigenvalue and eigenvector better by seeing the appearance of an eigenvector under transformation (Gol Tabaghi & Sinclair, 2013; Gueudet-Chartier, 2004). If the matrix has complex eigenvalues, it is not possible to show this situation to students with the help of graphics. For this reason, the definition of eigenvalue and eigenvector have been mentioned above can be given to students by carefully emphasizing the set of complex numbers. In addition to, real examples and matrix samples with complex eigenvalues can be solved in the course.

The research questions guiding this study are the following:

Q1: How do you express the definition of eigenvalue and eigenvector of a real matrix?

Q2: Let $B = \lambda I - A$ be an $n \times n$ matrix. Consider homogeneous equation $Bx = 0$. What is the dimension of the matrices x and 0?

$$B_{n\times n}x_{\square\times\square} = 0_{\square\times\square}$$

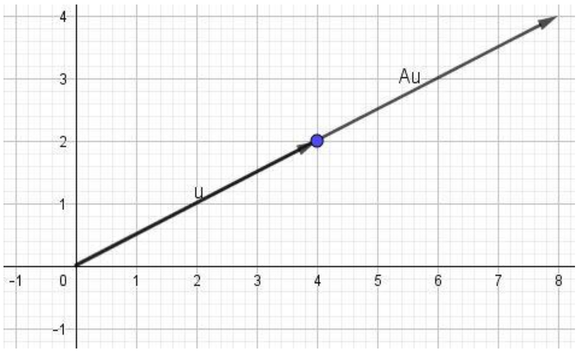
Q3: Let B be an n-dimensional square matrix and x be an n-dimensional vector. Does the equation $Bx = 0$ always have a solution?

☐ Yes ☐ No

Q4: Let B be an n-dimensional square matrix and x be an n-dimensional vector. What property is sufficient for matrix B to have an infinite number of solutions for the $Bx = 0$ system of equations?

☐ B is a square matrix ☐ $\text{trace}(B) = 0$
☐ $\det B = 0$ ☐ B is a diagonal matrix

Q5: For the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and $u \in \mathbb{R}^2$, the relationship between the vector u and the vector Au is given in the following figure.



Which of the following is exactly true according to this figure?

☐ A does not have an eigenvector ☐ Au is an eigenvector
☐ u is an eigenvector ☐ A does not have an eigenvalue

Q6: Which of the following is an eigenvalue of matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$?

☐ 2 ☐ −2 ☐ 3 ☐ −1

Q7: Consider the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$. For every vector u in the xy-plane, if the vector Au is not a multiple of the vector u, can you say "The matrix A does not have an eigenvector"?

☐ Yes ☐ No

Q8: Which of the following is true about the eigenvalues of an $n \times n$ dimensional matrix A?

☐ Infinite number ☐ At least one ☐ It may not have any eigenvalues

Method

The research design is quantitative, it is based on a descriptive analysis method since it puts forward the existing situation. Knowing linear algebra demands that the student starts thinking about the objects and operations of algebra not in terms of relations between particular matrices, vectors and operators but in terms of whole structures of such things: vector spaces over fields (real or complex), algebras, classes of linear operators, which can be transformed, represented in different ways, considered as isomorphic or not, etc. (Hillel & Sierpinska, 1994). Since the aim of this study is to measure the level of knowledge about the concepts of eigenvalue and eigenvector, a scale that includes the basic concepts mentioned above and related to these concepts is proposed. The suggestions in Gol Tabaghi and Sinclair (2013) were also used in the preparation of the scale.

The study group consists of 95 randomly selected third and fourth year students from Mehmet Akif Ersoy University Mathematics Education Department students who were taken a linear algebra course in Turkey in the 2021-2022 academic year.

In the study, a test consisting of 8 open-ended and multiple choice questions has been applied to determine the status of students who took the linear algebra course in which the concept of eigenvalue had been given before, in perceiving these concepts. In the preparation of the test, linear algebra textbooks and three field experts teaching linear algebra at these universities have been considered. In the light of the comments received, three of the 10 questions in the draft have been omitted and one question has been added. The test includes the definitions of eigenvalue and eigenvector concepts and questions about the calculation of eigenvalues and eigenvectors of a matrix.

