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SHARP WEAK BOUNDS FOR p-ADIC HARDY OPERATORS ON p-ADIC LINEAR SPACES

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Abstract. The current paper establishes the sharp weak bounds of p-adic fractional Hardy operator. Furthermore, optimal weak type estimates for p-adic Hardy operator on central Morrey space are also acquired.

1. Introduction

For every non-zero rational number x there is a unique $k = k(x) \in \mathbb{Z}$ such that $x = p^k s/t$, where $p \geq 2$ is a fixed prime number which is coprime to $s, t \in \mathbb{Z}$. We define a mapping $|.|_p : \mathbb{Q} \to \mathbb{R}_+$ as follows:

$$|x|_p = \begin{cases} p^{-k} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1}$$

The p-adic norm $|\cdot|_p$ undergoes many properties of the usual real norm $|\cdot|$ with an additional non-Archimedean property,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \tag{2}$$

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of rational numbers with respect to the p-adic norm $|\cdot|_p$. A p-adic number $x \in \mathbb{Q}_p$ can be written in the formal power series as (see [30]):

$$x = p^k(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + ...) \tag{3}$$

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where $\alpha_i, k \in \mathbb{Z}, \alpha_0 \neq 0, \alpha_i \in \{0, 1, 2, ..., p-1\}, i = 1, 2, \cdots$. The p-adic norm ensures the convergence of series (3) in $\mathbb{Q}_p$, because $|p^k \alpha_i p^i|_p \leq p^{-k-i}$.

The n-dimensional vector space $\mathbb{Q}_p^n$, $n \geq 1$, consists of tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$ and $j = 1, 2, \ldots, n$. The norm on this space is given by

$$|\mathbf{x}|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

In non-Archimedean geometry, the ball and its boundary are defined, respectively, as:

$$B_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^k\}, \;\; S_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^k\}.$$

For convenience we denote $B_k(\mathbf{0})$and $S_k(\mathbf{0})$ by B_k and S_k, respectively.

The local compactness and commutativity of the group $\mathbb{Q}_p^n$ under addition implies the existence of Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0} d\mathbf{x} = |B_0|_H = 1,$$

where the notation $|B|_H$ refers to the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. Furthermore, it is not hard to see that $|B_k(\mathbf{a})|_H = p^{nk}$, $|S_k(\mathbf{a})|_H = p^{nk}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Let $w(\mathbf{x})$ be a nonnegative locally integrable function on $\mathbb{Q}_p^n$ and $w(E)$ the weighted measure of measurable subset $E \subset \mathbb{Q}_p^n$, that is $w(E) = \int_E w(x)dx$ respectively. The space of all complex-valued functions f with norm conditions:

$$\|f\|_{L^r(w;\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^r w(\mathbf{x}) d\mathbf{x} \right)^{1/r} < \infty,$$

is denoted by $L^r(w, \mathbb{Q}_p^n), (0 < r < \infty)$, and is known as weighted Lebesgue space. Note that $L^r(1, \mathbb{Q}_p^n) = L^r(\mathbb{Q}_p^n)$.

In [22], authors have defined the weighted p-adic weak Lebesgue space $L^{r,\infty}(w; \mathbb{Q}_p^n)$ by

$$\|f\|_{L^{r,\infty}(w,\mathbb{Q}_p^n)} = \sup_{\mu > 0} \mu w\Big(\{\mathbf{x} \in \mathbb{Q}_p^n : |f(\mathbf{x})| > \mu\} \Big)^{1/r} < \infty.$$

When $w = 1$, we get the weak Lebesgue space $L^{r,\infty}(\mathbb{Q}_p^n)$ defined in [32]. Next, we give the relevant p-adic function spaces.

