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# Approximation of Fourier Series of a Function of Class $W\left(L^p,\xi(t)\right)$ by Product Means

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**Abstract:** Lipchitz class of function had been introduced by McFadden[7}.Recently dealing with degree of approximation of Fourier series of a function of Lipchitz class Nigam[4] and Misra et al [2],[3] have established certain theorems. Extending their results in this paper, a theorem on degree of approximation of a function $f \in W\left(L^p,\xi(t)\right)$ by product summability $(E,q)\left(\overline{N},p_n\right)$ of Fourier series associated with f , has been established.

*Keywords*: *Degree of Approximation,* $W\left(L^p,\xi(t)\right)$ *class of function,* $(E,q)\left(\overline{N},p_n\right)$ *product mean, Fourier series.*

## 1. Introduction and Preliminaries

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty \quad , \quad n \to \infty \ , (P_{-i} = p_{-i} = 0 \ , i \geq 0). \tag{1}$$

The sequence –to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon \ , \tag{2}$$

defines the sequence $\{t_n\}$ of the $\left(\overline{N},p_n\right)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s \quad ,\text{as} \quad n \to \infty \ , \tag{3}$$

then the series $\sum a_n$ is said to be $\left(\overline{N},p_n\right)$ summable to $s$ .

The conditions for regularity of $\left(\overline{N},p_n\right)$- summability are easily seen to be

$$\begin{cases} (i)\ P_n \to \infty, as\ n \to \infty, \\ (ii)\ \sum_{i=0}^{n} p_i \leq C\left|P_n\right| , as\ n \to \infty. \end{cases} \tag{4}$$

The sequence –to-sequence transformation [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \quad , \tag{5}$$

defines the sequence $\{T_n\}$ of the $(E,q)$ mean of the sequence $\{s_n\}$.
If

$$T_n \to s \text{ , as } \quad n \to \infty \text{ ,} \tag{6}$$

then the series $\sum a_n$ is said to be $(E,q)$ summable to $s$.
Clearly $(E,q)$ method is regular. Further, the $(E,q)$ transform of the $(\overline{N}, p_n)$ transform of $\{s_n\}$ is defined by

$$\begin{aligned} \tau_n &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_\upsilon s_\upsilon \right\} \quad . \end{aligned} \tag{7}$$

If

$$\tau_n \to s \text{ , as } \quad n \to \infty , \tag{8}$$

then $\sum a_n$ is said to be $(E,q)(\overline{N}, p_n)$-summable to $s$.
Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over (-π,π) . Then the Fourier series associated with $f$ at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \tag{9}$$

Let $s_n(f;x)$ be the n-th partial sum of (1.9). The $L_\infty$-norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\} \tag{10}$$

and the $L_\upsilon$-norm is defined by

$$\|f\|_\upsilon = \left( \int_0^{2\pi} |f(x)|^\upsilon \right)^{\frac{1}{\upsilon}} \quad , \upsilon \geq 1. \tag{11}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree $n$ under norm $\| \, . \, \|_\infty$ is defined by [5]

$$\|P_n - f\|_\infty = \sup\{|p_n(x) - f(x)| : x \in R\} \tag{12}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_\upsilon$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon . \tag{13}$$

This method of approximation is called Trigonometric Fourier approximation[6].
A function $f(x) \in Lip\,\alpha$ , if

$$|f(x+t) - f(x)| = O\left(|t|^\alpha\right) \text{ , } 0 < \alpha \leq 1,\ t > 0 \tag{14}$$

and $f(x) \in Lip(\alpha, r)$, for $0 \le x \le 2\pi$ , if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r \, dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right) \; ,0<\alpha \le 1,\; r \ge 1,\; t>0 . \tag{15}$$

For a given positive increasing function $\xi(t)$, the function $f(x) \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^r \, dx\right)^{\frac{1}{r}} = O(\xi(t)),\; r \ge 1,\; t>0. \tag{16}$$

For a given positive increasing function $\xi(t)$ and an integer $p>1$ the function $f(x) \in W\left(L^p, \xi(t)\right)$, if

$$\left(\int_0^{2\pi} |f(x+t)-f(x)|^p \, (\sin x)^{p\beta} \, dx\right)^{\frac{1}{p}} = O(\xi(t)),\; \beta \ge 0. \tag{17}$$

We use the following notation throughout this paper:

$$\phi(t) = f(x+t)+f(x-t)-2f(x), \tag{18}$$

and

$$K_n(t) = \frac{1}{2\pi(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right\}. \tag{19}$$

Further, the method $(E,q)(\overline{N}, p_n)$ is assumed to be regular throughout the paper.

Dealing with The degree of approximation by the product $(E,q)(C,1)$-mean of Fourier series, Nigam et al [4] proved the following theorem.

**Theorem 1.** If a function $f$ ,$2\pi$ - periodic, belonging to class $Lip\,\alpha$ , then its degree of approximation by $(E,q)(C,1)$ summability mean on its Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by $\left\| E_n^q C_n^1 - f \right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right)$ ,$0<\alpha<1$ ,where $E_n^q C_n^1$ represents t*he* $(E,q)$ transform of $(C,1)$ transform of $s_n(f;x)$.

Misr*a* et al [2] proved the following theorem using $(E,q)(\overline{N}, p_n)$ mean of Fourier series.

**Theorem 2.** If $f$ is a $2\pi-$ periodic function of class $Lip\,\alpha$ , then degree of approximation by the product $(E,q)(\overline{N}, p_n)$ summability means of its Fourier series (9) of $f(x)$ is given by

$$\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right) ,0<\alpha<1 ,$$ where $\tau_n$ is as defined in (7) .

Recently, Misra et al [3] proved the following theorem using $(E,q)(\overline{N},p_n)$ mean of the Fourier series using a $2\pi-$ periodic function of class $Lip(\xi(t),r)$.

