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Approximate solutions of the hyperchaotic Rössler system by using the Bessel collocation scheme

Suayip Yuzbasi

Department of Mathematics, Faculty of Science, Akdeniz University, 07058 Antalya, Turkey

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Abstract: The purpose of this study is to give a Bessel polynomial approximation for the solutions of the hyperchaotic Rössler system. For this purpose, the Bessel collocation method applied to different problems is developed for the mentioned system. This method is based on taking the truncated Bessel expansions of the functions in the hyperchaotic Rössler systems. The suggested secheme converts the problem into a system of nonlinear algebraic equations by means of the matrix operations and collocation points, The accuracy and efficiency of the proposed approach are demonstrated by numerical applications and performed with the help of a computer code written in Maple. Also, comparison between our method and the differential transformation method is made with the accuracy of solutions.

Keywords: Hyperchaotic Rössler system, Bessel collocation method, approximate solution, Nonlinear differential equation systems, Bessel functions of first kind, Collocation points.

1 Introduction

A variety of physical and engineering phenomena can generally be modeled by chaotic or hyperchaotic systems of ODEs. Most of these systems do not have precise analytical solution (or a closed form), so approximation and numerical techniques are needed.

Several authors have used numeric-analytic methods to solve these chaotic systems. Some commonly used techniques for some chaotic systems are the adomian decomposition method [1-4], the variation iteration method [5-9], the homotopy analysis method [10], the homotopy perturbation method [11] and the differential transformation method [12]. In addition, Park [13], Plienpanich et. al [14] and Yassen [15,16] worked on chaotic dynamical systems.

In this study, we consider the hyperchaotic Rössler system, was constructed by Rössler [17], which is a three-dimensional system of differential equations given by:

$$\begin{cases} \frac{dx}{dt} = -y - z, \;\; x(0) = \gamma_1 \\ \frac{dy}{dt} = x + ay + w, \;\; y(0) = \gamma_2 \\ \frac{dz}{dt} = b + xz, \;\; z(0) = \gamma_3 \\ \frac{dw}{dt} = -cz + dw, \;\; w(0) = \gamma_4 \end{cases} \tag{1}$$

where x, y, z and w are state variables, a, b, c, d and R are positive constants, γ_1, γ_2 and γ_3, γ_4 are appropriate constants.

* Corresponding author e-mail: syuzbasi@akdeniz.edu.tr

Lately, Yüzbaşı et.al. [20-22] have worked the Bessel collocation method to solve singular differential equations, pantograph equations, nonlinear Lan-Emden differential equations and system of linear differential equations.

In this paper, the basic ideas of the above-mentioned studies are developed and applied the hyperchaotic Rössler system which is a class of nonlinear differential equation systems.

Our aim in this article is to obtain approximate solutions of system (1) expressed in the truncated Bessel series form

$$x(t)=\sum_{n=0}^{N}a_{1,n}J_n(t),\ y(t)=\sum_{n=0}^{N}a_{2,n}J_n(t),\ z(t)=\sum_{n=0}^{N}a_{3,n}J_n(t) \text{ and } w(t)=\sum_{n=0}^{N}a_{4,n}J_n(t) \tag{2}$$

so that $a_{1,n}$, $a_{2,n}$, $a_{3,n}$and $a_{4,n}$ $(n=0,1,2,...,N)$ are the unknown Bessel coefficients and $J_n(t)$, $n=0,1,2,\ldots,N$ are the Bessel polynomials of first kind defined by

$$J_n(t)=\sum_{k=0}^{[\![\frac{N-n}{2}]\!]}\frac{(-1)^k}{k!(k+n)!}\left(\frac{t}{2}\right)^{2k+n} \tag{3}$$

