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Regularization Short-time Fourier transform in extended Colmbeau algebra

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Abstract: This paper describes the moderateness of Short-time Fourier transform via the Caputo fractional derivative. Moreover, we consider some properties of the generalized STFT in extended Colmbeau algebra.

Keywords: Colombeau algebra, STFT, Caputo fractional derivative.

1 Introduction

Fractional calculus has been emerging as a very interesting tool for an increasing number of scientific fields, namely, in the areas of electromagnetism, control engineering, and signal processing. However, an interested scientist or engineer has to face the problem created by the somewhat chaotic state of the art due to the existence of several definitions that lead to different results: Riemann-Liouville, Caputo, Grunwald-Letnikov, Hadamard, Marchaud, are some of the known definitions.

One of the central objects in the theory of time-frequency analysis is short-time Fourier transform (STFT). The STFT represents a sort of compromise between the time- and frequency-based views of a signal. It provides some information about both when and at what frequencies a signal event occurs. However, you can only obtain this information with limited precision, and that precision is determined by the size of the window. The Fourier transform of the resulting signal is taken as the window is slid along the time axis, resulting in a two-dimensional representation of the signal.
Algebras of generalized functions have been developed by many authors mainly inspired by the work of J. F. Colombeau (cf.[1],[2],[3]) and have proved valuable as a tool for treating partial differential equations with singular data or coefficients (cf.[6]). Fractional derivatives of Colombeau generalized functions are introduced in (cf.[5]). The reason for introducing fractional derivatives into the Colombeau algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it.

We investigate the properties of the generalized STFT with generalized function f and generalized window function g. We ensure that the basic classical regularity properties carry over to the generalized case. The paper is organized as follows: After the introduction we give some basic preliminaries such as notations and definitions of the objects. Then, we show the moderateness results for STFT drived by the Caputo fractional derivative in the extended algebra of generalized functions. We prove the moderateness and the negligibility for the fractional derivatives for STFT. A generalized version of the regularity properties of classical short-time Fourier transform is obtained.

2 Preliminaries

2.1. Colombeau algebra

In this paper Ω denotes an open subset of $\mathbb{R}^d$ $(d \in \mathbb{N})$. The elements of Colombeau algebras $\mathscr{G}$ are equivalence classes of regularization, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Throughout this paper, for elements of the space $C^\infty(\Omega)^I$ of sequences of smooth functions indexed by $\varepsilon \in I = (0,1]$ we

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shall use the notation $(u_\varepsilon)_\varepsilon \in (0,1]$ (so $u_\varepsilon \in C^\infty(\Omega)$.) We set $\mathscr{G}(\Omega) = \mathscr{E}_M(\Omega)/\mathscr{N}(\Omega)$, where

$$\mathscr{E}_M(\Omega) = \{(u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^{(0,1]} |\ \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N}\ s.t\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{-N}),\ \varepsilon \to 0\},$$

$$\mathscr{N}(\Omega) = \{(u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))^{(0,1]} |\ \forall K \subset\subset \Omega, \forall \alpha \in \mathbb{N}_0^n\ \forall s \in \mathbb{N}\ s.t\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{s}),\ \varepsilon \to 0\}.$$

Element of $\mathscr{E}_M(\Omega)$ and $\mathscr{N}(\Omega)$ are called moderate, resp. negligible functions. Families $(r_\varepsilon)_\varepsilon$ of complex numbers such that $|r_\varepsilon| = O(\varepsilon^{-p})$ as $\varepsilon \to 0$ for some $p \geq 0$ are called moderate, those for which $|r_\varepsilon| = O(\varepsilon^{q})$ for every $q \geq 0$ are termed negligible. The ring $\tilde{\mathbb{R}}$ of Colombeau generalized numbers is obtained by factoring moderate families of complex numbers with respect to negligible families.

The algebra $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ of generalized functions based on $\mathscr{S}(\mathbb{R}^d)$ belongs to the family of spaces of generalized functions based on a locally convex topological vector space.

