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Korteweg-de Vries flow equations from Manakov equation by multiple scale method

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Abstract: We perform a multiple scales analysis on the modified nonlinear Schrödinger (MNLS) equation in the Hamiltonian form. We derive, as amplitude equations, Korteweg-de Vries (KdV) flow equations in the bi-Hamiltonian form.

Keywords: Manakov Equation; Multiple Scale method; Flow Equation.

1 Introduction

We consider a system of two coupled nonlinear Schrödinger (NLS) equations,

$$\begin{aligned} iq_{1t}+q_{1xx}+\left(\alpha\left|q_1\right|^2+\beta\left|q_2\right|^2\right)q_1=0,\\ iq_{2t}+q_{2xx}+\left(\beta\left|q_1\right|^2+\delta\left|q_2\right|^2\right)q_2=0, \end{aligned} \tag{1}$$

where α, β, δ are some constants. The integrability of this system was proved by Manakov only for the case α=β=δ, which we shall refer as the integrable Manakov system [1].

Equation (1) is important for a number of physical applications when α is positive and all remaining constants are set equal to 1. For example, for two-mode optical fibres, $\alpha=2$ [2]; for propagation of two modes in fibres with strong birefringence, $\alpha=\frac{2}{3}$ [3] and in the general case $\frac{2}{3}\leq\alpha\leq 2$ for elliptical eigenmodes. The special value $\alpha=1$ (1) corresponds to at least two possible physcal cases, namely the case of a purely electrostrictive nonlinearity or in the elliptical birefringence case, when the angle between the major and minor axes of the birefringence ellipse is $ca.35^o$. The experimental observation of Manakov solitons in crystals has been reported by [4]. Recently the Manakov model has appeared in a Kerr-type approximation of photorefractive crystals [5]. The pulse-pulse collision between wavelength-division-multiplexed channels of optical fibre transmission systems is described by (1) with $\alpha=2$ ([6], [7], [8], [9]). Wavelength division-multiplexing is one means of increasing the bandwidth in optical communication systems. This technique is limited by the finite bandwidth of the Er-doped fibre amplifiers which are now incorporated into most, if not all, such systems. General quasi-periodic solutions in terms of n-phase theta functions for the integrable Manakov system are derived by Adams [10], while a series of special solutions are given in ([11], [12], [13], [14]). Quasi-periodic and periodic solutions showed for coupled nonlinear Schrödinger equations of Manakov type, also mention the method

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of constructing elliptic finite-gap solutions of the stationary KdV and AKNS hierarchy [15].

The Manakov equations are a system of two coupled nonlinear Schrödinger (NLS) equations of the form

$$\begin{aligned} iq_{1t} + q_{1xx} + 2\mu\left(|q_1|^2 + |q_2|^2\right) q_1 = 0, \\ iq_{2t} + q_{2xx} + 2\mu\left(|q_1|^2 + |q_2|^2\right) q_2 = 0, \end{aligned} \tag{2}$$

and known to be a useful model for the study of pulse propagation in birefringent optical fibers. Here $q_1 = q_1(x,t)$ and $q_2 = q_2(x,t)$ are the complex amplitudes of two interacting components, μ is positive parameter, and x and t are normalized space and time. Note that our variables x and t are interchanged with those of [16], in order to represent the propagation variable, the one associated with the first-order derivative in the Manakov equation, by t. [This is consistent with Manakov's original paper [1], Eq. (3).]

In this paper we apply a multiple scales method following Zakharov and Kuznetsov [17] to derive the KdV flow equations from the integrable Manakov equations (2). This is an important derivation since the KdV flow equations comes out from the integrable Manakov equations. Comparing our derivation to the KdV-integrable Manakov equations derivation, the equations for the coefficients at each order in epsilon, contain no secular terms. Thus no freedom is left in choosing coefficients and the expansion is uniquely determined.

In section 2 we present some background materials on the integrable Manakov equations, KdV flow equations. In section 3 we first give a multiple scales method. Then we apply the method to the integrable Manakov equations with derivation of the KdV flow equations.

Throughout the paper we make extensive use of Reduce to calculate and simplify our results.

2 Background Materials

In this section we present some background materials on the integrable Manakov equations, and KdV flow equations.