2 Bessel collocation scheme for the hyperchaotic Rössler system

Firstly, let us consider the approximate solutions $x(t)$, $y(t)$, $z(t)$ and $w(t)$ of system (1) defined by the truncated Bessel series (3). We can write the functions defined in relation (2) in the matrix forms,

$$x(t)=J(t)A_1,\ y(t)=J(t)A_2,\ z(t)=J(t)A_3 \text{ and } w(t)=J(t)A_4 \tag{4}$$

where

$$J(t)=\begin{bmatrix}J_0(t)\ J_1(t)\ \cdots\ J_N(t)\end{bmatrix},\ A_1=\begin{bmatrix}a_{1,0} & a_{1,1} & \cdots & a_{1,N}\end{bmatrix}^T,\ A_2=\begin{bmatrix}a_{2,0} & a_{2,1} & \cdots & a_{2,N}\end{bmatrix}^T$$
$$A_3=\begin{bmatrix}a_{3,0} & a_{3,1} & \cdots & a_{3,N}\end{bmatrix}^T \quad \text{and} \quad A_4=\begin{bmatrix}a_{4,0} & a_{4,1} & \cdots & a_{4,N}\end{bmatrix}^T.$$

Also, we can show the relations (4) in matrix forms

$$x(t)=T(t)D^TA_1,\ y(t)=T(t)D^TA_2,\ z(t)=T(t)D^TA_3,\ \text{and } w(t)=T(t)D^TA_4 \tag{5}$$

so that

$$T(t)=\begin{bmatrix}1 & t & t^2 & \cdots & t^N\end{bmatrix}$$

and

if N is odd,

$$D=\begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N}} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)},$$

if N is even,

$$D=\begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!2^{N}} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}..$$

On the other hand, the relation between the matrix $T(t)$ and its derivative $T^{(1)}(t)$ is given by

$$T^{(1)}(t)=T(t)B^T \tag{6}$$

where B,

$$B^T=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

y help of the relations (5) and (6) , we have the matrix relations

$$x^{(1)}(t)=T(t)B^TD^TA_1,\ y^{(1)}(t)=T(t)B^TD^TA_2,\ z^{(1)}(t)=T(t)B^TD^TA_3 \ \text{ and } \ w^{(1)}(t)=T(t)B^TD^TA_4. \tag{7}$$

We can express the matrices $u(t)$ and $u^{(1)}(t)$ as follows:

$$u(t)=\bar{T}(t)\bar{D}A$$

so that

$$u(t)=\left[x(t)\ y(t)\ z(t)\ w(t)\right]^T,\ u^{(1)}(t)=\left[x^{(1)}(t)\ y^{(1)}(t)\ z^{(1)}(t)\ w^{(1)}(t)\right]^T$$

$$\bar{T}(t)=\begin{bmatrix} T(t) & 0 & 0 & 0 \\ 0 & T(t) & 0 & 0 \\ 0 & 0 & T(t) & 0 \\ 0 & 0 & 0 & T(t) \end{bmatrix},\ \bar{D}=\begin{bmatrix} D^T & 0 & 0 & 0 \\ 0 & D^T & 0 & 0 \\ 0 & 0 & D^T & 0 \\ 0 & 0 & 0 & D^T \end{bmatrix},\ \bar{B}=\begin{bmatrix} B^T & 0 & 0 & 0 \\ 0 & B^T & 0 & 0 \\ 0 & 0 & B^T & 0 \\ 0 & 0 & 0 & B^T \end{bmatrix} \text{ and } A=\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}.$$

Thus, we can indicate the system (1) with the matrix form

$$u^{(1)}(t)-Pu(t)-R\bar{u}(t)=g \tag{8}$$

where

$$P=\begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -c & d \end{bmatrix},\ u(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix},\ R=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \bar{u}(t)=[x(t)z(t)] \quad \text{and} \quad g=\begin{bmatrix} 0 \\ 0 \\ b \\ 0 \end{bmatrix}.$$