Here the definition of $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ holds for extended Colombeau algebra to the spatial variable x and for a Caputo fractional derivative. The Colombeau algebra generalized functions is the set $\mathscr{G}^e_{\mathscr{S}}(\mathbb{R}^d) = \mathscr{E}^e_{\mathscr{S}}(\mathbb{R}^d)/\mathscr{N}^e_{\mathscr{S}}(\mathbb{R}^d)$ where

$$\mathscr{E}^e_{\mathscr{S}}(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} |\ \forall \alpha, \beta \in \mathbb{N}_0^d, \exists N \in \mathbb{N},\ s.t\ \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{-N})\ as\ \varepsilon \to 0\},$$

$$\mathscr{N}^e_{\mathscr{S}}(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} |\ \forall \alpha, \beta \in \mathbb{N}_0^d, \forall q \in \mathbb{N},\ s.t\ \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{q})\ as\ \varepsilon \to 0\}.$$

In the setting of Colombeau theory, the notion of regularity is based on the subalgebra $\mathscr{G}^\infty(\mathbb{R}^d)$ of regular generalized functions in $\mathscr{G}(\mathbb{R}^d)$. It is defined by those elements which have a representative $(u_\varepsilon)_\varepsilon$ satisfying

$$\mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathscr{S}(\mathbb{R}^d)^{(0,1]} |\exists N \in \mathbb{N}\ \forall \alpha, \beta \in \mathbb{N}_0^d\ s.t\ \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{-N})\ as\ \varepsilon \to 0\}.$$

$\mathscr{G}^\infty_{\mathscr{S}}(\mathbb{R}^d)$ can be viewed as the quotient $\mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)/\mathscr{N}(\mathbb{R}^d)$.
The algebra $\mathscr{G}_\tau(\mathbb{R}^d)$ of tempered generalized functions was introduced by J. F. Colombeau in (cf. [3]) in order to develop a theory of Fourier transform in algebras of generalized functions. We have

$$\mathscr{O}_M(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) | \forall \alpha \in \mathbb{N}_0^d\ \exists N \in \mathbb{N} :\ \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |D^\alpha u_\varepsilon(x)| < \infty\}.$$

The Colombeau algebra of tempered generalized functions is $\mathscr{G}^e_\tau(\mathbb{R}^d) = \mathscr{E}^e_\tau(\mathbb{R}^d)/\mathscr{N}^e_\tau(\mathbb{R}^d)$. where

$$\mathscr{E}^e_\tau(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathscr{O}_M(\mathbb{R}^d)^{(0,1]} |\ \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \exists N \geq 0 :\ \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |D^\alpha u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{-N})\},$$

$$\mathscr{N}^e_\tau(\mathbb{R}^d) = \{(u_\varepsilon)_\varepsilon \in \mathscr{O}_M(\mathbb{R}^d)^{(0,1]} |\ \forall \alpha \in \mathbb{R}_+ \cup \{0\}, \exists N \geq 0\ \forall s \geq 0 :\ \sup_{x\in\mathbb{R}^d} \langle x\rangle^{-N} |D^\alpha u_\varepsilon(x)| = \mathscr{O}(\varepsilon^{s})\}.$$

Here, D^α, $m-1 \leq \alpha < m$, $m \in \mathbb{N}$, is the Caputo fractional derivative. Imbedding $S'(\mathbb{R})$ into $\mathscr{G}^e_\tau(\mathbb{R})$ is given by the map: $i : v \to [(v * \varphi_\varepsilon)_{\varepsilon>0}]$, where

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon}),\ \varphi(x) \in C_0^\infty(\mathbb{R}), \varphi(x) \geq 0, \int \varphi(x)dx = 1, \int x^\alpha \varphi(x)dx = 0, \forall \alpha \in \mathbb{N}, |\alpha| > 0.$$

2.2. Fractional derivatives

Let f_ε be a representative of a Colombeau generalized function. The fractional derivatives of order $\alpha > 0$ in the Caputo sense on the interval $[0,t)$ is defined by:

$$D_t^\alpha f_\varepsilon(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f_\varepsilon^m(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f_\varepsilon(t), & \alpha = m \end{cases},$$

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for $m \in \mathbb{N}$ and $\varepsilon \in (0,1]$.