Definition 1. *[34] Suppose* $1 < r < \infty$ *and* $\mu \in \mathbb{R}$. *The p-adic space* $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ *is the set of all measurable functions* $f \colon \mathbb{Q}_p^n \to \mathbb{R}$ *which satisfy*

$$\|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |f(\mathbf{x})|^r d\mathbf{x} \right)^{1/r} < \infty.$$

When $\mu = -1/r$, *then*

$\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = L^r(\mathbb{Q}_p^n)$. *It is easy to see that* $\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ *is reduced to* $\{0\}$ *whenever* $\mu < -1/r$.

Definition 2. *[35] Suppose $\mu \in \mathbb{R}$ and $1 < r < \infty$. The p-adic space $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ is defined as*

$$W\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = \{f : \|f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} |B_\gamma|_H^{-\mu-1/r} \|f\|_{WL^r(B_\gamma)},$$

and $\|f\|_{WL^r(B_\gamma)}$ is the local p-adic L^r-norm of $f(x)$ restricted to the ball B_γ, that is

$$\|f\|_{WL^r(B_\gamma)} = \sup_{\mu>0} |\{\mathbf{x} \in B_\gamma : |f(\mathbf{x})| > \mu\}|^{1/r}.$$

Evidently, if $\mu = -1/r$, then $W\dot{B}^{r,\mu}(\mathbb{Q}_p^n) = L^{r,\infty}(\mathbb{Q}_p^n)$. Also, $\dot{B}^{r,\mu}(\mathbb{Q}_p^n) \subseteq W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)$ for $-1/r < \mu < 0$ and $1 \leq r < \infty$.

In the last several decades, a growing interest to p-adic models have been seen because p-adic analysis is a natural base for development of various models of ultrametric diffusion energy landscape [4]. It also attracts great deal of interest towards quantum mechanics [30], theoretical biology [11], quantum gravity [1, 7], string theory [31], spin glass theory [3, 26]. In [4], it was shown that the p-adic analysis can be efficiently applied both to relaxation in complex speed systems and processes combined with the relaxation of a complex environment. Besides, the applications of p-adic analysis can be found in harmonic analysis and pseudo-differential equations, see for example [5, 9, 10, 21, 28, 29].

The one-dimensional Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0, \tag{4}$$

has been introduced by Hardy in [18] for measurable functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$. This operator satisfies the inequality

$$\|Hf\|_{L^r(\mathbb{R}^+)} \leq \frac{r}{r-1}\|f\|_{L^r(\mathbb{R}^+)}, \quad 1 < r < \infty, \tag{5}$$

where the constant $r/(r-1)$ is sharp.

In [12], Faris has proposed an extension of the Hardy operator H on higher dimensional Euclidean space $\mathbb{R}^n$ by

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{t}|\leq|\mathbf{x}|} f(\mathbf{t})d\mathbf{t}. \tag{6}$$

where $|\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{1/2}$ for $\mathbf{x} = (x_1, \cdots, x_n)$. In addition, Christ and Grafakos [8] have obtained the exact value of the norm of (6). For more details related to Hardy type operators and, in particular, to boundedness of these operators, we refer to publications [6, 13, 19, 23, 24, 27, 36, 39].

On the other hand, the fractional Hardy operator is obtained by merely writing $|\cdot|^{n-\alpha}$ $(0 \leq \alpha < n)$ instead of $|\cdot|^n$ with in (6). The weak type estimates for the fractional Hardy type operators has also spotlighted many researchers in the past, see for example [2, 13, 15, 16, 20, 37, 38].

In what follows, the higher dimensional fractional Hardy operator in the p-adic field

$$H^p_\alpha f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{t}|_p \le |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t}, \qquad \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}.$$

has been defined and studied for $0 \le \alpha < n$ and $f \in L_{\mathrm{loc}}(\mathbb{Q}_p^n)$ in [33]. When $\alpha = 0$, the operator H^p_α transfers to the p-adic Hardy operator (see [14]). Fu et al. in [14] have acquired the optimal bounds of p-adic Hardy operator on Lebesgue spaces. For more details, we refer the publications [17, 22, 25, 34] and the references therein.

