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A Matrix scheme based on fractional finite difference method for solving fractional delay differential equations with boundary conditions

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Abstract: In this paper, the method of fractional finite difference presents and used for solving a number of famous fractional order version of scientific models. The proposed method besides being simple is so exact which is sensible in the solved problems.

Keywords: Finite difference method; Caputo derivative; Fractional delay differential equations; Boundary conditions.

1 Introduction

Ordinary differential equations (ODEs) and delay differential equations (DDEs) are used to describe some models in many applied sciences such as mechanics, biomathematics, etc. While ODEs contain derivatives which depend on the solution at the present value of the independent variable ("time"), DDEs also contain derivatives which depend on the solution at previous times. DDEs arise in models throughout the sciences [4, 10].

The mathematical field that deals with derivatives of any real order is called fractional. For a long time, it was only considered as a pure mathematical branch. Nevertheless, during the last decades, fractional calculus has attracted the attention of many researchers; it has been successfully applied in various areas like computational biology [8], economy [3], etc.

In recent years, Fractional Delay differential Equations (FDDEs) have attracted the attention of many researchers. These equations have many applications in various areas like physics [9], biology [8], sociology [10], chaos [2], Agriculture [5] and so on.

The goal of this paper is to provide a matrix scheme for solving fractional delay differential equations of the general form:

$$D_*^{\beta} y(t) = f\left(t, y\left(t\right), y\left(t-\tau\right), D_*^{\alpha} y(t), D_*^{\alpha} y(t-\tau)\right) \tag{1}$$

on $a \le t \le b, 0 < \alpha \le 1, 1 < \beta \le 2$ and under the following interval and boundary conditions:

$$\begin{aligned} y\left(t\right) &= \varphi\left(t\right) \qquad -\tau \le t \le a, \\ y(b) &= \gamma, \end{aligned}$$

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where $D_*^{\beta}y(t), D_*^{\alpha}y(t)$ and $D_*^{\alpha}y(t-\tau)$ are the standard Caputo fractional derivatives, $\varphi(t)$is the initial value and γ is a smooth function.

This paper is organized as follows; we recall some necessary definitions of the fractional calculus in section 2. In section 3, we present a matrix method based on the fractional finite difference method and use it for solving FDDEs with boundary conditions. In section 4, the adaptation of a variety of differential equations in the mathematical modeling process of difference applications will be considered; for example, the problem of the mechanical engineering [12], the combustion engines control problem [1], the problem of the pendulum [7]. Finally, we give some brief conclusions in section 5.

2 Basic definitions

For Riemann - Liouville fractional integration and the Caputo fractional derivative we have the following properties [6]:

1. For real values of $\alpha > 0$, the Caputo fractional derivative provides the operation inverse to the Riemann - Liouville integration from the left

$$ {}_0^C D_t^{\alpha} y(t) {}_0I_t^{\alpha} = f(t), \qquad \alpha > 0, \qquad f(t) \in C[0,1]. $$

2. If $f(x) \in C^{\lceil \alpha \rceil}[0,1]$, then

$$ {}_0I_t^{\alpha} {}_0^C D_t^{\alpha} y(t) = f(t) - \sum_{j=0}^{\lceil \alpha \rceil - 1} \frac{t^j}{j!} (\frac{d^j}{dt^j} f)(0), \qquad n-1 < \alpha \le n, $$

where $C^{\lceil \alpha \rceil}[0,1]$ is the space of $\lceil \alpha \rceil$ times, continuously differentiable functions.

3. The Riemann – Liouville fractional integral operator of order$\alpha \ge 0$, of a function$f \in C_{\mu}$, $\mu \ge -1$, is defined as

$$ \begin{array}{l} {}_0I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)\,dx, \qquad \alpha > 0, \qquad t > 0, \\ {}_0I_t^{0} f(t) = f(t). \end{array} $$

Some Properties of the operator${}_0I_t^{\alpha}$, which are need here, we mention only the following:

For $f \in C_{\mu}$, $\mu \ge -1$, $\alpha, \beta \ge 0$ and $\gamma > -1$:

i. ${}_0I_t^{\alpha} {}_0I_t^{\beta} f(t) = {}_0I_t^{\alpha+\beta} f(t)$, **ii.** ${}_0I_t^{\alpha} {}_0I_t^{\beta} f(t) = {}_0I_t^{\beta} {}_0I_t^{\alpha} f(t)$, **iii.** ${}_0I_t^{\alpha} t^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