For imbedding of the fractional derivatives of distributions into $\mathscr{G}^e([0,T])$, $T > 0$ we use the convolution with mollifier $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$.
Let v be the distribution. Then $v_\varepsilon = v * \varphi_\varepsilon(t)$, according to the classical Colombeau theory for entire derivatives. Consider $\tilde{D}^\alpha v_\varepsilon$, $\alpha \in \mathbb{R}_+ \cup \{0\}$. For $0 < \alpha < 1$ we have:

$$\begin{aligned}\tilde{D}^\alpha v_\varepsilon = D^\alpha v_\varepsilon * \varphi_\varepsilon(t) &= \frac{1}{\Gamma(1-\alpha)}\Big(\int_0^t \frac{\dot{v}_\varepsilon(\tau)}{(t-\tau)^\alpha}d\tau\Big) * (\varphi_\varepsilon(t)) \\ &\leqslant \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} \Big|\int_0^t \frac{\dot{v}_\varepsilon(\tau)}{(t-\tau)^\alpha}d\tau\Big| \cdot \|\varphi_\varepsilon(t)\|_{L^1} \\ &\leqslant \frac{C}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} |\dot{v}_\varepsilon(t)| \frac{T^{1-\alpha}}{1-\alpha} \leqslant C_{T,\alpha}\varepsilon^{-N}, \quad \exists N > 0,\ \alpha \in \mathbb{R}_+ \cup \{0\}.\end{aligned}$$

In a similar way, for higher fractional derivatives we apply the semigroup property of fractional differentiation.

3 Main Results

3.1. Short-time Fourier transform in extended Colombeau algebra of generalized functions.

In (cf. [4]), in addition to some properties of the generalized STFT, the class of Short-time Fourier transform as a well-defined element of $\mathscr{G}_\tau(\mathbb{R}^{2d})$ and $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^{2d})$ was thoroughly investigated. Furthermore, the above mentioned study used entire derivatives to obtain moderateness results. The present study, however, implements Caputo fractional derivative to achieve the same results.
As one of the proof for lemma, we use this inequality that for every $\gamma \in \mathbb{R}$, $n \in \mathbb{N}_0$ there exists a constant $C \geq 0$ such that

$$|D^\alpha g(y-x)| \leq \frac{C}{\langle y-x\rangle^m}.$$

This together with Peetre's inequality gives

$$|D^\alpha g(y-x)| \leq C \cdot \langle y\rangle^{\pm m}\langle x\rangle^{\mp m}. \tag{1}$$

Lemma 3.1. Let $(f_\varepsilon)_\varepsilon \in \mathscr{G}_\tau(\mathbb{R}^{2d})$, $g \in \mathscr{S}(\mathbb{R}^{2d})$. Then Short-time Fourier transform

$$(V_g f)_\varepsilon(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\omega}\overline{g(y-x)} f_\varepsilon(y)dy$$

is a well-defined element of $\mathscr{G}_\tau(\mathbb{R}^{2d})$.