The purpose of the current paper is to study the sharp weak bounds for fractional Hardy operator in the p-adic field on p-adic Lebesgue space. Moreover, we also discuss the optimal weak type estimates for Hardy operator in the p-adic field on central Morrey spaces.

2. Sharp weak bounds for p-adic fractional Hardy Operator on Lebesgue spaces

Our main result for this section is as follows.

Theorem 1. *Suppose $0 < \alpha < n$ and $n + \gamma > 0$. If $f \in L^1(\mathbb{Q}_p^n)$, then*

$$\|H^p_\alpha f\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma;\mathbb{Q}_p^n)} \le C \|f\|_{L^1(\mathbb{Q}_p^n)},$$

where the constant

$$C = \left(\frac{1 - p^{-n}}{1 - p^{-(n+\gamma)}} \right)^{(n-\alpha)/(n+\gamma)}$$

is optimal.

Proof. We have

$$\begin{aligned} |H^p_\alpha f(\mathbf{x})| &= \left| \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{t}|_p \le |\mathbf{x}|_p} f(\mathbf{t}) d\mathbf{t} \right| \\ &\le |\mathbf{x}|_p^{-(n-\alpha)} \|f\|_{L^1(\mathbb{Q}_p^n)}. \end{aligned} \tag{7}$$

Let $C_1 = \|f\|_{L^1(\mathbb{Q}_p^n)}$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H^p_\alpha f(\mathbf{x})| > \mu\} \subset \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x}|_p < (C_1/\mu)^{1/(n-\alpha)}\}.$$

Thus,

$$
\begin{aligned}
&\|H^p_\alpha f\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|x|_p^\gamma;\mathbb{Q}_p^n)}\\
\leq&\sup_{\mu>0}\mu\bigg(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|H^p_\alpha f(\mathbf{x})|>\mu\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
\leq&\sup_{\mu>0}\mu\bigg(\int_{\mathbb{Q}_p^n}\chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<\left(C_1/\mu\right)^{1/(n-\alpha)}\right\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{\mu>0}\mu\bigg(\int_{|\mathbf{x}|_p<\left(C_1/\mu\right)^{1/(n-\alpha)}}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{\mu>0}\mu\bigg(\sum_{j=-\infty}^{\log_p\left(C_1/\mu\right)^{1/(n-\alpha)}}\int_{S_j}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{\mu>0}\mu\bigg(\sum_{j=-\infty}^{\log_p\left(C_1/\mu\right)^{1/(n-\alpha)}}p^{j(n+\gamma)}d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\sup_{\mu>0}\mu\left(\frac{C_1}{\mu}\right)\\
\leq&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}\|f\|_{L^1(|\mathbf{x}|_p^\beta)}. \qquad (8)
\end{aligned}
$$

To show that the constant

$$\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)},$$

appeared in (8) is optimal, we proceed as, consider

$$f_0(\mathbf{x})=\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p\leq 1\}}(\mathbf{x}),$$

then

$$\|f_0\|_{L^1(\mathbb{Q}_p^n)}=1.$$

Also,

$$
\begin{aligned}
H^p_\alpha f_0(\mathbf{x})=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}f_0(\mathbf{t})d\mathbf{t}\\
=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{t}|_p\leq 1\}}(\mathbf{t})d\mathbf{t}\\
=&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\begin{cases}\int_{|\mathbf{t}|_p\leq|\mathbf{x}|_p}d\mathbf{t}, & |\mathbf{x}|_p\leq 1;\\ \int_{|\mathbf{t}|_p\leq 1}d\mathbf{t}, & |\mathbf{x}|_p>1.\end{cases}
\end{aligned}
$$

Since $|B_{\log_p|\mathbf{x}|_p}|_H = |\mathbf{x}|_p^n|B_0|_H$, therefore,

$$H_\alpha^p f_0(\mathbf{x}) = \begin{cases} |\mathbf{x}|_p^\alpha, & |\mathbf{x}|_p \leq 1; \\ |\mathbf{x}|_p^{\alpha-n}, & |\mathbf{x}|_p > 1. \end{cases}$$