3 Numerical Method

In this section we will present a numerical method for solving the boundary value problems. We will consider a modeling problem for boundary value problems for fractional delay differential equations as follows:

$$ D_*^{\beta} y(t) + a(t) D_*^{\alpha} y(t) + b(t) y(t) + c(t) y(t-\tau) = f(t) \tag{2} $$

such that if $a \le t \le b$, $0 < \alpha \le 1$ and $1 < \beta \le 2$, subject to the interval and boundary conditions we have:

$$ \begin{array}{ll} y(t) = \varphi(t) & -\tau \le t \le a, \\ y(b) = \gamma, & \end{array} \tag{3} $$

where $\varphi(t)$ is the initial value, γ is a smooth function and τ is the amount of delay. In which $a(t)$,$b(t)$, $c(t)$, $f(t)$ and $\varphi(t)$ are the smooth. If $y(t)$ is to have smooth answer in problem (2), it should be satisfied in the boundary problems of (2) and (3); also they should be continuous in the interval [a,b] and e continuously differentiable in the interval (a,b).This numerical method includes the finite difference operator on a specific consistent mesh. Consider a uniform grid,

$$\{t_n = nh : n = -m, -m+1, \ldots, -1, 0, 1, \ldots, N\}$$

where m and N are integers such that $h = \frac{b-a}{N}$ and $h = \frac{\tau}{m}$ in which $m = pq$ and p is an affirmative integer and q is theτ Mantissa.

On simplification the discrete problem (2), (3) reduces to a system of (N+1) linear difference equations given by

$$Ay = f, \tag{4}$$

therefore

$$y = A^{-1} f, \tag{5}$$

where $y =< y_0, ..., y_N >^t$, $f =< f_0, ..., f_N >^t$ and $A = [a_{i,p}]$, the nonzero entries of the system matrix being given by

$$
\begin{aligned}
& a_{0,0} = a_{N,N} = 1 \\
& For \quad i = 1 \\
& \quad a_{1,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(2^{2-\beta} - 2\times 3^{2-\beta} - 1) - a(t_1)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 1) \\
& \quad a_{1,1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + a(t_1)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + b(t_1) \\
& \quad a_{1,2} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} \\
& For \quad i = 2 \\
& \quad a_{2,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(-2\times 4^{2-\beta} + 3^{2-\beta} + 2^{3-\beta} - 1) - a(t_2)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{1-\alpha}) \\
& \quad a_{2,1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(4^{2-\beta} - 2\times 3^{2-\beta} - 2) + a(t_2)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \\
& \quad a_{2,2} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + a(x_2)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + b(x_2) \\
& \quad a_{2,3} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + a(t_2)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)} \\
& For \quad i = \{3, \ldots, m-1\} \ , \ p = \{1, ..., i-2\} \\
& \quad a_{i,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha}) \\
& \quad a_{i,p} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] \\
& \qquad + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha}) \\
& \quad a_{i,i-1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \\
& \quad a_{i,i} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + b(t_i) \\
& \quad a_{i,i+1} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)} \\
& For \quad i = m \\
& \quad a_{i,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha}) + c(t_i) \\
& \quad a_{i,p} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] \\
& \qquad + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha}) \\
& \quad a_{i,i-1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \\
& \quad a_{i,i} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + b(t_i) \\
& \quad a_{i,i+1} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}
\end{aligned}
$$

