Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 52 (4) (2023), 923 – 930
DOI : 10.15672/hujms.1076176

Research Article

Asymptotic behavior of solutions of N-th order forced integro-differential equations with β-Laplacian

Said Rezk Grace[1], John R. Graef*[2], Ercan Tunç[3]

[1] *Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12221, Egypt*
[2] *Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA*
[3] *Department of Mathematics, Faculty of Arts and Sciences, Tokat Gaziosmanpasa University, 60240, Tokat, Turkey*

Abstract

The authors prove some new results on the asymptotic behavior of solutions of nth order forced integro-differential equations with a β-Laplacian. The main goal is to investigate when all solutions behave at infinity like certain nontrivial nonlinear functions. They apply a technique involving Young's inequality. The paper concludes with two examples illustrating the applicability of the main results.

Mathematics Subject Classification (2020). 34E10, 34A34

Keywords. higher order integro-differential equations, asymptotic behavior, β-Laplacian, forced equation

1. Introduction

Consider the forced integro-differential equation

$$\left(a(t)\,(x'(t))^{\beta}\right)^{(n-1)} = e(t) + \int_c^t (t-s)^{\alpha-1} f(s, x(s))ds, \tag{1.1}$$

where $\beta \geq 1$ is the ratio of odd positive integers, $\alpha \in (0,1)$, $c > 1$, and $n \in \mathbb{N}$. We assume that:

(i) $a : [c, \infty) \to (0, \infty)$ and $e : [c, \infty) \to \mathbb{R}$ are continuous functions;
(ii) $f : [c, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exist a continuous function $m : [c, \infty) \to (0, \infty)$ and positive numbers γ and τ with $\gamma \leq \beta$ such that

$$xf(t, x) \leq m(t)t^{\tau-1}\,|x|^{\gamma+1}.$$

*Corresponding Author.
Email addresses: saidgrace@yahoo.com (S. R. Grace), John-Graef@utc.edu (J. R. Graef), ercantunc72@yahoo.com (E. Tunç)
Received: 19.02.2022; Accepted: 27.12.2022

A function $x : [c, \infty) \to \mathbb{R}$ is a *solution* of equation (1.1) if $x \in C^1([c,\infty),\mathbb{R})$, $a(x')^\beta \in C^{n-1}([c,\infty),\mathbb{R})$, and x satisfies equation (1.1). Oscillation and nonoscillation of such solutions are defined in the usual way.

In the last few decades, integral equations have gained considerably more attention due to their applications in many engineering and scientific disciplines; they appear as the mathematical models for systems and processes in areas such as physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex media.

Oscillation and other asymptotic results for integral as well as integro-differential equations are relatively scarce in the literature; some recent results on various types of integral equations can be found in [1, 3–12, 15–17]. It appears that there are no such results for integral equations of the type (1.1). The main objective of this paper then is to establish some new criteria for the asymptotic behavior of all solutions of equation (1.1). We also investigate some new criteria on the asymptotic behavior of the nonoscillatory solutions of equation (1.1) with $\tau = 1$ in condition (ii).

2. Main results

To obtain our results in this paper, we shall make use of the following lemmas.

Lemma 2.1 (Young's inequality [13]). *If X and Y are nonnegative, $\delta > 1$, and $1/\delta + 1/\delta^* = 1$, then*

$$XY \le \frac{1}{\delta}X^\delta + \frac{1}{\delta^*}Y^{\delta^*}, \tag{2.1}$$

where equality holds if and only if $Y = X^{\delta-1}$.

Lemma 2.2 ([14, 18]). *Let β, γ, and p be positive constants such that*

$$p(\beta-1)+1>0 \quad \textit{and} \quad p(\gamma-1)+1>0.$$

Then

$$\int_0^t (t-s)^{p(\beta-1)}\, s^{p(\gamma-1)} ds = t^\theta B, \quad t \ge 0,$$

where

$$B := B\left[p(\gamma-1)+1, p(\beta-1)+1\right] \quad \textit{and} \quad B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds,$$

for $\xi > 0$, $\eta > 0$, and $\theta = p(\beta+\gamma-2)+1$.