2.1 The Manakov equation

The Manakov equation, which known to be a useful model for the study of pulse propagation in birefringent optical fibers, is given by the equation (1) together with the complex conjugate. This Manakov equation (α=β=$\delta = 1$) (1) can be written in the following solution of (1) in the form,

$$\begin{aligned} q_1(x,t) = v_1(x)\exp\{ia_1t + iC_1\int^x dx v_1^{-2}(x)\}, \\ q_2(x,t) = v_2(x)\exp\{ia_2t + iC_2\int^x dx v_2^{-2}(x)\}, \end{aligned} \tag{3}$$

where the $v_{1,2}(x)$ are real fonctions and a_1, a_2, C_1, C_2 are real constant. Substituting (1) into (3) reduce the system to the equations,

$$
\begin{aligned}
&\frac{d^2 v_1}{dx^2} + \delta v_1^3 + \beta v_1 v_2^2 - a_1 v_1 - C_1^2 v_1^{-3} = 0, \\
&\frac{d^2 v_2}{dx^2} + \alpha v_2^3 + \beta v_2 v_1^2 - a_2 v_2 - C_2^2 v_2^{-3} = 0.
\end{aligned} \tag{4}
$$

The system (4) is a natural Hamiltonian two-particle system with a Hamiltonian of the form,

$$
\mathscr{H} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{4}(\delta v_1^2 + 2\beta v_1^2 v_2^2 + \alpha v_2^4) - \frac{1}{2}a_1 v_1^2 - \frac{1}{2}a_2 v_2^2 + \frac{1}{2}C_1^2 v_1^{-2} + \frac{1}{2}C_2^2 v_2^{-2}, \tag{5}
$$

where $p_i(x) = dq_i(x)/dx$, $i = 1,2$ [15].

2.2 The KdV Flow Equations

KdV flow equations are in the following form ([18], [19]):

$$
u_{t_n} = R^n[u]u_x = B_0[u]\delta_u \mathscr{H}_{n+1} = B_1[u]\delta_u \mathscr{H}_n, \quad n = 0, 1, \ldots, \tag{6}
$$

where δ_u variational derivative, the recursion and Hamiltonian operators, and Hamiltonian functions are respectively given by

$$
R = B_1[u]\partial^{-1}, \quad B_0[u] = \partial, \quad B_1[u] = \partial^3 + 4u\partial + 2u_x, \tag{7}
$$

$$
\begin{aligned}
\mathscr{H}_0 &= \tfrac{1}{2}u, \\
\mathscr{H}_1 &= \tfrac{1}{2}u^2, \\
\mathscr{H}_2 &= u^3 - \tfrac{1}{2}u_x^2, \\
&\ \vdots
\end{aligned} \tag{8}
$$

3 The Multiple Scales Method

Following Zakharov and Kuznetsov [17] we use a multiple scales method to derive the KdV flow equations (6) from the integrable Manakov equations (2). We also apply the method to derive Hamiltonian functions for the KdV flow equations (6) from that of the Manakov equation (2).

We now consider integrable Manakov equations (2) and seek a solution in the form by separating the phase and amplitude:

$$
q_1(\xi,\tau) = e^{i\theta(\xi,\tau)}\sqrt{N(\xi,\tau)}, \; q_2(\xi,\tau) = e^{i\theta(\xi,\tau)}\sqrt{M(\xi,\tau)} \tag{9}
$$

with complex conjugates. Inserting this assumed solution into the integrable Manakov equations (2) and grouping the real and imaginary parts, we respectively obtain the following system:

$$
\begin{aligned}
N_\tau &= -2(NV)_\xi, \\
V_\tau &= \left(V^2 - 2\mu N - 2\mu M - \tfrac{N_{\xi\xi}}{2N} + \tfrac{N_\xi^2}{4N^2}\right)_\xi \\
M_\tau &= -2(MV)_\xi, \\
V_\tau &= \left(V^2 - 2\mu N - 2\mu M - \tfrac{M_{\xi\xi}}{2M} + \tfrac{M_\xi^2}{4M^2}\right)_\xi
\end{aligned} \tag{10}
$$

where $\theta(\xi,\tau)_\xi = V(\xi,\tau)$. Then we assume the following series expansions for solutions:

$$\begin{aligned} M &= 1+\sum_{n=1}^{\infty}\varepsilon^{2n}N_n(x,t_1,t_2,\dots,t_n),\\ N &= 1+\sum_{n=1}^{\infty}\varepsilon^{2n}N_n(x,t_1,t_2,\dots,t_n),\\ V &= \sum_{n=1}^{\infty}\varepsilon^{2n}V_n(x,t_1,t_2,\dots,t_n). \end{aligned} \tag{11}$$