$$\begin{aligned}
&For \quad i=\{m+1,\dots,N-1\}\ ,\ p=\{1,...,i-2\}-\{i-m\}\\
&\quad a_{i,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta}+(i+1)^{2-\beta}+2i^{2-\beta}-(i-1)^{2-\beta}]-a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha}-i^{1-\alpha})\\
&\quad a_{i,i-m}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(m+3)^{2-\beta}-2(m+2)^{2-\beta}+2m^{2-\beta}-(m-1)^{2-\beta}]\\
&\qquad +a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((m+2)^{1-\alpha}-(m+1)^{1-\alpha}-m^{1-\alpha})+c(t_{i-m})\\
&\quad a_{i,p}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta}-2(i-p+2)^{2-\beta}+2(i-p)^{2-\beta}-(i-p-1)^{2-\beta}]\\
&\qquad +a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha}-(i-p+1)^{1-\alpha}-(i-p)^{1-\alpha})\\
&\quad a_{i,i-1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-2(i-1)^{2-\beta}+2(i-3)^{2-\beta}]+a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{2-\alpha}+1)\\
&\quad a_{i,i}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta}-2^{3-\beta}-1]+a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+b(t_i)\\
&\quad a_{i,i+1}=\tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)}+a(t_i)\tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}\\
&f_0=\phi_0\\
&For \quad i=1,\dots,m-1 \quad f_i=f(t_i)+\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-(i+2)^{2-\beta}]\phi_{-1}-c(t_i)\phi_{i-m},\\
&For \quad i=m+1,\dots,N-1 \quad f_i=f(t_i)+\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-(i+2)^{2-\beta}]\phi_{-1},\\
&f_N=\gamma
\end{aligned}$$

4 Numerical Results

In this section, the examples are considered and solved by means of the proposed method. Then, the graphs and tables will be shown according to different amounts of α and β.

Model 1. Gantry cranes can lift several hundred tons and can have spans of well over 50 meters. See Figure1. Park and Huang in [12] presented a model for that,

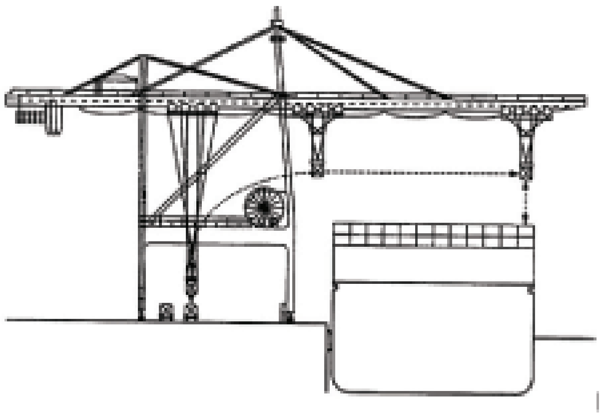

Fig. 1: Container crane and ship.

and its equation is as:

$$D_*^{\beta}y(t)+\varepsilon D_*^{\alpha}y(t)+(1-k)y(t)+ky(t-\tau)=0, \tag{6}$$

under the boundary conditions:

$$y(t)=1 \quad -\tau\le t\le 0, \qquad y(20)=0,$$

where $\varepsilon=0.1$ and $k=-0.15$.

The difference method algorithm for this boundary problem is:

$$L^N y_i = \begin{cases} for\ i=1,...,N \\ (\frac{h^{2-\beta}}{2\Gamma(3-\beta)}\sum_{j=0}^{i}[(j+2)^{2-\beta}-j^{2-\beta}]D_+D_-y_{i-j})+(\frac{\varepsilon h^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{i}[(j+1)^{1-\alpha}-j^{1-\alpha}]D_+y_{i-j}) \\ +(1-k)y_i+ky_{i-m}=0 \end{cases}$$

$$for\ i=-m,-m+1,...,0 \qquad y_i=1$$

Based on the relations presented in section 3 we have:

$$\begin{aligned}
&a_{0,0}=a_{N,N}=1\\
&For\quad i=1\\
&\quad a_{1,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(2^{2-\beta}-2\times 3^{2-\beta}-1)-\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-1)\\
&\quad a_{1,1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta}-2^{3-\beta}-1)+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+1-k\\
&\quad a_{1,2}=\tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)}\\
&For\quad i=2\\
&\quad a_{2,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(-2\times 4^{2-\beta}+3^{2-\beta}+2^{3-\beta}-1)-\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{1-\alpha})\\
&\quad a_{2,1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(4^{2-\beta}-2\times 3^{2-\beta}-2)+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{2-\alpha}+1)\\
&\quad a_{2,2}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta}-2^{3-\beta}-1)+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+1-k\\
&\quad a_{2,3}=\tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)}+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}\\
&For\quad i=\{3,\dots,m-1\}\ ,\ p=\{1,...,i-2\}\\
&\quad a_{i,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta}+(i+1)^{2-\beta}+2i^{2-\beta}-(i-1)^{2-\beta}]-\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha}-i^{1-\alpha})\\
&\quad a_{i,p}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta}-2(i-p+2)^{2-\beta}+2(i-p)^{2-\beta}-(i-p-1)^{2-\beta}]\\
&\qquad +\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha}-(i-p+1)^{1-\alpha}-(i-p)^{1-\alpha})\\
&\quad a_{i,i-1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-2(i-1)^{2-\beta}+2(i-3)^{2-\beta}]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{2-\alpha}+1)\\
&\quad a_{i,i}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta}-2^{3-\beta}-1]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+1-k\\
&\quad a_{i,i+1}=\tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)}+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}\\
&For\quad i=m\\
&\quad a_{i,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta}+(i+1)^{2-\beta}+2i^{2-\beta}-(i-1)^{2-\beta}]-\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha}-i^{1-\alpha})+k\\
&\quad a_{i,p}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta}-2(i-p+2)^{2-\beta}+2(i-p)^{2-\beta}-(i-p-1)^{2-\beta}]\\
&\qquad +\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha}-(i-p+1)^{1-\alpha}-(i-p)^{1-\alpha})\\
&\quad a_{i,i-1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-2(i-1)^{2-\beta}+2(i-3)^{2-\beta}]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{2-\alpha}+1)\\
&\quad a_{i,i}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta}-2^{3-\beta}-1]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+1-k\\
&\quad a_{i,i+1}=\tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)}+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}\\
&For\quad i=\{m+1,\dots,N-1\}\ ,\ p=\{1,...,i-2\}-\{i-m\}\\
&\quad a_{i,0}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta}+(i+1)^{2-\beta}+2i^{2-\beta}-(i-1)^{2-\beta}]-\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha}-i^{1-\alpha})\\
&\quad a_{i,i-m}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(m+3)^{2-\beta}-2(m+2)^{2-\beta}+2m^{2-\beta}-(m-1)^{2-\beta}]\\
&\qquad +\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((m+2)^{1-\alpha}-(m+1)^{1-\alpha}-m^{1-\alpha})+k\\
&\quad a_{i,p}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta}-2(i-p+2)^{2-\beta}+2(i-p)^{2-\beta}-(i-p-1)^{2-\beta}]\\
&\qquad +\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha}-(i-p+1)^{1-\alpha}-(i-p)^{1-\alpha})\\
&\quad a_{i,i-1}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta}-2(i-1)^{2-\beta}+2(i-3)^{2-\beta}]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha}-2^{2-\alpha}+1)\\
&\quad a_{i,i}=\tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta}-2^{3-\beta}-1]+\tfrac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha}-2)+1-k
\end{aligned}$$

$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}$

$f_0 = \phi_0$

$For \quad i = 1,\dots,m-1 \quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1} - k\phi_{i-m},$

$For \quad i = m+1,\dots,N-1 \quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1},$

$f_N = \gamma$

Table 1: Approximate solutions with different values α, β and $\tau = 12$ for Model 1.

t	$\alpha = 0.5, \beta = 1.5$	$\alpha = 0.75, \beta = 1.75$	$\alpha = 0.95, \beta = 1.95$
0	1	1	1
5	0.008121	0.008472	0.008670
10	-0.002108	-0.001939	-0.001281
15	0.001287	0.001353	0.001397
20	0.000772	0.000752	0.000738

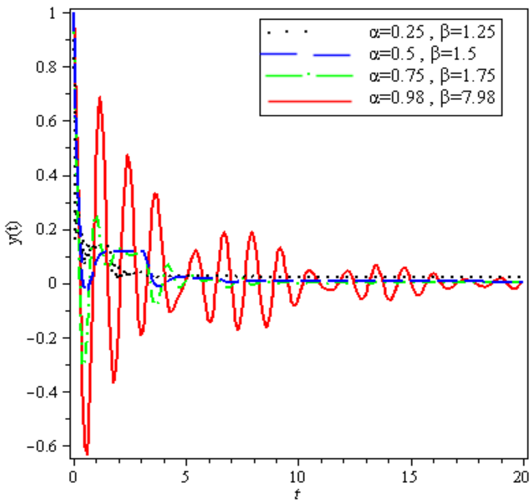

Fig. 2: The numerical solution of Model 1 for τ=12.