Lemma 2.3 ([2]). *Let α and p be positive constants such that $p(\alpha-1)+1>0$. Then,*

$$\int_0^t (t-s)^{p(\alpha-1)} e^{ps} ds \le Q e^{pt}, \quad t \ge 0,$$

where

$$Q = \frac{\Gamma\left(1+p(\alpha-1)\right)}{p^{1+p(\alpha-1)}},$$

and

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds, \quad x>0,$$

is the Euler-Gamma function.

For notational purpose, for any continuous function $b : [c,\infty) \to (0,\infty)$, it will be convenient to set

$$g_b(t) = (\beta-\gamma)\left(\frac{\gamma^\gamma}{\beta^\beta}\right)^{\frac{1}{\beta-\gamma}}\left(\frac{m^\beta(t)}{b^\gamma(t)}\right)^{\frac{1}{\beta-\gamma}}, \tag{2.2}$$

and let

$$I(t,c) := \int_c^t a^{-1/\beta}(s)ds.$$

We now give our first result on the asymptotic behavior of the nonoscillatory solutions of equation (1.1).

Theorem 2.4. *Let conditions (i)–(ii) hold and assume that there exist real numbers* $p > 1$, $0 < \alpha < 1$, *and* $\tau = 2 - \alpha - 1/p$ *such that* $p(\alpha - 1) + 1 > 0$ *and* $p(\tau - 1) + 1 > 0$. *If there is a continuous function* $b : [c, \infty) \to (0, \infty)$ *such that*

$$\int_c^\infty b^q(t) t^{(n-1)q} I^{\beta q}(t, c) dt < \infty, \quad \text{where } \; q = \frac{p}{p-1}, \tag{2.3}$$

$$\lim_{t\to\infty} |e(t)| < \infty, \tag{2.4}$$

and

$$\lim_{t\to\infty} \int_c^t (t-s)^{\alpha-1} s^{\tau-1} g_b(s) ds < \infty, \tag{2.5}$$

then every nonoscillatory solution $x(t)$ *of equation* (1.1) *satisfies*

$$\limsup_{t\to\infty} \frac{|x(t)|}{t^{(n-1)/\beta} I(t, c)} < \infty. \tag{2.6}$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1), say $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. It follows from (i)–(ii) and (1.1) that

$$\begin{aligned}\left(a(t)\left(x'(t)\right)^\beta\right)^{(n-1)} \leq & \int_c^{t_1} (t-s)^{\alpha-1} |f(s, x(s))| ds + |e(t)| \\ & + \int_{t_1}^t (t-s)^{\alpha-1} s^{\tau-1} \left[m(s) x^\gamma(s) - b(s) x^\beta(s)\right] ds \\ & + \int_{t_1}^t (t-s)^{\alpha-1} s^{\tau-1} b(s) x^\beta(s) ds. \end{aligned} \tag{2.7}$$

Applying (2.1) to $\left[m(t)x^\gamma(t) - b(t)x^\beta(t)\right]$ with

$$\delta = \frac{\beta}{\gamma} > 1, \quad X = x^\gamma(t), \quad Y = \frac{\gamma}{\beta}\frac{m(t)}{b(t)}, \quad \text{and} \quad \delta^* = \frac{\beta}{\beta - \gamma},$$

we obtain

$$\begin{aligned} m(t)x^\gamma(t) - b(t)x^\beta(t) &= \frac{\beta}{\gamma} b(t) \left[x^\gamma(t) \frac{\gamma}{\beta}\frac{m(t)}{b(t)} - \frac{\gamma}{\beta}(x^\gamma(t))^{\beta/\gamma}\right] \\ &= \frac{\beta}{\gamma} b(t)\left[XY - \frac{1}{\delta}X^\delta\right] \leq \frac{\beta}{\gamma} b(t) \left(\frac{1}{\delta^*} Y^{\delta^*}\right) \\ &= \left(\frac{\beta-\gamma}{\gamma}\right)\left[\frac{\gamma}{\beta} m(t)\right]^{\beta/(\beta-\gamma)} b^{\gamma/(\gamma-\beta)}(t) := g_b(t), \end{aligned} \tag{2.8}$$