We substitute the collocation points, defined by

$$t_s=\frac{R}{N}s \tag{9}$$

into Eq. (9), and thus, we have the matrix equation system

$$u^{(1)}(t_s)-Pu(t_s)-R\bar{u}(t_s)=g$$

or briefly the fundamental matrix equation is

$$U^{(1)}-\bar{P}U-\bar{R}\bar{U}=G \tag{10}$$

where

$$U^{(1)}=\begin{bmatrix} u^{(1)}(t_0) \\ u^{(1)}(t_1) \\ \vdots \\ u^{(1)}(t_N) \end{bmatrix},\ \bar{P}=\begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & P & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P \end{bmatrix}_{(N+1)\times(N+1)},\ U=\begin{bmatrix} u(t_0) \\ u(t_1) \\ \vdots \\ u(t_N) \end{bmatrix},\ G=\begin{bmatrix} g \\ g \\ \vdots \\ g \end{bmatrix}_{(N+1)\times 1},$$

$$\bar{R}=\begin{bmatrix} R & 0 & \cdots & 0 \\ 0 & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{bmatrix}_{(N+1)\times(N+1)} \text{ and } \bar{U}=\begin{bmatrix} \bar{u}(t_0) \\ \bar{u}(t_1) \\ \vdots \\ \bar{u}(t_N) \end{bmatrix}.$$

By using the relations (8) and the collocation points (10), we obtain recurrence relations

$$u(t_s)=\bar{T}(t_s)\bar{D}A \text{ and } u^{(1)}(t_s)=\bar{T}(t_s)\bar{B}\bar{D}A$$

which can be written as

$$U=T\bar{D}A \tag{11}$$

where

$$T=\left[\bar{T}(t_0)\ \bar{T}(t_1)\ \ldots\ \bar{T}(t_N)\right]^T,\bar{T}(t_s)=\begin{bmatrix}T(t_s)&0&0&0\\0&T(t_s)&0&0\\0&0&T(t_s)&0\\0&0&0&T(t_s)\end{bmatrix}\quad\text{and}\quad s=0,1,\ldots,N.$$

By substituting the collocation points (10) into the matrix $\bar{u}(t)$ given Eq. (9), we find the matrix form

$$\bar{U}=\begin{bmatrix}\bar{u}(t_0)\\\bar{u}(t_1)\\\vdots\\\bar{u}(t_N)\end{bmatrix}=\begin{bmatrix}z(t_0)x(t_0)\\z(t_1)x(t_1)\\\vdots\\z(t_N)x(t_N)\end{bmatrix}=\bar{Z}\bar{X}\tag{12}$$

where

$$\bar{Z}=\tilde{T}\tilde{D}\bar{A}_3\tag{13}$$

so that

$$\tilde{T}=\begin{bmatrix}T(t_0)&0&\cdots&0\\0&T(t_1)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&T(t_N)\end{bmatrix},\tilde{D}=\begin{bmatrix}D^T&0&\cdots&0\\0&D^T&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&D^T\end{bmatrix}_{(N+1)\times(N+1)},\bar{A}_3=\begin{bmatrix}A_3&0&\cdots&0\\0&A_3&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&A_3\end{bmatrix}_{(N+1)\times(N+1)},\tilde{\tilde{T}}=\begin{bmatrix}T(t_0)\\T(t_1)\\\vdots\\T(t_N)\end{bmatrix},$$

$$\tilde{\tilde{D}}=\left[D^T\ \tilde{Z}\ \tilde{Z}\ \tilde{Z}\right],\tilde{Z}=\begin{bmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{bmatrix}_{(N+1)\times(N+1)}\quad\text{and}\quad A=\begin{bmatrix}A_1\\A_2\\A_3\\A_4\end{bmatrix}.$$

When we substitute relations (12)-(14) into Eq. (11), we have fundamental matrix equation

$$\left\{T\bar{B}\bar{D}-\bar{P}T\bar{D}-\bar{R}\tilde{T}\tilde{D}\bar{A}_3\tilde{\tilde{T}}\tilde{\tilde{D}}\right\}A=G\tag{14}$$