Model 2. Averina et al. [1] considered a simple mathematical model for the combustion engines control problem that takes into account the signal coming from the UEGO sensor, and its equation is as:

$$D_*^{\beta} y(t) + D_*^{\alpha} y(t) + y(t) + ky(t-\tau) = 0, \tag{7}$$

under the boundary conditions:

$$y(t) = 10 \quad -\tau \leq t \leq 0, \qquad y(25) = 13.8,$$

where $k = 1$.

The difference method algorithm for this boundary problem is:

$$L^N y_i = \begin{cases} for\ i = 1, ..., N \\ \left(\frac{h^{2-\beta}}{2\Gamma(3-\beta)} \sum_{j=0}^{i} [(j+2)^{2-\beta} - j^{2-\beta}] D_+ D_- y_{i-j}\right) + \left(\frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] D_+ y_{i-j}\right) \\ + y_i + k y_{i-m} = 0 \end{cases}$$

$$for\ i = -m, -m+1, ..., 0 \qquad y_i = 10$$

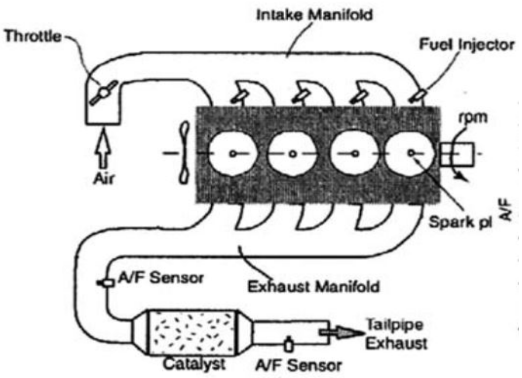

Fig. 3: Gasoline engine. The delay varies with engine speed, mass flow rate through the engine, exhaust pressure, and exhaust temperature.

Based on the relations presented in section 3 we have:

$$\begin{aligned}
&a_{0,0} = a_{N,N} = 1 \\
&For \quad i = 1 \\
&\quad a_{1,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(2^{2-\beta} - 2\times 3^{2-\beta} - 1) - \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 1) \\
&\quad a_{1,1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + 1 - k \\
&\quad a_{1,2} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} \\
&For \quad i = 2 \\
&\quad a_{2,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(-2\times 4^{2-\beta} + 3^{2-\beta} + 2^{3-\beta} - 1) - \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{1-\alpha}) \\
&\quad a_{2,1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(4^{2-\beta} - 2\times 3^{2-\beta} - 2) + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \\
&\quad a_{2,2} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + 1 \\
&\quad a_{2,3} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)} \\
&For \quad i = \{3, \ldots, m-1\} \ , \ \ p = \{1, ..., i-2\} \\
&\quad a_{i,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha}) \\
&\quad a_{i,p} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] \\
&\qquad + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha}) \\
&\quad a_{i,i-1} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \\
&\quad a_{i,i} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + 1 \\
&\quad a_{i,i+1} = \tfrac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)} \\
&For \quad i = m \\
&\quad a_{i,0} = \tfrac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \tfrac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha}) + 1
\end{aligned}$$

$$a_{i,p} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}]$$
$$+\frac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha})$$
$$a_{i,i-1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1)$$
$$a_{i,i} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + 1$$
$$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

For $i = \{m+1,\ldots,N-1\}$, $p = \{1,...,i-2\} - \{i-m\}$

$$a_{i,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \frac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha})$$
$$a_{i,i-m} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(m+3)^{2-\beta} - 2(m+2)^{2-\beta} + 2m^{2-\beta} - (m-1)^{2-\beta}]$$
$$+\frac{h^{-\alpha}}{\Gamma(2-\alpha)}((m+2)^{1-\alpha} - (m+1)^{1-\alpha} - m^{1-\alpha}) + 1$$
$$a_{i,p} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}]$$
$$+\frac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha})$$
$$a_{i,i-1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1)$$
$$a_{i,i} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) + 1$$
$$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

$f_0 = \phi_0$

For $i = 1,\ldots,m-1$ $\quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1} - \phi_{i-m},$