where $g_b(t)$ is as in (2.2). Using (2.8) in (2.7) gives

$$\begin{aligned}\left(a(t)\left(x'(t)\right)^\beta\right)^{(n-1)} \leq & \int_c^{t_1} (t_1-s)^{\alpha-1} |f(s, x(s))| ds + |e(t)| \\ & + \int_{t_1}^t (t-s)^{\alpha-1} s^{\tau-1} g_b(s) ds \\ & + \int_{t_1}^t (t-s)^{\alpha-1} s^{\tau-1} b(s) x^\beta(s) ds. \end{aligned} \tag{2.9}$$

In view of (2.4) and (2.5), it follows from (2.9) that

$$\left(a(t)\left(x'(t)\right)^\beta\right)^{(n-1)} \leq M_1 + \int_{t_1}^t (t-s)^{\alpha-1} s^{\tau-1} b(s) x^\beta(s) ds := w(t)$$

for some constant $M_1 > 0$. Integrating this inequality $(n-1)$-times from t_1 to t gives

$$a(t)\left(x'(t)\right)^\beta \leq M_2 t^{n-2} + M_3 t^{n-1} w(t) := \Omega(t)$$

for some positive constants $M_2 > 0$ and $M_3 > 0$. This can be written as

$$x'(t) \leq \left(\frac{\Omega(t)}{a(t)}\right)^{1/\beta}. \tag{2.10}$$

Noting that $\Omega(t)$ is an increasing function, it follows from (2.10) that

$$\begin{aligned} x(t) &\leq x(t_1) + \Omega^{1/\beta}(t)\int_{t_1}^{t} a^{-1/\beta}(s)ds = x(t_1) + \Omega^{1/\beta}(t)I(t,t_1) \\ &= \left[\frac{x(t_1)}{I(t,t_1)} + \Omega^{1/\beta}(t)\right] I(t,t_1) \\ &\leq \left[\frac{x(t_1)}{I(t_2,t_1)} + \Omega^{1/\beta}(t)\right] I(t,t_1) \end{aligned} \tag{2.11}$$

for $t \geq t_2$ and all $t_2 > t_1$. From (2.11), we obtain

$$\frac{x(t)}{I(t,t_1)} \leq M_4 + \Omega^{1/\beta}(t) \quad \text{for } t \geq t_2, \tag{2.12}$$

where $M_4 = x(t_1)/I(t_2,t_1) > 0$. Applying the elementary inequality

$$(A+B)^{\mu} \leq 2^{\mu-1}(A^{\mu}+B^{\mu}), \quad A,B \geq 0 \quad \text{and} \quad \mu \geq 1, \tag{2.13}$$

to (2.12) gives

$$\left(\frac{x(t)}{I(t,t_1)}\right)^{\beta} \leq 2^{\beta-1}(M_4)^{\beta} + 2^{\beta-1}\Omega(t) \quad \text{for } t \geq t_2. \tag{2.14}$$