Proof. For the $\tau-$moderateness it is shown that for every $\alpha, \beta > 0$, $m-1 < \alpha, \beta < m$ there exists $N \in \mathbb{N}$ such that

$$|D_x^\alpha D_\omega^\beta V_g f_\varepsilon(x,\omega)| \leq C \cdot \varepsilon^{-N}\langle x\rangle^N\langle\omega\rangle^N$$

holds.
We investigate

$$|D_x^\alpha D_\omega^\beta V_g f_\varepsilon(x,\omega)| = |D_x^\alpha D_\omega^\beta \int e^{-2\pi i y\omega}\overline{g(y-x)} f_\varepsilon(y)dy|. \tag{2}$$

Analyzing $D_\omega^\beta e^{-2\pi i y\omega}$, $\beta \in \mathbb{R}_+ \cup \{0\}$. For $m-1 < \beta < m$, one can find:

$$|D_\omega^\beta e^{-2\pi i y\omega}| = |\frac{1}{\Gamma(m-\beta)}\int_0^\omega \frac{D_\tau^m(e^{-2\pi i y\tau})}{(\omega-\tau)^{\beta+1-m}}d\tau|$$

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$$\leq \frac{(2\pi y)^m}{\Gamma(m-\beta)} \int_0^{\omega} \frac{d\tau}{(\omega-\tau)^{\beta+1-m}} d\tau \leq \frac{(2\pi y)^m}{\Gamma(m-\beta)} \omega^{m-\beta}.$$

For the computation of $D_x^{\alpha} g(y-x)$, $\alpha \in \mathbb{R}_+ \cup \{0\}$, $m-1<\alpha<m$, one can get:

$$|D_x^{\alpha} g(y-x)| = |\frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{D_x^m g(y-\tau)}{(x-\tau)^{\alpha+1-m}} d\tau| \leq \frac{C}{\Gamma(m-\alpha)} \int_0^x \frac{\langle y\rangle^{-m_1}\langle \tau\rangle^{m_1}}{(x-\tau)^{\alpha+1-m}} d\tau$$
$$\leq C\langle y\rangle^{-m_1}\langle x\rangle^{m-\alpha-1} \int_0^x \langle \tau\rangle^{m+m_1-\alpha-1} d\tau \leq C\langle y\rangle^{-m_1}\langle x\rangle^{2m-2\alpha+m_1-1}.$$

For $(f_\varepsilon)_\varepsilon$ there exists $N_1 \in \mathbb{N}$, $C>0$ such that

$$|f_\varepsilon(y)| \leq C\langle y\rangle^{N_1}\varepsilon^{-N_1}.$$

Now, we replace the above computation in integral (2)

$$|D_x^{\alpha} D_\omega^{\beta} V_g f_\varepsilon(x,\omega)| \leq \int |f_\varepsilon(y)| \cdot |D_\omega^{\beta} e^{-2\pi i y\omega}| \cdot |D_x^{\alpha}\overline{g(y-x)}| dy$$
$$\leq \frac{(2\pi)^m \varepsilon^{-N}}{\Gamma(m-\beta)\Gamma(m-\alpha)} \langle\omega\rangle^{m-\beta}\langle x\rangle^{2m-2\alpha+m_1-1} \int \langle y\rangle^{m+N_1-m_1} dy.$$

Choosing the negative sign in the power of $\langle y\rangle$, the convergence of integral is ensured. Take $m_1 = m+N_1+l+1$, so for every $\alpha,\beta>0$, $m-1<\alpha,\beta<m$, there exists $N := 2m-2\alpha+m_1-1$ and $C>0$ such that

$$|D_x^{\alpha} D_\omega^{\beta} V_g f_\varepsilon(x,\omega)| \leq C\varepsilon^{-N}\langle\omega\rangle^N\langle x\rangle^N$$

holds.
Now suppose that f_ε^1 and f_ε^2 are representatives of an element $f \in \mathscr{G}_\tau(\mathbb{R}^d)$. We have to show that $(V_g f_\varepsilon^1)_\varepsilon = (V_g f_\varepsilon^2)_\varepsilon$.

$$(V_g f_\varepsilon^2)_\varepsilon - (V_g f_\varepsilon^1)_\varepsilon = \int (f_\varepsilon^2 - f_\varepsilon^1)(y) \cdot \overline{g(y-x)} e^{-2\pi i y\omega} dy \in \mathscr{N}_\tau(\mathbb{R}^{2d}).$$