Briefly, we can write Eq. (15) as

$$WA=G\text{ or }[W;G];\ W=T\bar{B}\bar{D}-\bar{P}T\bar{D}-\bar{R}\tilde{T}\tilde{D}\bar{A}_3\tilde{\tilde{T}}\tilde{\tilde{D}}\tag{15}$$

which corresponds to a system of the $4(N+1)$ nonlinear algebraic equations with the unknown Bessel coefficients $a_{1,n}$, $a_{2,n}$, $a_{3,n}$, $a_{4,n}$, $(n=0,1,2,...,N)$.
By using the relation $u(t)$ given in Eq.(8), the matrix form for conditions given in system (1) can be written as

$$VA=[\gamma]\text{ or }[V;\gamma]\tag{16}$$

so that

$$V=\bar{T}(0)\bar{D}A$$

and

$$\gamma=\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\end{bmatrix}.$$

Lastly, by replacing the rows of the matrix $[V;\gamma]$ by two rows of the augmented matrix $[W;G]$, we have the new augmented matrix

$$[\tilde{W};\tilde{G}] \tag{17}$$

which is a nonlinear algebraic system. The unknown the Bessel coefficients can be found by solving this system. The unknown Bessel coefficients $a_{i,0},a_{i,1},\ldots,a_{i,N},\ (i=1,2,3,4)$ are substituted in Eq. (5). Thus we obtain the Bessel polynomial solutions

$$x_N(t)=\sum_{n=0}^{N}a_{1,n}J_n(t),\ \ y_N(t)=\sum_{n=0}^{N}a_{2,n}J_n(t),\ \ z_N(t)=\sum_{n=0}^{N}a_{3,n}J_n(t),\ \text{and } w_N(t)=\sum_{n=0}^{N}a_{4,n}J_n(t).$$

We can easily check the accuracy of this solutions as follows:

Since the truncated Bessel series (2) are approximate solutions of system (1), when the function $x_N(t)$, $y_N(t)$, $z_N(t)$, $w_N(t)$ and theirs derivatives are substituted in system (1), the resulting equation must be satisfied approximately; that is, for $t=t_q\in[0,R]$ $q=0,1,2,\ldots$

$$\begin{cases} E_{1,N}(t_q)=\left|x^{(1)}(t_q)+y(t_q)+z(t_q)\right|\cong 0,\\ E_{2,N}(t_q)=\left|y^{(1)}(t_q)-x(t_q)-ay(t_q)-w(t_q)\right|\cong 0,\\ E_{3,N}(t_q)=\left|z^{(1)}(t_q)-b-x(t_q)z(t_q)\right|\cong 0,\\ E_{4,N}(t_q)=\left|w^{(1)}(t_q)+cz(t_q)-dw(t_q)\right|\cong 0 \end{cases} \tag{18}$$

and $E_{i,N}(t_q)\le 10^{-k_q}$, $i=1,2,3$ (k_q positive integer). If max $10^{-k_q}=10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{i,N}(t_q)$, $(i=1,2,3,4)$ at each of the points becomes smaller than the prescribed 10^{-k}, see [20-24].

3 Numerical applications

In this section, we apply the presented method for solutions of the hyperchaotic Rössler system given in Eq.(1). Bessel collocation algorithm were coded in the computer algebra package Maple. In this application, all the calculations were made with Digits = 30. We also fix the values of the parameters $a=0.25$, $b=3$, $c=0.5$, $d=0.05$, $R=1$, and take the initial conditions $x(0)=-20$, $y(0)=0$, $z(0)=0$, $w(0)=15$.