In view of the definition of $\Omega(t)$, it follows from (2.14) that

$$\left(\frac{x(t)}{I(t,t_1)}\right)^{\beta} \leq 2^{\beta-1}(M_4)^{\beta} + 2^{\beta-1}\left[M_2 t^{n-2} + M_3 t^{n-1} w(t)\right],$$

from which we see that

$$\left(\frac{x(t)}{t^{(n-1)/\beta}I(t,t_1)}\right)^{\beta} \leq M_5 + 2^{\beta-1}M_3 w(t) \tag{2.15}$$

for some constant $M_5 > 0$. In view of the definition of $w(t)$, it follows from (2.15) that

$$\left(\frac{x(t)}{t^{(n-1)/\beta}I(t,t_1)}\right)^{\beta} \leq M_6 + M_7\int_{t_1}^{t}(t-s)^{\alpha-1}\, s^{\tau-1}b(s)x^{\beta}(s)ds, \tag{2.16}$$

where $M_6 = M_5 + 2^{\beta-1}M_1M_3$ and $M_7 = 2^{\beta-1}M_3$. Applying Hölder's inequality and Lemma 2.2 to the integral on the right in (2.16), we obtain

$$\begin{aligned} \int_{t_1}^{t}(t-s)^{\alpha-1}\, s^{\tau-1}b(s)x^{\beta}(s)ds &\leq \left(\int_{t_1}^{t}(t-s)^{p(\alpha-1)}\, s^{p(\tau-1)}ds\right)^{1/p}\left(\int_{t_1}^{t} b^{q}(s)x^{\beta q}(s)ds\right)^{1/q} \\ &\leq \left(\int_{0}^{t}(t-s)^{p(\alpha-1)}\, s^{p(\tau-1)}ds\right)^{1/p}\left(\int_{t_1}^{t} b^{q}(s)x^{\beta q}(s)ds\right)^{1/q} \\ &\leq (Bt^{\theta})^{1/p}\left(\int_{t_1}^{t} b^{q}(s)x^{\beta q}(s)ds\right)^{1/q} \\ &= B^{1/p}\left(\int_{t_1}^{t} b^{q}(s)x^{\beta q}(s)ds\right)^{1/q}, \end{aligned} \tag{2.17}$$

where

$$B = B\left[p(\tau-1)+1, p(\alpha-1)+1\right], \quad \text{and} \quad \theta = p(\tau+\alpha-2)+1 = 0.$$

Using (2.17) in (2.16), we obtain

$$z(t) := \left(\frac{x(t)}{t^{(n-1)/\beta}I(t,t_1)}\right)^{\beta} \leq 1 + M_6 + M_8\left(\int_{t_1}^{t} b^{q}(s)x^{\beta q}(s)ds\right)^{1/q}, \tag{2.18}$$

where $M_8 = M_7 B^{1/p} > 0$. Employing again inequality (2.13), we obtain from (2.18) that

$$z^q(t) \le 2^{q-1}(1+M_6)^q + 2^{q-1}M_8^q \int_{t_1}^t b^q(s)x^{\beta q}(s)ds,$$

which, in view of the left hand side of (2.18), can be written as

$$z^q(t) \le 2^{q-1}(1+M_6)^q + 2^{q-1}M_8^q \int_{t_1}^t b^q(s)s^{(n-1)q}I^{\beta q}(s,t_1)z^q(s)ds. \tag{2.19}$$

Setting $P_1 = 2^{q-1}(1+M_6)^q$, $Q_1 = 2^{q-1}M_8^q$, and $w(t) = z^q(t)$ so that $z(t) = w^{1/q}(t)$, inequality (2.19) becomes

$$w(t) \le P_1 + Q_1 \int_{t_1}^t b^q(s)s^{(n-1)q}I^{\beta q}(s,t_1)w(s)ds.$$

By Gronwall's inequality and (2.3), we see that $w(t)$ is bounded. Thus,

$$\limsup_{t\to\infty} \frac{x(t)}{t^{(n-1)/\beta}I(t,t_1)} < \infty,$$

which is what we wanted to show.

The proof in case $x(t)$ is eventually negative is similar. This completes the proof of the theorem. □

We now give our second result on the asymptotic behavior of nonoscillatory solutions of (1.1).