We also define slow variables with respect to the scaling parameter $\varepsilon>0$ respectively as follows:

$$x=\varepsilon(\xi+2\tau),\quad t_n=\varepsilon^{2n+1}\tau\ ,\ n=1,2,\dots \tag{12}$$

We now substitute series expansions (11) with (12) into the system (10) and equate coefficients at the powers of ε to zero separately. Then we end up an infinite set of equations for N_n and M_n in the powers of ε for each n. If we let $\varepsilon\to 0$ and vanish the terms at minimal powers of ε, by considering the case $n\geq 1$ we obtain the following:

(i) For the coefficients of ε^3, we find

$$\begin{aligned} &2N_{1x}+2V_{1x}=0,\\ &4\mu N_{1x}+4\mu M_{1x}+4V_{1x}=0,\\ &2M_{1x}+2V_{1x}=0,\\ &4\mu N_{1x}+4\mu M_{1x}+4V_{1x}=0. \end{aligned} \tag{13}$$

(ii) For the coefficient of ε^5, we find

$$\begin{aligned} &2(N_{2x}+V_{2x})+N_{1t_1}+2N_{1x}V_1+2V_{1x}N_1=0,\\ &4\mu(N_{2x}+M_{2x})+N_{1xxx}+12\mu(N_{1x}N_1+M_{1x}N_1)\\ &+4V_{2x}+2V_{1t_1}+12V_{1x}N_1-4V_{1x}V_1=0,\\ &2(M_{2x}+V_{2x})+M_{1t_1}+2M_{1x}V_1+2V_{1x}M_1=0,\\ &4\mu(N_{2x}+M_{2x})+M_{1xxx}+12\mu(N_{1x}V_1+M_{1x}M_1)\\ &+4V_{2x}+2V_{1t_1}+12V_{1x}N_1-4V_{1x}V_1=0. \end{aligned} \tag{14}$$

(iii) For the coefficients of ε^7, we find

$$\begin{aligned} &2(N_{3x}+V_{3x})+N_{2t_1}+N_{1t_2}+2(N_{2x}V_1+N_{1x}V_2+V_{2x}N_1+V_{1x}N_2)=0,\\ &4(\mu N_{3x}+M_{3x}+V_{3x})+N_{2xxx}+12\mu(N_{2x}N_1+M_{2x}N_1)\\ &+12\mu(N_{1x}N_2+M_{1x}N_2)+12\mu(N_{1x}N_1^2+M_{1x}N_1^2)\\ &+2N_{1xxx}N_1-2N_{1xx}N_{1x}-4(V_{2x}N_1+-4V_{2x}V_1)\\ &+2(V_{2t_1}+V_{1t_2})+12V_{2x}N_1+6V_{1t_1}N_1\\ &+12V_{1x}N_2+12V_{1x}N_1^2-12V_{1x}N_1V_1-4V_{1_x}V_2=0,\\ &2(M_{3x}+V_{3x})+M_{2t_1}+M_{1t_2}+2(M_{2x}V_1+M_{1x}V_2+V_{2x}M_1+V_{1x}M_2)=0,\\ &4(\mu N_{3x}+M_{3x}+V_{3x})+M_{2xxx}+12\mu(N_{2x}M_1+M_{2x}M_1)\\ &+12\mu(N_{1x}M_2+M_{1x}M_2)+12\mu(N_{1x}M_1^2+M_{1x}N_1^2)\\ &+2M_{1xxx}M_1-2M_{1xx}M_{1x}-4(V_{2x}M_1+-4V_{2x}V_1)\\ &+2(V_{2t_1}+V_{1t_2})+12V_{2x}M_1+6V_{1t_1}N_1\\ &+12V_{1x}M_2+12V_{1x}M_1^2-12V_{1x}M_1V_1-4V_{1_x}V_2=0\\ &\vdots \end{aligned} \tag{15}$$

and so on. If we take $\mu = \frac{1}{2}$ in (13), we find

$$N_1 = -V_1 = M_1, \tag{16}$$

by considering the constants of integration as zero.