For $i = m+1,\ldots,N-1$ $\quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1},$

$f_N = \gamma$

Table2: Approximate solutions with different values $\alpha, \beta$and $\tau = 3$ for Model 2.

t	$\alpha = 0.5,\ \beta = 1.5$	$\alpha = 0.75,\ \beta = 1.75$	$\alpha = 0.95,\ \beta = 1.95$
0	10	10	10
5	-3.403033	-3.329171	-3.251407
10	-0.563973	-0.554394	-0.552335
15	3.523999	3.560785	3.595190
20	-0.188642	-0.208558	-0.250169
25	13.950548	13.919044	13.805702

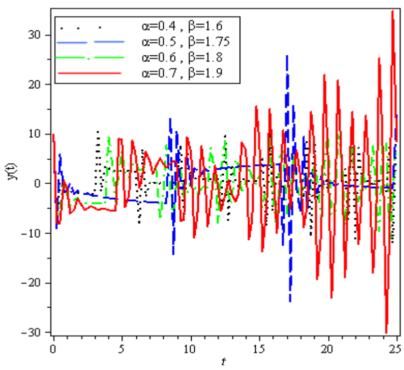

Fig. 4: The numerical solution of Model 2 for τ=3.

Model 3. Consider the delay damped pendulum equation that parameter m denotes the mass on the end of pendulum of length l, g is the gravitational and k is a measure of the damped [7],

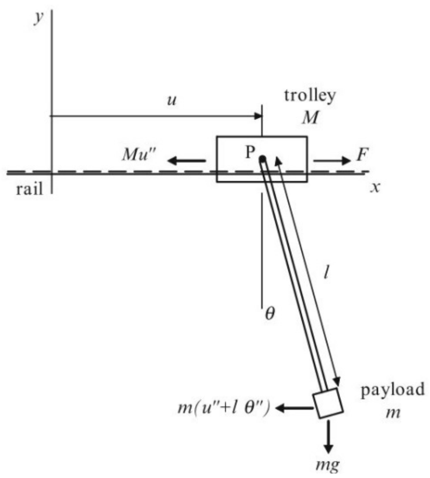

Fig. 5: Simplest pendulum model for a crane.

and its equation is as:

$$D_*^{\beta} y(t) + D_*^{\alpha} y(t) - 1.2y(t) + 1.1y(t-\tau) = 0, \tag{8}$$

under the boundary conditions:

$$y(t) = 0.05 \quad -\tau \le t \le 0, \quad y(100) = 1.$$

The difference method algorithm for this boundary problem is:

$$L^N y_i = \begin{cases} for\, i = 1, \dots, N \\ \left(\frac{h^{2-\beta}}{2\Gamma(3-\beta)} \sum_{j=0}^{i} [(j+2)^{2-\beta} - j^{2-\beta}] D_+ D_- y_{i-j}\right) + \left(\frac{h^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] D_+ y_{i-j}\right) \\ -1.2y_i + 1.1y_{i-m} = 0 \end{cases}$$

$$for\, i = -m, -m+1, \dots, 0 \qquad y_i = 0.05$$

Based on the relations presented in section 3 we have:

$$\begin{aligned} & a_{0,0} = a_{N,N} = 1 \\ & For \quad i = 1 \\ & \quad a_{1,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}(2^{2-\beta} - 2 \times 3^{2-\beta} - 1) - \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 1) \\ & \quad a_{1,1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) - 1.2 \\ & \quad a_{1,2} = \frac{2^{1-\beta} h^{-\beta}}{\Gamma(3-\beta)} \\ & For \quad i = 2 \\ & \quad a_{2,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}(-2 \times 4^{2-\beta} + 3^{2-\beta} + 2^{3-\beta} - 1) - \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{1-\alpha}) \\ & \quad a_{2,1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}(4^{2-\beta} - 2 \times 3^{2-\beta} - 2) + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1) \end{aligned}$$

$$a_{2,2} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}(3^{2-\beta} - 2^{3-\beta} - 1) + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) - 1.2$$
$$a_{2,3} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

$For \quad i = \{3,\dots,m-1\}\ ,\ p = \{1,...,i-2\}$

$$a_{i,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \frac{h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha})$$
$$a_{i,p} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha})$$
$$a_{i,i-1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1)$$
$$a_{i,i} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) - 1.2$$
$$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