There exists $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}_0$

$$|(V_g f)_\varepsilon(x,\omega)| \leq \int |\overline{g(y-x)}||f_\varepsilon(y)| dy \leq C\langle x\rangle^n \cdot \varepsilon^n \int \langle y\rangle^{k-n} dy \leq C\varepsilon^n\langle x\rangle^n.$$

We select $n = k+l+1$ so that it makes the integral convergent. For $N := n$ we may write

$$|(V_g f)_\varepsilon(x,\omega)| \leq C \cdot \langle x\rangle^N\langle\omega\rangle^N\varepsilon^n.$$

Theorem 3.2. Let $(f_\varepsilon)_\varepsilon \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^{2d})$, $g \in \mathscr{S}(\mathbb{R}^{2d})$. Then Short-time Fourier transform

$$(V_g f)_\varepsilon(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\omega}\overline{g(y-x)} f_\varepsilon(y) dy$$

is a well-defined element of $\mathscr{G}_{\mathscr{S}}(\mathbb{R}^{2d})$.

Proof. For the $\mathscr{S}-$moderateness indicate that for every $k_1,k_2 \in \mathbb{N}_0$ and for every $\alpha,\beta>0$, $m-1<\alpha,\beta<m$, there exists $N \in \mathbb{N}$ such that

$$|D_x^{\alpha} D_\omega^{\beta} V_g f_\varepsilon(x,\omega)| \leq C \cdot \varepsilon^{-N}\langle x\rangle^{-k_1}\langle\omega\rangle^{-k_2}$$

holds.
We suppose that $n \geq m$ and by integration by parts the following result is found:

$$|D_x^{\alpha} D_\omega^{\beta} V_g f_\varepsilon(x,\omega)| \leq C\frac{(2\pi)^{\beta}\omega^{m-\beta}}{\Gamma(m-\beta)} \int (1+\Delta_y)^n \left(f_\varepsilon(y)\overline{D^{\alpha} g(y-x)} y^m\right) dy$$

$$\leq C\frac{(2\pi)^{\beta}\omega^{m-\beta-2n}}{\Gamma(m-\beta)}\sum_{\substack{|\theta_1|\leq 2n\\|\theta_2|\leq 2n}}\int |D_y^{\theta_1}f_\varepsilon(y)|\cdot|D_y^{\theta_2}(D^{\alpha}g(y-x)\cdot y^m)|dy$$

$$\leq C\frac{(2\pi)^{\beta}\omega^{m-\beta-2n}}{\Gamma(m-\beta)}\sum_{\substack{|\theta_1|\leq 2n\\|\theta_2|\leq 2n}}\int |D_y^{\theta_1}f_\varepsilon(y)|\cdot|D_y^{\theta_2+\alpha-\theta_3}g(y-x)|\cdot y^{m-\theta_3}dy.$$

We estimate $D_y^{\theta_2+\alpha-\theta_3}g(y-x)$, and use the inequality (1)

$$D_y^{\theta_2-\theta_3+\alpha}g(y-x)=D_y^{\theta_2-\theta_3}(D_y^{\alpha}g(y-x))\leq D_y^{\theta_2-\theta_3}\Big(\frac{\langle x\rangle^{-m_1}\langle y\rangle^{2m+m_1-2\alpha-1}}{\Gamma(m-\alpha)}\Big)$$
$$\leq\frac{\langle x\rangle^{-m_1}\langle y\rangle^{2m+m_1-2\alpha-1-\theta_2+\theta_3}}{\Gamma(m-\alpha)}.$$