3.1 The Derivation of KdV Flow Equations

We now use (16) in the system (14) and take

$$N_2 = M_2,$$

$$M_2 = -V_2 + \frac{1}{8}V_{1xx} + \frac{3}{4}V_1^2, \tag{17}$$

so that we find the following equation

$$V_{1t_1} = \frac{1}{4}(V_{1xxx} - 4V_1V_{1x}), \tag{18}$$

or making the transformation

$$t_1 \to \frac{1}{4}t_1, \quad V_1 \to -\frac{3}{2}u$$

we derive the well known KdV equation

$$u_{t_1} = u_{xxx} + 6uu_x. \tag{19}$$

If we take in the equation (17) for V_2 as

$$V_2 = k_1V_1^2 + k_2V_{1xx}, \tag{20}$$

then insert this into the equation (15), we obtain the equation

$$V_{1t_2} = \frac{1}{32}\left(64(N_{3x}+V_{3x}) - (16k_1+96k_2-16)V_{1xxx}V_1 - (8k_2-1)V_{1xxxxx} - (32k_2+4)V_{1xx}V_{1x} - (32k_1-96)V_{1x}V^2\right), \tag{21}$$

$$N_3 = M_3.$$

Now chosing

$$N_3 = -V_3 + \frac{1}{1280}(-171V_{1xxxx} - 1356V_{1xx}V_{1x} + 516V_{1x}^2 + 3136V_1^3), \tag{22}$$

with

$$k_1 = -\frac{82}{27}, \quad k_2 = -\frac{133}{18},$$

from the equation (21), and using an appropriate transformation for $t_2 \to -\frac{1}{64}t_2$, we derive the Lax's fifth order KdV equation as t_2 KdV flow equation:

$$u_{t_2} = \left(u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3\right)_x. \tag{23}$$

We now insert (20) and (22) the latter into the system and choose

$$V_3 = k_3V_{1xxxx} + k_4V_{1xx}V_{1x} + k_5V_{1x}^2 + k_6V_1^3, \tag{24}$$

$$N_4 = M_4$$

we finally obtain from the coefficients of ε^9, the seventh order KdV flow equation

$$u_{t_3} = u_{xxxxxxx} + 14uu_{xxxxx} + 42u_x u_{xxxx} + 70u_{xx}u_{xxx} + 70u^2 u_{xxx} + 280uu_x u_{xx} + 70u_x^3 + 140u^3 u_x \tag{25}$$

where we take

$$k_1 = \frac{101}{56}, \qquad k_2 = \frac{\sqrt{195361} - 815}{896},$$
$$k_3 = \frac{3(-219\sqrt{195361} - 498965)}{4064256}, k_4 = \frac{3(7\sqrt{195361} - 21997)}{92256},$$
$$k_5 = \frac{-17\sqrt{195361} - 151889}{25088}, k_6 = -\frac{4769}{784}$$

and make an appropriate transformation $t_3 \to \frac{1}{512}t_3$. In general, proceeding the calculation as before, we obtain KdV flow equations (6).

We therefore get the KdV flow equations (6) integrable Manakov equations (2) by using the multiple scales method. This fact is quite natural because the expression (11) as $\varepsilon \to 0$ represents a quasi monochromatic weakly nonlinear wave packet whose complex envelope should be described by the integrable Manakov equations (2) ([20], [21]).

4 Conclusion

We have used a multiple scales method to provide a new derivation of the KdV flow equations from the integrable Manakov equations. This derivation is not only on the level of equation but also on the level of the Hamiltonian densities. The equations for the coefficients at each order in epsilon, contain no secular terms in our derivation of KdV flow equations (6). Therefore no freedom is left in choosing coefficients at each order in epsilon and the expansion is uniquely determined. Thus there exists a relation between the integrable Manakov equations (2) with the KdV flow equations (6).Details are given in [22].

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