The questions Q3 and Q4, which measure the existence of a solution based on theoretical knowledge, are questions that corresponds to the same learning outcome. Except for these two questions, each question correspond to the different sub-outcomes respectively. The question Q1 evaluates first outcome, Q2 second outcome, Q3 and Q4 third outcome, Q5 fourth outcome, Q6 fifth outcome, Q7 sixth outcome and Q8 seventh outcome.

A general evaluation of the students' answers has been made as "correct, incorrect, none". The student's answers have been examined in detail. It is determined that difficulty indexes of the questions are between 0.17 and 0.76. For the reliability of the scale, Kuder-Richardson 20 (KR-20) reliability analysis was done. The KR-20 reliability coefficient of the test is calculated as 0.72. According to Fraenkel and Wallen (2008), the obtained measurements are reliable. Also, data processing was carried out by using the chi-square test to determine the relationship between learning outcomes and student academic achievement with alpha degrees = 0.05 through the SPSS program. In the line with the results obtained from the analysis, inferences have been made on the perceptions of the students about eigenvalue-eigenvector and the reasons for the problems experienced.

Findings and Discussions

The percentage of correct answers given to the questions prepared by considering the solution of the equation system, the existence of the solution of the system, the root number of the polynomial, the basic theorems and the concepts have been mentioned above, are evaluated.

In question Q1 the students are asked to describe the definition of eigenvalue and vector of a real matrix. The set of complex numbers has not been mentioned in most of the students' answers. Our study shows that students think that the scalar number is a real number. However, the concept of scalar number represents the number belonging to the field of linear vector space, and this can be a real or complex number depending on the problem being studied. Student T has described it as (Figure 1):

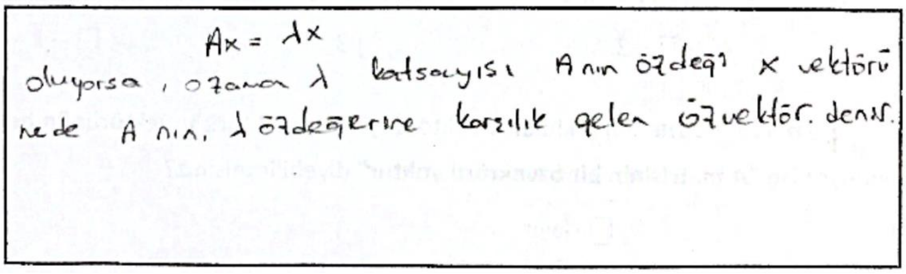

$Ax = \lambda x$
oluyorsa, o zaman λ katsayısı A nın özdeğri x vektörü ne de A nın λ özdeğerine karşılık gelen özvektör denir.

Figure 1 The Answer of the Student T to Question 1.

As it is seen, the student T has only written the equation and stated the scalar number λ is an eigenvalue and the vector x is an eigenvector. In the definition has not been given information from which space to take of x and λ. Student B has replied as (Figure 2):

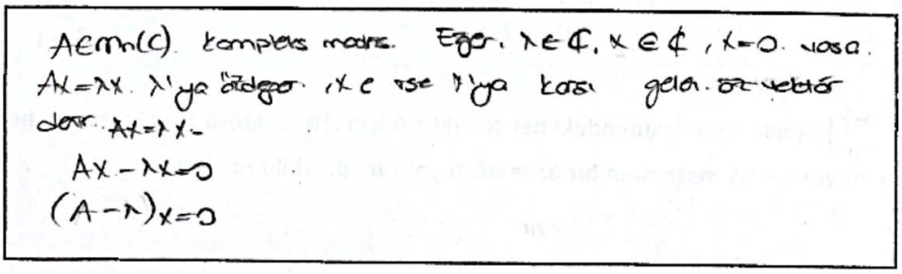

$A \in M_n(\mathbb{C})$ kompleks matris. Eğer $\lambda \in \mathbb{C}$, $x \in \mathbb{C}$, $x \neq 0$ ise
$Ax = \lambda x$ λ'ya özdeğer, $x \in$ ise λ'ya karşı gelen öz vektör denir. $Ax = \lambda x$
$Ax - \lambda x = 0$
$(A - \lambda)x = 0$

Figure 2 The Answer of the Student B to Question 1.