By following for $N=5,8,11$ the procedure given in Section 2, we obtain the approximate solutions of the non-chaotic system, respectively, as

$$x_5(t)=-20-7.34344x+36.24520x^2-42.82178x^3+34.61182x^4-11.05533x^5,$$

$$y_5(t) = -20.63247x - 4.47669x^2 + 6.11875x^3 - 1.81177x^4 + .512261x^5,$$

$$z_5(t) = 15 - 139.27022x + 491.84285x^2 - 822.48976x^3 + 655.25642x^4 - 199.90159x^5,$$

$$w_5(t) = -3.67253x + 12.87728x^2 - 21.95501x^3 + 17.81419x^4 - 5.54332x^5,$$

$$x_8(t) = -20 - 11.40590x + 85.37562x^2 - 278.87712x^3 + 636.99958x^4 - 906.28709x^5 + 783.30501x^6 - 376.75801x^7 + 77.27979x^8,$$

$$y_8(t) = -20.27545x - 8.09341x^2 + 23.05418x^3 - 44.45475x^4 + 63.39137x^5 - 54.76511x^6 + 26.26364x^7 - 5.37448x^8,$$

$$z_8(t) = 15 - 228.62169x + 1518.61182x^2 - 5641.84690x^3 + 12772.34268x^4 - 18012.49169x^5 + 15450.62769x^6 - 7374.29005x^7 + 1500.88294x^8,$$

$$w_8(t) = -5.70114x + 37.45674x^2 - 140.05004x^3 + 319.24424x^4 - 453.57086x^5 + 392.04308x^6 - 188.57241x^7 + 38.67907x^8,$$

$$x_{11}(t) = -20 - 13.91372x + 132.99998x^2 - 667.83178x^3 + 2451.79490x^4 - 6287.28040x^5 + 11423.69080x^6 - 14651.70371x^7 + 12966.21122x^8 - 7529.10733x^9 + 2579.84712x^{10} - 395.07219x^{11},$$

$$y_{11}(t) = -20.08195x - 11.58408x^2 + 51.30291x^3 - 175.52774x^4 + 450.58900x^5 - 818.48071x^6 + 1049.12062x^7 - 927.89361x^8 + 538.49190x^9 - 184.40921x^{10} + 28.22415x^{11},$$

$$z_{11}(t) = 15 - 282.00267x + 2512.02769x^2 - 13671.05684x^3 + 49938.15803x^4 - 127456.79128x^5 + 230538.19488x^6 - 294325.46664x^7 + 259269.03850x^8 - 149860.50502x^9 + 51116.22487x^{10} - 7792.56106x^{11},$$

$$w_{11}(t) = -6.95627x61.29578x^2 - 334.73483x^3 + 1227.60749x^4 - 3146.92250x^5 + 5717.86806x^6 - 7333.58839x^7 + 6489.94832x^8 - 3768.52571x^9 + 1291.28416x^{10} - 197.74439x^{11}.$$

The hyperchaotic Rössler system has been solved by using the differential transformation method (DTM) with the 9-term in [12]). Figure 1 shows the error functions gained by Eq. (19) for the present method and DTM. It is seen from Figure 1 that that the results obtained by the present method is better than obtained by the DTM. Plots of the error functions obtained the accuracy of the solution imply the effectiveness of the present method. From Figure 1, we see the errors decrease as N increases.