Theorem 2.5. *Let condition (i) and condition (ii) with* $\tau = 1$ *hold, and assume that there exist* $p > 1$ *and* $0 < \alpha < 1$ *such that* $p(\alpha - 1) + 1 > 0$. *If, in addition to* (2.3) *and* (2.4), *there is a continuous function* $b : [c,\infty) \to (0,\infty)$ *such that*

$$\lim_{t\to\infty} \int_c^t (t-s)^{\alpha-1} g_b(s)ds < \infty,$$

then every nonoscillatory solution $x(t)$ *of equation* (1.1) *satisfies*

$$\limsup_{t\to\infty} \frac{e^{-t/\beta}|x(t)|}{t^{(n-1)/\beta}I(t,c)} < \infty.$$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge c$. Proceeding exactly as in the proof of Theorem 2.4, we again arrive at (2.16) with $\tau = 1$, namely,

$$\left(\frac{x(t)}{t^{(n-1)/\beta}I(t,t_1)}\right)^\beta \le M_6 + M_7 \int_{t_1}^t (t-s)^{\alpha-1} b(s)x^\beta(s)ds. \tag{2.20}$$

Applying Hölder's inequality and Lemma 2.3 to the integral on the right hand side, we obtain

$$\begin{aligned}
\int_{t_1}^t (t-s)^{\alpha-1} b(s)x^\beta(s)ds &= \int_{t_1}^t \left[(t-s)^{\alpha-1}e^s\right]\left[e^{-s}b(s)x^\beta(s)\right]ds \\
&\le \left(\int_{t_1}^t (t-s)^{p(\alpha-1)}e^{ps}ds\right)^{1/p}\left(\int_{t_1}^t e^{-qs}b^q(s)x^{\beta q}(s)ds\right)^{1/q} \\
&\le \left(\int_0^t (t-s)^{p(\alpha-1)}e^{ps}ds\right)^{1/p}\left(\int_{t_1}^t e^{-qs}b^q(s)x^{\beta q}(s)ds\right)^{1/q} \\
&\le (Qe^{pt})^{1/p}\left(\int_{t_1}^t e^{-qs}b^q(s)x^{\beta q}(s)ds\right)^{1/q} \\
&= Q^{1/p}e^t\left(\int_{t_1}^t e^{-qs}b^q(s)x^{\beta q}(s)ds\right)^{1/q}.
\end{aligned} \tag{2.21}$$

Using (2.21) in (2.20), we obtain

$$z(t) := \left(\frac{x(t)}{t^{(n-1)/\beta} e^{t/\beta} I(t,t_1)}\right)^{\beta} \le 1 + M_9 + M_{10}\left(\int_{t_1}^{t} e^{-qs} b^q(s) x^{\beta q}(s) ds\right)^{1/q}, \qquad (2.22)$$

where M_9 an upper bound for $M_6 e^{-t}$, and $M_{10} = M_7 Q^{1/p} > 0$. Employing inequality (2.13) again, (2.22) becomes

$$z^q(t) \le 2^{q-1}(1+M_9)^q + 2^{q-1} M_{10}^q \int_{t_1}^{t} e^{-qs} b^q(s) x^{\beta q}(s) ds,$$

which can be written as

$$z^q(t) \le 2^{q-1}(1+M_9)^q + 2^{q-1} M_{10}^q \int_{t_1}^{t} b^q(s) s^{(n-1)q} I^{\beta q}(s,t_1) z^q(s) ds. \qquad (2.23)$$

Setting $P_2 = 2^{q-1}(1+M_9)^q$, $Q_2 = 2^{q-1} M_{10}^q$, and $w(t) = z^q(t)$, inequality (2.23) becomes

$$w(t) \le P_2 + Q_2 \int_{t_1}^{t} b^q(s) s^{(n-1)q} I^{\beta q}(s,t_1) w(s) ds.$$

The conclusion follows from Gronwall's inequality and (2.3), that is,

$$\limsup_{t\to\infty} \frac{e^{-t/\beta} x(t)}{t^{(n-1)/\beta} I(t,t_1)} < \infty.$$

This completes the proof of the theorem. □

We conclude this paper with two examples to illustrate our results.