In this answer, λ have been taken from $\mathbb{C}$ but the vector x has not been taken from $\mathbb{C}^n$.

It is seen that students answer the questions Q3, Q4 and Q6 correctly in the range of 40 to 65 percent. They are relatively successful in analytical arithmetic operations. It is observed that they are less successful in problems that require "analytical-structural thinking" (Sinclair & Gol Tabaghi, 2010) skills, namely in Q1, Q2 and Q8.

Question Q8 is quite confusing for students. In fact, they have to state that a matrix has at least one eigenvalue. 24 percent of the answers to question Q8 are "It may not have any eigenvalues". It can be considered that a student who gives this answer comes to the conclusion that "if the polynomial has no real root, it has no root". Students who cannot give the correct answer correspond to a rate of 61 percent. On the other hand, the definition of eigenvalue is requested in Q1. It is seen that complex numbers are not expressed in 66 percent of the answers given for this definition. Based on this, it can be said that most students ignore the fact that polymomas can have a complex root.

In Q5, the real eigenvalue of a matrix and its corresponding eigenvector are given graphically. This question has two purposes, to see if students could carry out the state given by graph (visually) and whether they could write an equation system form. From the answers given to Q5, it is seen that 51 percent of the students are able to connect arithmetic calculation and visual representation. Only a few of the 48 students who correctly answer this question interprete the figure directly. Student A is able to link the arithmetic calculation and visual representation (see Figure 3).

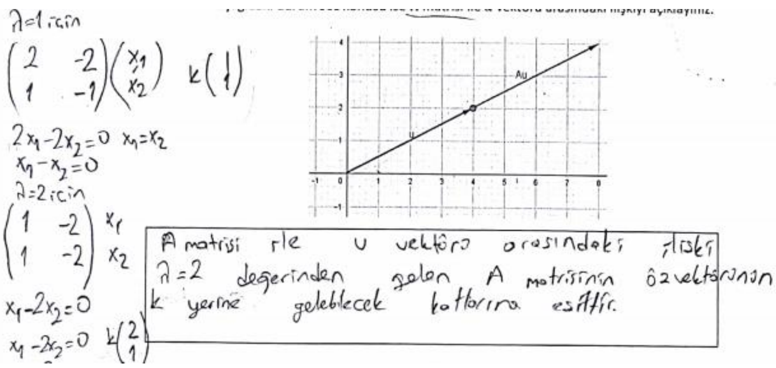


Figure 3 The Answer of the Student A to Question 5.

The most blatant feature of the students’ practical thinking is their tendency to base their understanding of an abstract concept on ‘prototypical examples’ rather than on its definition. For example, linear transformations were understood as ‘rotations, dilations, shears and combinations of these’. This way of understanding made it very difficult for them to see how a linear transformation could be determined by its value on a basis, and consequently, their notion of the matrix of a linear transformation remained at the level of procedure only (Dorier, 2002).

In the question Q7, the fact that a situation like Q5 does not occur is discussed. The eigenvalues of the matrix given in this question are complex. It can be evaluated that the graph given in Q5 is effective in the correctness of 56% of the answers given by the students. Accordingly, it can be thought that the visual handling of the concepts contributes positively to learning and reaching conclusions.

It is remarkable that students generally have answered question Q7 correctly, but many students have answered question Q6 wrong. Student H has not answered the question correctly; but has tried to link the information to the discriminant concept such that if the discriminant is less than zero, it has no eigenvalue (Figure 4).

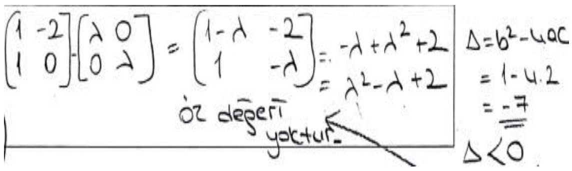


Figure 4 The Answer of the Student H to Question 6.