Now with replacing $N:=2m+m_1-2\alpha-1-\theta_2+\theta_3$ the following inequality is obtained:

$$|D_x^{\alpha}D_\omega^{\beta}V_gf_\varepsilon(x,\omega)|\leq C\frac{(2\pi)^{\beta}\omega^{m-\beta-2n}\langle x\rangle^{-m_1}}{\Gamma(m-\alpha)\Gamma(m-\beta)}\sum_{\substack{|\theta_1|\leq 2n\\|\theta_2|\leq 2n\\\theta_3\leq\theta_2}}\int |D_y^{\theta_1}f_\varepsilon(y)|\cdot\langle y\rangle^{|m-\theta_3|+N}dy.$$

Using the moderateness of $(f_\varepsilon)_\varepsilon$, leads to the existence of $N\in\mathbb{N}$ and a non-negative constant C such that

$$\langle y\rangle^{l+1+|m-\theta_3|+N}|D_y^{\theta_1}f_\varepsilon(y)|\leq C\cdot\varepsilon^{-N},\quad\forall y\in\mathbb{R}^d,$$

which causes the estimate

$$|D_x^{\alpha}D_\omega^{\beta}V_gf_\varepsilon(x,\omega)|\leq C\langle\omega\rangle^{m-\beta-2n}\langle x\rangle^{-m_1}\varepsilon^{-N}\leq C\langle\omega\rangle^{-n}\langle x\rangle^{-m_1}\varepsilon^{-N},\quad m-1<\alpha,\beta<m,\ \forall m,n\in\mathbb{N}.$$

Now suppose that f_ε^1 and f_ε^2 are representatives of an element $f\in\mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$. We need to show that $(V_gf_\varepsilon^1)_\varepsilon=(V_gf_\varepsilon^2)_\varepsilon$.

$$(V_gf_\varepsilon^2)_\varepsilon-(V_gf_\varepsilon^1)_\varepsilon=\int(f_\varepsilon^2-f_\varepsilon^1)(y)\cdot\overline{g(y-x)}e^{-2\pi iy\omega}dy\in\mathscr{N}_{\mathscr{S}}(\mathbb{R}^{2d}).$$

For any $n,n_1\in\mathbb{N}_0$,

$$|f_\varepsilon(y)|\leq\langle y\rangle^{-n_1}\cdot\varepsilon^n.$$

Since $g\in\mathscr{S}$ there exists some constant C such that $\|g\|_\infty<C$. So

$$|(V_gf)_\varepsilon(x,\omega)|\leq\int|\overline{g(y-x)}||f_\varepsilon(y)|dy\leq C\cdot\varepsilon^n\int\langle y\rangle^{-n_1}dy\leq C\varepsilon^n.$$

For $n_1=l+1$.

3.2. Basic classical relations of the STFT

In this part we prove some properties of the generalized STFT in which we use the Caputo derivative. These relations show the characteristics of the regular generalized moderate nets via STFT with fractional derivatives. First, the Proposition used for next lemma is proved.

Proposition 3.3. If $g\in\mathscr{S}(\mathbb{R}^d)$, then

$$X^{\beta}D^{\alpha}(M_\omega T_xg)\leq\sum_{k\leq\alpha}\sum_{\gamma\leq\beta}C_{T,\alpha}\binom{\alpha}{k}\binom{\beta}{\gamma}(2\pi i\omega)^m x^{\gamma}T_x(X^{\beta-\gamma}D^kg).$$

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Proof. If the below equality holds, as shown in (cf. [7]).

$$(X^{\beta}T_x g)(t) = \Big(\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}x^{\gamma}T_x X^{\beta-\gamma}g\Big)(t). \tag{3}$$

By an explicit computation and the use of usual Caputo derivative defined on the interval $[0,t)$, $t<T$, $T>0$ we obtain:

$$\begin{aligned} X^{\beta}D^{\alpha}(M_{\omega}T_x g) &= X^{\beta}D^{\alpha}(e^{-2\pi it\omega}(T_x g)) = X^{\beta}\sum_{k\leq\alpha}\binom{\alpha}{k}\Big(D^k(T_x g)D^{\alpha-k}(e^{-2\pi it\omega})\Big) \\ &\leq X^{\beta}\Big(\sum_{k\leq\alpha}\binom{\alpha}{k}\frac{1}{\Gamma(m-(\alpha-k))}(2\pi i\omega)^m t^{m-\alpha}D^k(T_x g)\Big) \\ &\leq \Big(\sum_{k\leq\alpha}\binom{\alpha}{k}\frac{1}{\Gamma(m-(\alpha-k))}(2\pi i\omega)^m t^{m-\alpha}X^{\beta}T_x(D^k g)\Big). \end{aligned}$$