Example 2.6. Consider the fourth order integro-differential equation

$$\left(t(x'(t))^3\right)''' = e^{-t}\sin 3t + \int_8^t (t-s)^{-1/2} e^{-4s} s^{-1/6} x^{5/3}(s) ds, \quad t \ge 8. \qquad (2.24)$$

Here we have $\alpha = 1/2$, $c = 8$, $\beta = 3$, $a(t) = t$, $e(t) = e^{-t}\sin 3t$, $f(t, x(t)) = e^{-4t} t^{-1/6} x^{5/3}(t)$, and $\gamma = 5/3$. Then

$$I(t,c) = I(t,8) = \int_8^t s^{-1/3} ds = \frac{3}{2}(t^{2/3} - 4).$$

Letting $p = 3/2$, we see that $q = 3$, $p(\alpha - 1) + 1 = 1/4 > 0$, $\tau = 2 - \alpha - 1/p = 5/6$ and $p(\tau - 1) + 1 = 3/4 > 0$. Letting $m(t) = b(t) = e^{-4t}$, we see that (ii) holds and $g_b(t) = ke^{-4t}$ with $k > 0$. Since

$$\int_c^\infty b^q(t) t^{(n-1)q} I^{\beta q}(t,t_1) dt \le \left(\frac{3}{2}\right)^9 \int_8^\infty \frac{t^{15}}{e^{12t}} dt < \infty,$$

condition (2.3) holds. It can be easily seen that condition (2.4) is satisfied. To see that (2.5) holds, note that letting $u = t - s + 8$, the integral becomes

$$\begin{aligned}
&\int_8^t (t-s)^{-1/2} s^{-1/6} k e^{-4s} ds \\
&\qquad \le -k8^{-1/6} \int_t^8 (u-8)^{-1/2} e^{4u-4t-32} du \\
&\qquad \le \frac{k}{\sqrt{2}e^{4t+32}} \int_8^t (u-8)^{-1/2} e^{4u} du \\
&\qquad = \frac{k}{\sqrt{2}e^{4t+32}} \left[\int_8^{16} (u-8)^{-1/2} e^{4u} du + \int_{16}^t (u-8)^{-1/2} e^{4u} du \right] \\
&\qquad = \frac{k}{\sqrt{2}e^{4t+32}} \left[\lim_{b \to 8^+} \int_b^{16} (u-8)^{-1/2} e^{4u} du \right] + \frac{k}{\sqrt{2}e^{4t+32}} \left[\int_{16}^t (u-8)^{-1/2} e^{4u} du \right] \\
&\qquad \le \frac{ke^{64}}{\sqrt{2}e^{4t+32}} \lim_{b \to 8^+} \int_b^{16} (u-8)^{-1/2} du + \frac{k\,(16-8)^{-1/2}}{\sqrt{2}e^{4t+32}} \int_{16}^t e^{4u} du \\
&\qquad = \frac{k2^{5/2}e^{64}}{\sqrt{2}e^{4t+32}} + \frac{k2^{-7/2}}{\sqrt{2}e^{4t+32}} \left(e^{4t} - e^{64} \right) < \infty \quad \text{as } t \to \infty,
\end{aligned}$$

so (2.5) holds. Since all conditions of Theorem 2.4 are satisfied, we may conclude that every nonoscillatory solution $x(t)$ of equation (2.24) satisfies (2.6), that is,

$$\limsup_{t \to \infty} \frac{|x(t)|}{t^{(n-1)/\beta} I(t,c)} = \limsup_{t \to \infty} \frac{|x(t)|}{\frac{3}{2} t (t^{2/3} - 4)} < \infty.$$

Example 2.7. Consider the fourth order integro-differential equation

$$\left(t(x'(t))^3 \right)''' = e^{-4t} \cos 3t + \int_8^t (t-s)^{-1/2} e^{-4s} x^{5/3}(s) ds, \quad t \ge 8. \tag{2.25}$$