In the definition, the eigenvalue must be taken as a complex number. Since students generally think that the root of a polynomial is only a real number, these eigenvalues of a matrix with complex eigenvalues are ignored. In fact, if the eigenvalues of a matrix are desired to be two-dimensional and only complex eigenvalues, the answer is “this matrix has no eigenvalue”.

According to the data obtained as a result of the analysis of the answers given to the questions with the Chi-square test, there is a significant relationship between the questions Q1 and Q4; Q1 and Q8; Q2 and Q3; Q5 and Q7; Q7 and Q8.

The questions aiming to evaluate seven learning outcomes are evaluated. The following Table 1 reflects students’ percentage of correct answers and their standard deviations. Since Q3 and Q4 target the same learning outcome, the percentage of correct answers to Q5 and Q7 is calculated taking an average of correct answers of these questions.

Table 1 *Descriptive Statistics of Students (n=95)*

Learning Outcomes	**Questions**	**Percentage of Correct Answers**	**Standard Deviation**
Defines the concepts of eigenvalue and eigenvector	Q1	34%	0.724
Expresses the explicit state of the system of equations given in implicit form	Q2	17%	0.729
Uses the theoretical knowledge about the existence of the solution of the system	Q3 and Q4	64%	0.495

Graphically interprets a system of linear equations	Q5	51%	0.614
Algebraically calculates the eigenvalues of the two-dimensional square matrix	Q6	41%	0.597
Knows that if the eigenvalues of a square matrix of dimension 2 are complex, the eigenvectors corresponding to these eigenvalues cannot be represented in the Euclidean plane	Q7	56%	0.581
Knows that a square matrix has at least one eigenvalue	Q8	39%	0.509

As can be seen from Table 1, the percentage of correct answers to the question about the existence of at least one eigenvalue of a square matrix is 39%. In addition, the percentage of students who fully expressed the definition of eigenvalue is obtained as 34%. These two values are the lower percentages in the table.

The percentage of correct answers to the question about implicit representation of linear equations via matrix form which is one of the basic subjects of Linear Algebra course is 17. One reason why the ratio is low can be interpreted as the lack of understanding of the concept of dimension. On the other hand, the percentage of correct answers given to the questions about theoretical knowledge of linear equations system, namely basic theorems and concepts of Linear Algebra course and the visual representation of equations system are 64 and 51 respectively. The image of an eigenvector corresponding to a real eigenvalue under a linear transformation is different from the images of other vectors. The relatively high percentage of this problem, such as 51, may be a result of the graph given. The effect of graphic can also be seen in the percentage of correct answers given to the question about the eigenvector of the matrix with the complex eigenvalue, which is 56 percent. In general, students investigate the solution of the problem based on the formula, and if the result of the formula cannot make sense, they choose the negative one among the answers (Figure 4). This is why the percentage of correct answers given to this question is 39 percent. Their success is low as can be seen from the percentage of correct answers to the questions in the analytical structure. This result is consistent with the conclusion Sinclair and Gol Tabaghi (2010) has stated

> Analytic-arithmetic thinking involves describing a proper set-up to carry out computations and specifies an object by a formula. Synthetic-geometric thinking involves using geometric descriptions to visualize mathematical objects in two and three-

dimensional space. Analytic-structural thinking involves thinking about an object in terms of its properties (p. 149).

When dealing with the solution of a mathematical problem, students avoid evaluating the problem geometrically (Gueudet-Chartier, 2004). In one of the questions given to the students, there is a vector in the plane and the image of that vector under a linear transformation. An eigenvalue of the matrix corresponding to the linear transformation and an eigenvector corresponding to this eigenvalue is expected to be easily and accurately expressed by the students. Whereas the students have preferred to convert the given graph to an algebraic equation so they could not reach a solution. In calculus or geometry lessons, students examine many different examples such as straight-line graphs, quadratic equations, function graphs, and solutions of linear equation systems, etc. In this question, it was seen that these students had difficulties with representing a vector by point or arrow (Gueudet-Chartier, 2004; Brousseau, 1998). Gol (2012) concluded that use of the software stimulated both dynamic imagery and different communication strategies. The experience allowed the students to see vectors' direction and position on the plane; to analyze the behavior of vector x and its transformation under the matrix A.