Now,

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \mu\} = \{|\mathbf{x}|_p \leq 1 : |\mathbf{x}|_p^\alpha > \mu\} \cup \{|\mathbf{x}|_p > 1 : |\mathbf{x}|_p^{\alpha-n} > \mu\}.$$

Since $0 < \alpha < n$, therefore, when $\mu \geq 1$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \mu\} = \emptyset,$$

and when $0 < \mu < 1$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \mu\} = \{\mathbf{x} \in \mathbb{Q}_p^n : (\mu)^{1/\alpha} < |\mathbf{x}|_p < (1/\mu)^{1/n-\alpha}\}.$$

Ultimately we are down to:

$$\begin{aligned}
&\|H_\alpha^p f_0\|_{L^{(n+\gamma)/(n-\alpha))),\infty}(|\mathbf{x}|_p^\gamma;\mathbb{Q}_p^n)} \\
&= \sup_{0<\mu<1} \mu\biggl(\int_{\mathbb{Q}_p^n} \chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:(\mu)^{1/\alpha}<|\mathbf{x}|_p<(1/\mu)^{1/(n-\alpha)}\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&= \sup_{0<\mu<1} \mu\biggl(\int_{(\mu)^{1/\alpha}<|\mathbf{x}|_p<(1/\mu)^{1/(n-\alpha)}} |\mathbf{x}|_p^\gamma d\mathbf{x}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=(1-p^{-n})^{(n-\alpha)/(n+\gamma)} \sup_{0<\mu<1} \mu\biggl(\sum_{j=\log_p\mu^{1/\alpha}+1}^{\log_p\mu^{1/(\alpha-n)}} p^{j(n+\gamma)}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=(1-p^{-n})^{(n-\alpha)/(n+\gamma)} \sup_{0<\mu<1} \mu\biggl(\frac{p^{(\log_p\mu^{1/\alpha}+1)(n+\gamma)} - p^{(\log_p\mu^{1/(\alpha-n)}+1)(n+\gamma)}}{1-p^{(n+\gamma)}}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=(1-p^{-n})^{(n-\alpha)/(n+\gamma)} \sup_{0<\mu<1} \mu\biggl(\frac{\mu^{(n+\gamma)/\alpha} - \mu^{(n+\gamma)/(\alpha-n)}}{p^{-(n+\gamma)}-1}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=(1-p^{-n})^{(n-\alpha)/(n+\gamma)} \sup_{0<\mu<1} \biggl(\frac{1-\mu^{(n+\gamma)/\alpha}\mu^{(n+\gamma)/(n-\alpha)}}{1-p^{-(n+\gamma)}}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=\biggl(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\biggr)^{(n-\alpha)/(n+\gamma)} \sup_{0<\mu<1} \Bigl(1-\mu^{(n+\gamma)/\alpha}\mu^{(n+\gamma)/(n-\alpha)}\Bigr)^{(n-\alpha)/(n+\gamma)} \\
&=\biggl(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\biggr)^{(n-\alpha)/(n+\gamma)} \\
&=\biggl(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\biggr)^{(n-\alpha)/(n+\gamma)} \|f_0\|_{L^1(\mathbb{Q}_p^n)}.
\end{aligned} \tag{9}$$

We thus conclude from (8) and (9) that

$$\|H^p_\alpha\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma;\mathbb{Q}_p^n)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/q}.$$

□

3. Optimal Weak Type Estimates for p-adic Hardy Operator on Weak Central Morrey Spaces

In the current section we investigate the boundedness of p-adic Hardy operator on p-adic weak central Morrey spaces. It is shown the constant obtained in this case is also optimal.