Here we have $\alpha = 1/2$, $c = 8$, $\beta = 3$, $a(t) = t$, $e(t) = e^{-4t} \cos 3t$, and $f(t, x(t)) = e^{-4t} x^{5/3}(t)$. Proceeding as in Example 2.1, we can easily see that all conditions of Theorem 2.5 are satisfied, and so every nonoscillatory solution of equation (2.25) satisfies

$$\limsup_{t \to \infty} \frac{e^{-t/\beta} |x(t)|}{t^{(n-1)/\beta} I(t,c)} = \limsup_{t \to \infty} \frac{e^{-t/3} |x(t)|}{\frac{3}{2} t (t^{2/3} - 4)} < \infty.$$

References

[1] M. Bohner, S. R. Grace and N. Sultana, *Asymptotic behavior of nonoscillatory solutions of higher-order integro-dynamic equations*, Opuscula Math. **34**, 5–14, 2014.

[2] E. Brestovanská and M. Medveď, *Asymptotic behavior of solutions to second-order differential equations with fractional derivative perturbations*, Electron. J. Differ. Eq. **2014** (201), 1–10, 2014.

[3] B. C. Dhage, A. V. Deshmukh and J. R. Graef, *On asymptotic behavior of a nonlinear functional integral equation*, Commun. Appl. Nonlinear Anal. **15**, 55–67, 2008.

[4] S. R. Grace, J. R. Graef, S. Panigrahi and E. Tunç, *On the oscillatory behavior of Volterra integral equations on time-scales*, PanAmer. Math. J. **23**, 35–41, 2013.

[5] S. R. Grace, J. R. Graef and E. Tunç, *Asymptotic behavior of solutions of certain integro-differential equations*, PanAmer. Math. J. **29**, 45–60, 2019.

[6] S. R. Grace, J. R. Graef and E. Tunç, *On the asymptotic behavior of solutions of certain integro-differential equations*, J. Appl. Anal. Comput. **9**, 1305–1318, 2019.

[7] S. R. Grace, J. R. Graef and A. Zafer, *Oscillation of integro-dynamic equations on time scales*, Appl. Math. Lett. **26**, 383–386, 2013.

[8] S. R. Grace and A. Zafer, *Oscillatory behavior of integro-dynamic and integral equations on time scales*, Appl. Math. Lett. **28**, 47–52, 2014.

[9] J. R. Graef and S. R. Grace, *On the asymptotic behavior of solutions of certain forced third order integro-differential equations with d-Laplacian*, Appl. Math. Lett. **83**, 40–45, 2018.

[10] J. R. Graef, S. R. Grace and E. Tunç, *On the oscillation of certain integral equations*, Publ. Math. Debrecen **90**, 195–204, 2017.

[11] J. R. Graef and C. Tunç, *Continuability and boundedness of multi-delay functional integro-differential equations of the second order*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **109**, 169–173, 2015.

[12] J. R. Graef and O. Tunç, *Asymptotic behavior of solutions of Volterra integro-differential equations with and without retardation*, J. Integral Equations Appl. **33** (3), 289–300, 2021.

[13] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.

[14] Q.-H. Ma, J. Pečarić and J.-M. Zhang, *Integral inequalities of systems and the estimate for solutions of certain nonlinear two-dimensional fractional differential systems*, Comput. Math. Appl. **61**, 3258–3267, 2011.

[15] M. Medveď, *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions*, J. Math. Anal. Appl. **214**, 349–366, 1997.

[16] M. Medveď, *Integral inequalities and global solutions of semilinear evolution equations*, J. Math. Anal. Appl. **267**, 643–650, 2002.

[17] M. Medveď and M. Pospíšil, *Asymptotic integration of fractional differential equations with integrodifferential right-hand side*, Math. Modelling Anal. **20**, 471–489, 2015.

[18] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integral and Series: Elementary Functions*, Vol. 1, Nauka, Moscow, 1981 [in Russian].