Whether the answers to the questions are dependent on each other is measured by the Pearson's Chi-Square test. The results of the Pearson's Chi-Square test:

Table 2 *Answers to Q1 and Q4*

		Q4			
		empty	*wrong*	*correct*	*Total*
Q1	*empty*	2	5	12	19
	wrong	0	15	29	44
	correct	0	1	31	32
	Total	2	21	72	95

For Q1 and Q4, p value is 0.001 so there is a meaningful relationship between the answers to these questions. The equation for eigenvalue-eigenvector is a homogeneous equation system. The students who are aware of this relationship gave generally correct answers.

Table 3 *Answers to Q1 and Q8*

		Q8			
		empty	*wrong*	*correct*	*Total*
Q1	*empty*	1	14	4	19
	wrong	0	29	15	44
	correct	0	14	18	32
	Total	1	57	37	95

The *p* value is obtained as 0.032. Students who have paid attention to complex numbers into account in the definition have answered the question Q8 correctly.

Table 4 *Answers to Q2 and Q3*

		Q3			
		empty	*wrong*	*correct*	*Total*
Q2	*empty*	23	17	40	23
	wrong	20	19	39	20
	correct	2	14	16	2
	Total	45	50	95	45

For Q2 and Q3, the *p* value is obtained as 0.008. The number of wrong answers to both questions is high. This result shows that the students can not make sense of the equations given in matrix form.

Table 5 *Answers to Q5 and Q7*

		Q7			
		empty	*wrong*	*correct*	*Total*
Q5	*empty*	2	1	3	6
	wrong	0	24	17	41
	correct	2	13	33	48
	Total	4	38	53	95

The p value is 0.000 for these questions Q5 and Q7. The general tendency of the correct and wrong answer is the same. The number of correct answers to Q7 is very high. Here, it is seen that the graph given in Q5 has a positive effect on the interpretation of the question. The visual handling of the concepts contributes positively to learning and reaching conclusions.

Table 6 *Answers to Q7 and Q8*

		Q8			
		empty	*wrong*	*correct*	*Total*
Q7	*empty*	1	1	2	4
	wrong	0	29	9	38
	correct	0	27	26	53
	Total	1	57	37	95

For Q7 and Q8, *p*=0.000. In Table 6, it is seen that students who have given the wrong answer to Q7 generally have given the wrong answer to Q8. More than half of the students do not think that it can be a complex eigenvalue-eigenvector of a matrix.

Conclusions and Suggestions

In the study, it is seen that the majority of the students can not define eigenvalue and eigenvectors. It is determined that 17 percent of the students are able to express the explicit state of the equation system given in an implicit form and the image of a vector under linear transformation, 64 percent of the students are able to use the theoretical knowledge about the existence of the solution of the system, 39 percent of students knew that a square matrix has at least one eigenvalue. In addition, it is seen that more than half of the students are able to interpret the linear system of equations graphically, and with the help of this graphic. They have determined that the eigenvectors of a square matrix with complex eigenvalues could not be represented in the plane.

The system of equations in the eigenvalue definition is a homogeneous system of equations. The problem of finding an eigenvalue and its corresponding eigenvector requires the existence of an infinite solution of the corresponding homogeneous system. It is seen that the students who know the definition of eigenvalue and eigenvector also know that the determinant of the coefficient matrix must be zero. When the answers to the first 6 questions are examined, it is seen that the students have basic knowledge about obtaining the characteristic polynomial of a matrix. However, it is determined that 61 percent of the students have not known how many roots a polynomial should have. As a result, there are deficiencies and mistakes in the questions about the existence of the complex root. In order to solve these problems, it has been taught that it would be useful to mention the complex situation, to give explanatory examples, and to interpret the solutions geometrically while discussing the concept of eigenvalue in the lectures. In addition, it emerges from the results that sufficient attention should be paid to the examples of matrices with complex eigenvalues in the books. The fact that students generally deal with real numbers and consider the set of complex numbers at a secondary or even superficial level may cause complex roots to be ignored in root calculations.