Theorem 2. *Suppose $-1/r \le \mu < 0$, $1 \le r < \infty$ and if $f \in \dot{B}^{r,\mu}(\mathbb{Q}_p^n)$, then*

$$\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)},$$

and the constant 1 *is optimal.*

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned}|H^p f(\mathbf{x})| \le& \frac{1}{|\mathbf{x}|_p^n}\left(\int_{B(0,|\mathbf{x}|_p)} |f(\mathbf{t})|^r d\mathbf{t}\right)^{1/r}\left(\int_{B(0,|\mathbf{x}|_p)} d\mathbf{t}\right)^{1/r'}\\ =&|\mathbf{x}|_p^{n\mu}\|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.\end{aligned}$$

Let $C_2 = \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}$. Since $\mu < 0$, we have

$$\begin{aligned}\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le& \sup_{\gamma\in\mathbb{Z}}\sup_{y>0} y|B_\gamma|_H^{-\mu-1/r}\left|\{\mathbf{x}\in B_\gamma : C_2|\mathbf{x}|_p^{n\mu} > y\}\right|^{1/r}\\ =& \sup_{\gamma\in\mathbb{Z}}\sup_{y>0} y|B_\gamma|_H^{-\mu-1/r}\left|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}\right|^{1/r}.\end{aligned}$$

If $\gamma \le \log_p(y/C_2)^{1/n\mu}$, then for $\mu < 0$, we obtain

$$\begin{aligned}&\sup_{y>0}\sup_{\gamma\le\log_p(y/C_2)^{1/n\mu}} y|B_\gamma|_H^{-\mu-1/r}\left|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}\right|^{1/r}\\ &\le \sup_{y>0}\sup_{\gamma\le\log_p(y/C_2)^{1/n\mu}} tp^{-\gamma n\mu}\\ &= C_2\\ &\le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.\end{aligned}$$

If $\gamma > \log_p(y/C_2)^{1/n\mu}$, then for $\mu + 1/r > 0$, we get

$$\begin{aligned}
&\sup_{y>0} \sup_{\gamma>\log_p(y/C_2)^{1/n\mu}} y|B_\gamma|_H^{-\mu-1/r}|\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (y/C_2)^{1/n\mu}\}|^{1/r} \\
&\le \sup_{y>0} \sup_{\gamma>\log_p(y/C_2)^{1/n\mu}} yp^{-\gamma n(\mu+1/r)}(y/C_2)^{1/r\mu} \\
&= C_2 \\
&\le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore,

$$\|H^p f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \le \|f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}. \tag{10}$$

Conversely, to prove that the constant 1 is optimal, consider

$$f_0(\mathbf{x}) = \chi_{\{|\mathbf{x}|_p \le 1\}}(\mathbf{x}),$$

then,

$$\|f_0\|_{\dot{B}^{q,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} \chi_{\{|\mathbf{x}|_p\le 1\}}(\mathbf{x}) d\mathbf{x}\right)^{1/r}.$$

If $\gamma < 0$, then

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} d\mathbf{x}\right)^{1/r} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} p^{-n\gamma\mu} = 1,$$

since $\mu < 0$. If $\gamma \ge 0$, then using the condition that $\mu + 1/r > 0$, we have

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\ge 0}} \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_0} d\mathbf{x}\right)^{1/r} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\ge 0}} p^{-n\gamma(\mu+1/r)} = 1.$$

Therefore,

$$\|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = 1.$$

Moreover,

$$H^p f_0(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}|_p \le 1; \\ |\mathbf{x}|_p^{-n}, & |\mathbf{x}|_p > 1, \end{cases}$$

which implies that $|H^p f_0(\mathbf{x})| \le 1$. Next, in order to construct weak central Morrey norm we divide our analysis into following two cases:
Case 1. When $\gamma \le 0$, then

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\le 1} y|\{\mathbf{x} \in B_\gamma : 1 > y\}|^{1/r} = p^{n\gamma/r},$$

and

$$\|H^p f_0\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = \sup_{\gamma\le 0} |B_\gamma|_H^{-\mu-1/r} \|f\|_{WL^r(B_\gamma)} = \sup_{\gamma\le 0} p^{-n\gamma\mu} = 1 = \|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}.$$