The assumption (3) yields:

$$\begin{aligned} X^{\beta}D^{\alpha}(M_{\omega}T_x g) &\leq \Big(\sum_{k\leq\alpha}\binom{\alpha}{k}\frac{1}{\Gamma(m-(\alpha-k))}(2\pi i\omega)^m T^{m-\alpha}\Big(\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}x^{\gamma}T_x X^{\beta-\gamma}(D^k g)\Big) \\ &\leq C_{T,\alpha}\Big(\sum_{k\leq\alpha}\sum_{\gamma\leq\beta}\binom{\alpha}{k}\binom{\beta}{\gamma}\frac{1}{\Gamma(m-(\alpha-k))}(2\pi i\omega)^m x^{\gamma}T_x\Big(X^{\beta-\gamma}(D^k g)\Big). \end{aligned}$$

Lemma 3.4. Let $g\in\mathscr{S}(\mathbb{R}^d)$ and $F_{\varepsilon}:\mathbb{R}^{2d}\to\mathbb{C}$, such that there exists $N\in\mathbb{N}$, $C\geq 0$ and for every $n\in\mathbb{N}_0$

$$|F_{\varepsilon}(x,\omega)|\leq C\cdot\langle x\rangle^{-n}\langle\omega\rangle^{-n}\varepsilon^{-N}.$$

Then

$$(f_{\varepsilon}(y))_{\varepsilon} = \Big(\int\int F_{\varepsilon}(x,\omega)M_{\omega}T_x g(y)dxd\omega\Big)_{\varepsilon}\in\mathscr{E}^{\infty}_{\mathscr{S}}(\mathbb{R}^d).$$

Proof. With proposition (3), we obtain the following inequality:

$$\begin{aligned} X^{\beta}D^{\alpha}f_{\varepsilon}(y) &= \int\int F_{\varepsilon}(x,\omega)X^{\beta}D^{\alpha}(M_{\omega}T_x g(y))dxd\omega \\ &\leq \sum_{k\leq\alpha}\sum_{\gamma\leq\beta}C\binom{\alpha}{k}\binom{\beta}{\gamma}\int\int(2\pi i\omega)^m x^{\gamma}T_x(X^{\beta-\gamma}D^k g)dxd\omega. \end{aligned}$$

Since $g\in\mathscr{S}(\mathbb{R}^d)$ then a non-negative constant C such that

$$\|X^{\beta-\gamma}D^k g\|_{L^{\infty}}\leq C.$$

It leads to,

$$\begin{aligned} \|X^{\beta}D^{\alpha}f_{\varepsilon}\|_{L^{\infty}} &\leq C\cdot\int\int|F_{\varepsilon}(x,\omega)||x^{\gamma}||\omega^m|dxd\omega \\ &\leq C\cdot\int\int\langle x\rangle^{\gamma}\langle\omega\rangle^m|F_{\varepsilon}(x,\omega)|dxd\omega \\ &\leq C\cdot\int\int\langle x\rangle^{\gamma+m}\langle\omega\rangle^{\gamma+m}|F_{\varepsilon}(x,\omega)|dxd\omega \\ &\leq C\cdot\int\int\langle x\rangle^{-n+\gamma+m}\langle\omega\rangle^{-n+\gamma+m}dxd\omega. \end{aligned}$$

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Take $n = |\gamma + m| + l + 1$ so the integrals converge.