Among the prepared questions, there are two questions about calculating the matrix eigenvalue. The first one has real eigenvalues and the second one has complex eigenvalues. The matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ have real eigenvalues and the equation $Au = \lambda u$ for this matrix is provided for example $\lambda = 1$ and $u = [2,1]$ vector. Thus, students can see the equation $Au = \lambda u$ in $\mathbb{R}^2$ plane. However, in another question, no matter which $u \in \mathbb{R}^2$ $(u \neq 0)$ vector is taken for the matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$, the equation $Au = \lambda u$ is not provided. In this case, students can naturally say that this matrix has no eigenvalue. From the evaluation results, it appears that while dealing with the concept of eigenvalue in the lectures, the complex situation should be mentioned, illustrative examples and geometric interpretation of the solutions, as well as more attention should be paid to such examples in the books. The fact that the complex numbers are not addressed sufficiently in the course causes the most basic structure in eigenvalue and eigenvector definitions to be neglected. Students working mostly on real numbers and leaving the complex number system at the second level or even the basic level may cause the roots to be ignored during calculation of the polynomial root.

The concreteness that seems to lack in linear algebra could be more efficiently provided by the use of drawings, especially drawings illustrating concepts and properties in abstract vector spaces (Gueudet-Chartier, 2004). In the case of eigenvectors, when students can compare eigenvectors related to different eigenvalues or when they can determine linear independence of different vectors associated to an eigenvalue, they show evidence of having constructed a process (Salgado, 2015). Larson et al. (2008) mentioned the difficulty involved in transforming $Av = \lambda v$ into $(A - \lambda I)v = 0$ and later into $|A - \lambda I| = 0$. Our results are consistent with results presented by the studies carried out by Salgado (2015), Stewart and Thomas (2011), Gol (2012), Gueudet-Chartier (2004), and Larson et al. (2008). In the research conducted by Gol Tabaghi and Sinclair (2013) it was determined that students had difficulties in making the formal definition of the concepts of eigenvalue and eigenvector, understanding the behavior of the eigenvector, and establishing the connection between these concepts with the concept of linear transformation, as well. Perhaps before the eigenvalue and eigenvector concepts are given, even before the solution of linear systems, the fundamental theorem of algebra about how many roots of an n^{th} order polynomial should be given. As Tabaghi and Sinclair (2013) concluded, the use of software contributes to a deeper understanding of abstract concepts. Approximate roots of high-order polynomials as well as second-order samples should be sampled with appropriate mathematical software (for example Maple, Matlab, GeoGebra, Sketchpad), and then matrices should be addressed.

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Lisans Matematik Eğitimi Öğrencilerinin Özdeğer ve Özvektör Algıları Üzerine

Özet:

Matrislerle ifade edilebilen sistemlerin analizi, uygulama alanında oldukça önemlidir. Böyle bir sistemin düzgün çalışıp çalışmadığı sistemi temsil eden matrisin özdeğerlerine göre belirlenmektedir. Lisans seviyesinde özdeğer ve özvektör kavramlarının öğretimi doğrusal cebir dersi kapsamında yapılmaktadır. Bu çalışmada doğrusal cebir dersini almış lisans öğrencilerinin özdeğer ve özvektör kavramları hakkındaki algıları araştırılmıştır. Araştırma Eğitim Fakültesi Matematik Eğitimi bölümü öğrencilerinden 95 öğrencinin katılımıyla gerçekleştirilmiştir. Öğrencilerin özdeğer teorisine yaklaşımlarını ölçen bir ölçek geliştirilmiştir. Ölçeğin güvenilirliği için Kuder-Richardson 20 (KR-20) güvenirlik analizi yapılmıştır ve 0,72 olarak bulunmuştur. Çalışmada öğrenme çıktıları ile akademik başarı arasındaki ilişkiyi görmek için ki-kare testi kullanılmış ve betimsel analiz yapılmıştır. Özdeğer ve özvektör kavramlarının algılanmasında ortaya çıkan problemler ve çözüm önerileri sunulmaktadır.

Anahtar kelimeler: lineer cebir, matrisler, özdeğer, özvektör.

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