Case 2. When $\gamma > 0$, we have

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y|\{\mathbf{x} \in B_{\mathbf{0}} : 1 > y\} \cup \{1 < |\mathbf{x}|_p < p^\gamma : |\mathbf{x}|_p^{-n} > y\}|^{1/r}.$$

For further analysis, this case is further divided into the following subcases:
Case 2(a). If $1 < \gamma < \log_p y^{-1/n}$, then

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y\{1 + p^{n\gamma} - 1\}^{1/r} = \sup_{0<t\leq 1} tp^{n\gamma/r}.$$

Case 2(b). If $1 < \log_p y^{-1/n} < \gamma$, then:

$$\|H^p f_0\|_{WL^r(B_\gamma)} = \sup_{0<y\leq 1} y(1 + y^{-1} - 1)^{1/r} = \sup_{0<y\leq 1} y^{1-1/r}.$$

Now, for $1 \leq r < \infty$ and $-1/r \leq \mu < 0$, from case 2(a) and 2(b), we obtain

$$\begin{aligned}
&\|H^p f_0\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \\
&= \max\left\{\sup_{0<y\leq 1}\sup_{1<\gamma\leq \log_p(1/y)^{-1/n}} yp^{-n\gamma\mu}, \sup_{0<y\leq 1}\sup_{1<\log_p(1/y)^{-1/n}<\gamma} y^{1-1/r}p^{-n\gamma(\mu+1/r)}\right\} \\
&= \max\left\{\sup_{0<y\leq 1} t^{1+\mu}, \sup_{0<y\leq 1} y^{1+\mu}\right\} \\
&= 1 = \|f_0\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)}. \qquad (11)
\end{aligned}$$

Finally, using (10) and (11), we arrive at:

$$\|H\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)\to W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} = 1.$$

□

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References

[1] Aref'eva, I. Ya., Dragovich, B., Frampton, P. H., Volovich, I. V., The wave function of the universe and p-adic gravity, *Internat. J. Modern Phys. A,* 6(24) (1991), 4341–4358. https://doi.org/10.1142/S0217751X91002094

[2] Asim, M, Hussain, A., Sarfraz, N., Weighted variable Morrey–Herz estimates for fractional Hardy operators, *J. Ineq. Appl.,* 2022(2) (2022), 12 pp. https://doi.org/10.1186/s13660-021-02739-z

[3] Avetisov, V. A., Bikulov, A. H., Kozyrev, S. V., Application of p-adic analysis to models of breaking of replica symmetry, *J. Phys. A: Math. Gen.,* 32(50) (1999), 8785–8791. https://doi.org/10.1088/0305-4470/32/50/301

[4] Avetisov, V. A., Bikulov, A. H., Kozyrev, S. V., Osipov, V. A., p-adic models of ultrametric diffusion constrained by hierarchical energy landscaapes, *J. Phys. A:Math.Gen.,* 35(2) (2002), 177–189. https://doi.org/10.1088/0305-4470/35/2/301

[5] Bandaliyev, R. A., Volosivets, S. S., Hausdorff operator on weighted Lebesgue and grand Lebesgue p-adic spaces, *p-Adic Numbers Ultrametric Anal. Appl.,* 11(2) (2019), 114–122. https://doi.org/10.1134/S207004661902002X

[6] Bliss, G. A., An integral inequality, *J. London Math. Soc.,* 5(1) (1930), 40–46. https://doi.org/10.1112/jlms/s1-5.1.40

[7] Brekke, L., Freund, Peter, G. O., p-adic numbers in Physics, *Phys. Rep.,* 233(1) (1993), 1–66. https://doi.org/10.1016/0370-1573(93)90043-D

[8] Christ, M., Grafakos, L., Best constants for two nonconvolution inequalities, *Proc. Amer. Math. Soc.*, 123(6) (1995), 1687–1693. https://doi.org/10.1090/S0002-9939-1995-1239796-6