$For \quad i = m$

$$a_{i,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha}) + 1.1$$
$$a_{i,p} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha})$$
$$a_{i,i-1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1)$$
$$a_{i,i} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) - 1.2$$
$$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

$For \quad i = \{m+1,\dots,N-1\}\ ,\ p = \{1,...,i-2\} - \{i-m\}$

$$a_{i,0} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[-2(i+2)^{2-\beta} + (i+1)^{2-\beta} + 2i^{2-\beta} - (i-1)^{2-\beta}] - \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i+1)^{1-\alpha} - i^{1-\alpha})$$
$$a_{i,i-m} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(m+3)^{2-\beta} - 2(m+2)^{2-\beta} + 2m^{2-\beta} - (m-1)^{2-\beta}] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((m+2)^{1-\alpha} - (m+1)^{1-\alpha} - m^{1-\alpha}) + 1.1$$
$$a_{i,p} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[(i-p+3)^{2-\beta} - 2(i-p+2)^{2-\beta} + 2(i-p)^{2-\beta} - (i-p-1)^{2-\beta}] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}((i-p+2)^{1-\alpha} - (i-p+1)^{1-\alpha} - (i-p)^{1-\alpha})$$
$$a_{i,i-1} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - 2(i-1)^{2-\beta} + 2(i-3)^{2-\beta}] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(3^{1-\alpha} - 2^{2-\alpha} + 1)$$
$$a_{i,i} = \frac{h^{-\beta}}{2\Gamma(3-\beta)}[3^{2-\beta} - 2^{3-\beta} - 1] + \frac{\varepsilon h^{-\alpha}}{\Gamma(2-\alpha)}(2^{1-\alpha} - 2) - 1.2$$
$$a_{i,i+1} = \frac{2^{1-\beta}h^{-\beta}}{\Gamma(3-\beta)} + \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$$

$f_0 = \phi_0$

$For \quad i = 1,\dots,m-1 \quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1} - 1.1\phi_{i-m},$

$For \quad i = m+1,\dots,N-1 \quad f_i = f(t_i) + \frac{h^{-\beta}}{2\Gamma(3-\beta)}[i^{2-\beta} - (i+2)^{2-\beta}]\phi_{-1},$

$f_N = \gamma$

Table 3: Approximate solutions with different values α, β and $\tau = 3$for Model 3.

t	$\alpha = 0.5, \beta = 1.5$	$\alpha = 0.75, \beta = 1.75$	$\alpha = 0.95, \beta = 1.95$
0	0.05	0.05	0.05
20	0.064957	0.066416	0.067952
40	0.114362	0.118258	0.122323
60	0.231902	0.240483	0.249385
80	0.477159	0.484398	0.4904881
100	0.955663	0.962917	0.992859

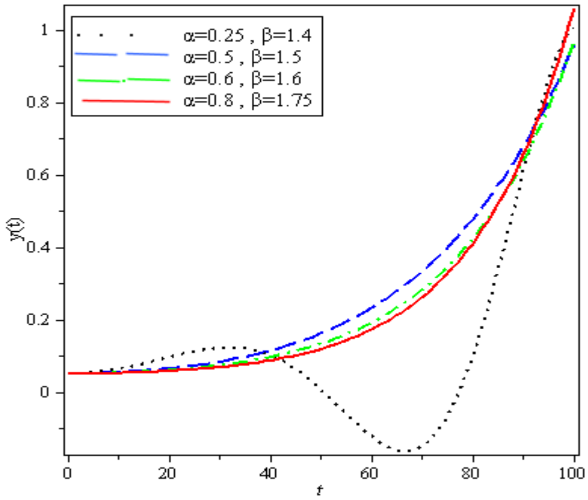

Fig. 6: The numerical solution of Model 3 for τ=3.

5 Conclusions

The fundamental goal of this work is constructed a numerical method for the numerical solution of linear and nonlinear delay differential equations of fractional order with boundary conditions. With this scheme which is based on finite difference method, an approximate solution for solving different types of boundary problems with fractional order is obtained. The graphs of the solution of the considered examples for different value of delay, α and β are plotted in figures 1-3 to examine the effect of delay in producing fluctuations is presented.

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