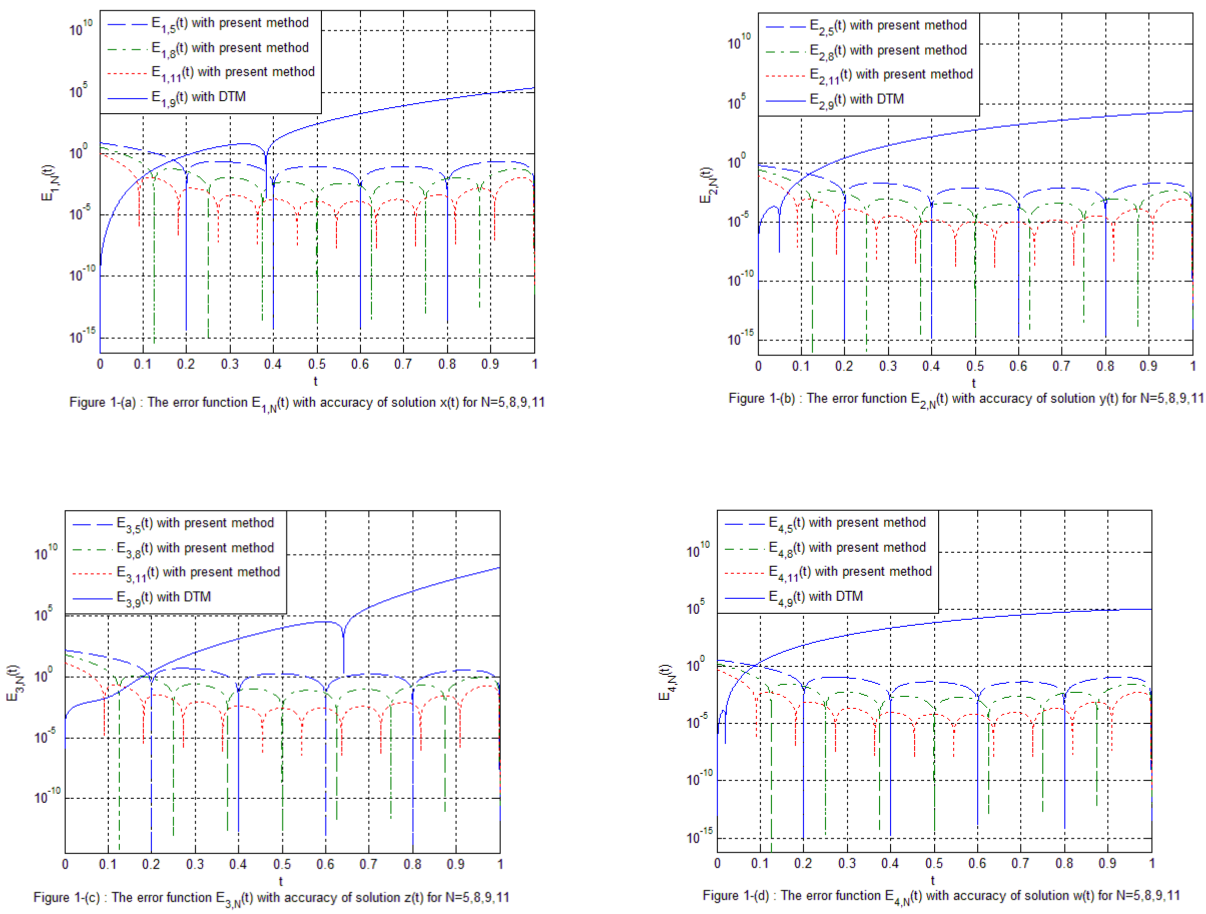

Figure 1. For the examples given above, **(a)** plot of the error functions obtained with accuracy of the solution $x_N(t)$ for $N = 5,8,9,11$, **(b)** plot of the error functions gained with accuracy of the solution $y_N(t)$ for $N = 5,8,9,11$, **(c)** plot of the error functions obtained with accuracy of the solution $z_N(t)$ for $N = 5,8,9,11$ and **(d)** plot of the error functions obtained with accuracy of the solution $w_N(t)$ for $N = 5,8,9,11$.

4 Conclusions

In this study, we have presented the Bessel collocation method for approximating solutions of the hyperchaotic Rössler system. We have demonstrated the accuracy and efficiency of the presented technique with an example. For a measure of confidence in the results, we assured the correctness of the obtained solutions by putting them back into the original system with the aid of Maple. We compare the error functions obtained by Eq.(19) for the presented method and DTM [12] for various values of$N$ in Figure 1. It is seen from Figure 1 that the results obtained by the present method is better than that obtained by the DTM. Also, it is seen from Figure 1 that the errors decrease as N increases. Plots of the error functions show that our scheme an accurate and reliable technique for the nonlinear hyperchaotic Rössler systems of ODEs. The approximate solutions of the system (1) by the suggested method can be found easily in shorter time with the computer programs such as Matlab, Maple and Mathematica. The computations associated with the example have been performed on a computer by help of a computer code written in Maple.

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