[9] Chuong, N. M., Egorov, Yu. V., Khrennikov, A., Meyer, Y., Mumford, D., Harmonic, wavelet and p-adic analysis. Including papers from the International Summer School held at Quy Nhon University of Vietnam, Quy Nhon, June 10–15, 2005. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007. x+381 pp. ISBN: 978-981-270-549-5; 981-270-549-X. https://doi.org/10.1142/6373

[10] Chuong, N. M., Hung, H. D., Maximal functions and weighted norm inequalities on local fields, *Appl. Comput. Harmon. Anal.*, 29(3) (2010), 272–286. https://doi.org/10.1016/j.acha.2009.11.002

[11] Dubischar, D., Gundlach, V.M., Steinkamp, O., Khrennikov, A., A p-adic model for the process of thinking disturbed by physiological and information noise, *J. Theor. Biol.*, 197(4) (1999), 451–467. DOI: 10.1006/jtbi.1998.0887

[12] Faris, W. G., Weak Lebesgue spaces and quantum mechanical binding, *Duke Math. J.*, 43(2) (1976), 365–373. DOI: 10.1215/S0012-7094-76-04332-5

[13] Fu, Z. W., Grafakos, L., Lu, S. Z., Zhao, F. Y., Sharp bounds for m-linear Hardy and Hilbert operators, *Houston. J. Math.,* 38(1) (2012), 225–244.

[14] Fu, Z. W., Wu, Q. Y., Lu, S. Z., Sharp estimates of p-adic Hardy and Hardy-Littlewood-Pólya operators, *Acta Math. Sin. (Engl. Ser.)*, 29(1) (2013), 137–150. https://doi.org/10.1007/s10114-012-0695-x

[15] Gao, G., Zhao, F. Y., Sharp weak bounds for Hausdorff operators, *Anal Math.*, 41(3) (2015), 163–173. https://doi.org/10.1007/s10476-015-0204-4

[16] Gao, G., Hu, X., Zhang, C., Sharp weak estimates for Hardy-type operators, *Ann. Funct. Anal.,* 7(3) (2016), 421–433. https://doi.org/10.1215/20088752-3605447

[17] Gao, G., Zhong, Y., Some estimates of Hardy operators and their commutators on Morrey-Herz spaces, *J. Math. Inequal.,* 11(1) (2017), 49–58. DOI: 10.7153/jmi-11-05

[18] Hardy, G. H., Note on a theorem of Hilbert, *Math. Z.*, 6(3-4) (1920), 314–317. https://doi.org/10.1007/BF01199965

[19] Ho, K.-P., Hardy's inequality on Hardy–Morrey spaces, *Georg. Math. J.*, 26(3) (2019), 405–413. https://doi.org/10.1515/gmj-2017-0046

[20] Hussain A., Asim, M., Aslam, M., Jarad, F., Commutators of the fractional Hardy operator on weighted variable Herz-Morrey spaces, *J. Funct. Spaces,* (2021), Art. ID 9705250, 10 pp. https://doi.org/10.1155/2021/9705250

[21] Hussain, A., Sarfraz, N., The Hausdorff operator on weighted p-adic Morrey and Herz type spaces, *p-Adic Numbers Ultrametric Anal. Appl.*, 11(2) (2019), 151–162. https://doi.org/10.1134/S2070046619020055

[22] Hussain, A., Sarfraz, N., Optimal weak type estimates for p-adic Hardy operators, *p-Adic Numbers Ultrametric Anal. Appl.,* 12(1) (2020), 29–38. https://doi.org/10.1134/S2070046620010033

[23] Hussain, A., Ahmed, M., Weak and strong type estimates for the commutators of Hausdorff operator, *Math. Inequal. Appl.,* 20(1) (2017), 49–56. DOI: 10.7153/mia-20-04