In the next theorem, we make use of the inversion of the class of Short-time Fourier transform whose formula is mentioned below.

Proposition 3.5. (Inversion formula for the STFT). Suppose that $g, h \in \mathscr{S}(\mathbb{R}^d)$ and $\langle g, h\rangle \neq 0$. Then for all $f = [(f_\varepsilon)_\varepsilon] \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$,

$$(f_\varepsilon)_\varepsilon = \frac{1}{\langle g, h\rangle}\Big[\Big(\int\int_{\mathbb{R}^{2d}} V_g f_\varepsilon(x,\omega) M_\omega T_x h dx d\omega\Big)_\varepsilon\Big].$$

Theorem 3.5. Let $f \in \mathscr{G}_{\mathscr{S}}(\mathbb{R}^d)$ or $\mathscr{G}_\tau(\mathbb{R}^d)$. Then the following conditions are equivalent:

(1) $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$;

(2) $(V_g f)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^{2d})$;

(3) There exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}_0$ there exists a non-negative constant C such that

$$|(V_g f)_\varepsilon(x,\omega)| \leq C \cdot \langle x\rangle^N \langle \omega\rangle^N \varepsilon^n.$$

Proof. (1) $\Rightarrow$ (2) By integration by parts we obtain:

$$|D^\alpha_x D^\beta_\omega V_g f_\varepsilon(x,\omega)| \leq C\frac{(2\pi)^\beta \omega^{m-\beta-2n}}{\Gamma(m-\beta)} \sum_{\substack{|\theta_1|\leq 2n \\ |\theta_2|\leq 2n \\ \theta_3 \leq \theta_2}} \int |D^{\theta_1}_y f_\varepsilon(y)| \cdot |D^{\theta_2+\alpha-\theta_3}_y g(y-x)| \cdot y^{m-\theta_3} dy.$$

Since $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$, there exists $N \in \mathbb{N}$ such that for all $\theta_1 \in \mathbb{N}^d_0$ and every $n_1 \in \mathbb{N}_0$ there is a non-negative constant C such that

$$|D^{\theta_1} f_\varepsilon(y)| \leq C \cdot \langle y\rangle^{-n_1} \varepsilon^{-N}.$$

For $\theta_2 - \theta_3 + \alpha$ it was in lemma 3

$$D^{\theta_2-\theta_3+\alpha}_y g(y-x) \leq C\langle x\rangle^{-m_1}\langle y\rangle^N.$$

Thus we may write

$$|D^\alpha_x D^\beta_\omega V_g f_\varepsilon(x,\omega)| \leq C\langle\omega\rangle^{-n}\langle x\rangle^{-m_1}\int \langle y\rangle^{|m-\theta_3|+N-n_1} dy.$$

And we select $n_1 = |m-\theta_3| + N + l + 1$ so that it makes the integral convergent.
Then we have the estimate

$$|D^\alpha_x D^\beta_\omega V_g f_\varepsilon(x,\omega)| \leq C\langle\omega\rangle^{-n}\langle x\rangle^{-m_1}\varepsilon^{-N}, \quad m-1 < \alpha, \beta < m, \ \forall m_1, n \in \mathbb{N}.$$

(2) $\Rightarrow$ (3) The estimate (3) results from the $\mathscr{E}^\infty_{\mathscr{S}}$-estimate for $(V_g f)_\varepsilon$ when $\alpha = \beta = 0$.

(3) $\Rightarrow$ (1) According to the Inversion formula we have

$$f_\varepsilon(y) = \frac{1}{\|g\|^2_{L^2}}\int\int_{\mathbb{R}^{2d}} V_g f_\varepsilon(x,\omega) M_\omega T_x g dx d\omega.$$

Set $F_\varepsilon := V_g f_\varepsilon$ in lemma 3 and with condition (3) results that $(f_\varepsilon)_\varepsilon \in \mathscr{E}^\infty_{\mathscr{S}}(\mathbb{R}^d)$.

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