[24] Hussain, A., Gao, G., Multidimensional Hausdorff operators and commutators on Herz-type spaces, *J. Inequal. Appl.*, 2013(594) (2013), 12 pp. https://doi.org/10.1186/1029-242X-2013-594

[25] Liu, R.H., Zhou, J., Sharp estimates for the p-adic Hardy type operator on higher-dimensional product spaces, *J. Inequal. Appl.*, 2017(219) (2017), 13 pp. https://doi.org/10.1186/s13660-017-1491-z

[26] Parisi, G., Sourlas, N., p-adic numbers and replica symmetry, *Eur. Phys. J. B Condens. Matter Phys.*, 14(3) (2000), 535–542. https://doi.org/10.1007/s100510051063

[27] Persson, L.-E., Samko, S. G., A note on the best constants in some hardy inequalities, *J. Math. Inequal.*, 9(2) (2015), 437–447. DOI:10.7153/jmi-09-37

[28] Sarfraz, N., Gürbüz, F., Weak and strong boundedness for p-adic fractional Hausdorff operator and its commutator, *Int. J. Nonlinear Sci. Numer. Simul.*, 2021 (2021), 12 pp. https://dx.doi.org/10.1515/ijnsns-2020-0290

[29] Sarfraz, N., Aslam, M., Some weighted estimates for the commutators of p-adic Hardy operator on two weighted p-adic Herz-type spaces. *AIMS Math.,* 6(9) (2021), 9633–9646. DOI:10.3934/math.2021561

[30] Vladimirov, V. S., Volovich, I. V., Zelenov, E. I., p-adic Analysis and Mathematical Physics, Series on Soviet and East European Mathematics, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 1994, xx+319 pp. ISBN: 981-02-0880-4. https://doi.org/10.1142/1581

[31] Vladimirov, V. S., Volovich, I. V., p-adic quantum mechanics, *Commun. Math. Phy.*, 123 (1989), 659–676. https://doi.org/10.1007/BF01218590

[32] Volosivets, S. S., Weak and strong estimates for rough Hausdorff type operator defined on p-adic linear space, *p-Adic Numbers Ultrametric Anal. Appl.*, 9(3) (2017), 236–241. https://doi.org/10.1134/S2070046617030062

[33] Wu, Q.Y., Boundedness for commutators of fractional p-adic Hardy operator, *J. Inequal. Appl.*, 2012(293) (2012), 12pp. https://doi.org/10.1186/1029-242X-2012-293

[34] Wu, Q. Y., Mi, L., Fu, Z. W., Boundedness of p-adic Hardy operators and their commutators on p-adic central Morrey and BMO spaces, *J. Funct. Spaces Appl.,* (2013), Art. ID 359193, 10 pp. https://doi.org/10.1155/2013/359193

[35] Wu, Q. Y., Fu, Z. W., Hardy-Littlewood-Sobolev inequalities on p-adic central Morrey spaces, *J. Funct. Spaces*, (2015), Art. ID 419532, 7 pp. https://doi.org/10.1155/2015/419532

[36] Xiao, J., L^p and BMO bounds of weighted Hardy-Littlewood averages, *J. Math. Anal. Appl.*, 262(2) (2001), 660–666. https://doi.org/10.1006/jmaa.2001.7594

[37] Yu, H., Li, J., Sharp weak bounds for n-dimensional fractional Hardy operators, *Front. Math. China*, 13(2) (2018), 449–457. https://doi.org/10.1007/s11464-018-0685-0

[38] Zhao, F. Y., Lu, S. Z., The best bound for n-dimensional fractional Hardy operator, *Math. Inequal Appl.*, 18(1) (2015), 233–240. DOI: 10.7153/mia-18-17

[39] Zhao, F. Y., Fu, Z. W., Lu, S. Z., Endpoint estimates for n-dimensional Hardy operators and their commutators, *Sci. China Math.*, 55(10) (2012), 1977–1990. https://doi.org/10.1007/s11425-012-4465-0