**Theorem 3.** Let $\xi(t)$ be a positive increasing function. If $f$ is a $2\pi-$ Periodic function of the class $Lip(\xi(t),r)$, $r\ge 1, t>0$, then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means of the Fourier series (9) of $f(x)$ is given by

$$\|\tau_n - f\|_\infty = O\left((n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right) ,r\ge 1.,$$

where $\tau_n$ is as defined in (7).

**2 Main Theorem**

In this paper, we have proved a theorem on degree of approximation by the product mean $(E,q)(\overline{N},p_n)$ of the Fourier series of a function of class $W(L^p,\xi(t))$ . We prove.

**Theorem 4.** Let $\xi(t)$ be a positive increasing function and $f$ a $2\pi-$ Periodic function of the class $W(L^p,\xi(t))$, $p>1, t>0$ . Then degree of approximation by the product $(E,q)(\overline{N},p_n)$ summability means of the Fourier series (9) of $f(x)$ is given by

$$\|\tau_n - f\|_r = O\left((n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right) ,r\ge 1 , \tag{20}$$

provided

$$\left(\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|\sin^\beta t}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{21}$$

and

$$\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^r dt\right)^{\frac{1}{r}} = O\left((n+1)^\delta\right) \tag{22}$$

hold uniformly in $x$ with $\frac{1}{r}+\frac{1}{s}=1$, where $\delta$ is an arbitrary number such that $s(1-\delta)-1>0$ and $\tau_n$ is as defined in (7).

## 3 Required Lemma

We require the following Lemma for the proof the theorem.

**Lemma 1.**

$$|K_n(t)| = \begin{cases} O(n) & ,0 \le t \le \dfrac{1}{n+1} \\ O\left(\dfrac{1}{t}\right), & \dfrac{1}{n+1} \le t \le \pi \end{cases} .$$

**Proof:** For $0 \le t \le \frac{1}{n+1}$, we have $\sin nt \le n \sin t$, then

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{(2\upsilon+1)\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}(2k+1)\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon\right\}\right|$$

$$\le \frac{(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|$$

$$= O(n)$$

For $\frac{1}{n+1} \le t \le \pi$, by Jordan's lemma we have, $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$, $\sin nt \le 1$.

Then

$$|K_n(t)| = \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi\left(1+q\right)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi\ p_\upsilon}{t}\right\}\right|$$

$$= \frac{1}{2\left(1+q\right)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right\}\right|.$$

$$= \frac{1}{2\left(1+q\right)^n t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

**4 Proof of Theorem 4**

Using Riemann –Lebesgue theorem, we have for the n-th partial sum $s_n\left(f;x\right)$ of the Fourier series (9) of $f(x)$,

$$s_n\left(f;x\right)-f(x)=\frac{1}{2\pi}\int_0^\pi \phi(t)\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt.$$

Following Titchmarch [5], the $\left(\overline{N},p_n\right)$ transform of $s_n\left(f;x\right)$ is given by

$$t_n-f(x)=\frac{1}{2\pi P_n}\int_0^\pi \phi(t)\sum_{k=0}^{n}p_k\,\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\,dt,$$

Writing the $\left(E,q\right)\left(\overline{N},p_n\right)$ transform of $s_n\left(f;x\right)$ by $\tau_n$, we have

$$\tau_n-f=\frac{1}{2\pi\left(1+q\right)^n}\int_0^\pi \varphi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\,\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt$$

$$=\int_0^\pi \phi(t)\ K_n(t)\,dt$$

$$=\left\{\int_{0}^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right\}\phi(t)\ K_n(t)\ dt$$

$$= \ I_1+I_2 \text{, say} \tag{23}$$

Now

$$|I_1|=\frac{1}{2\pi(1+q)^n}\left|\int_{0}^{1/n+1}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\sin\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}dt\right|$$

$$=\left|\int_{0}^{\frac{1}{n+1}}\phi(t)K_n(t)\,dt\right|$$

$$\leq\left(\int_{0}^{\frac{1}{n+1}}\left|\frac{t\,\phi(t)\sin^{\beta}t}{\xi(t)}\right|^{r}dt\right)^{\frac{1}{r}}\left(\int_{0}^{\frac{1}{n+1}}\left|\frac{\xi(t)K_n(t)}{t\sin^{\beta}t}\right|^{s}dt\right)^{\frac{1}{s}},$$

where $\frac{1}{r}+\frac{1}{s}=1$, using Hölder's inequality

$$=O(1)\left(\int_{0}^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^{s}dt\right)^{\frac{1}{s}},$$

using lemma-1 and (21),

$$=O(\xi\left(\frac{1}{n+1}\right))\left(\int_{\varepsilon}^{\frac{1}{n+1}}\frac{dt}{t^{(1+\beta)s}}\right)^{\frac{1}{s}},\ \text{for some}\ 0\leq\varepsilon\leq\frac{1}{n+1}$$

$$=O\left(\xi\left(\frac{1}{n+1}\right)\right)O\left((n+1)^{-\frac{1}{s}+1+\beta}\right)$$

$$= O\left( \xi\left( \frac{1}{n+1} \right) (n+1)^{\beta+\frac{1}{r}} \right) \tag{24}$$

Next,

$$|I_2| \le \left( \int_{\frac{1}{n+1}}^{\pi} \left( \left| \frac{t^{-\delta} |\phi(t)| \sin^{\beta} t}{\xi(t)} \right| \right)^r dt \right)^{\frac{1}{r}} \left( \int_{\frac{1}{n+1}}^{\pi} \left( \left| \frac{\xi(t) |K_n(t)|}{t^{-\delta} \sin^{\beta} t} \right| \right)^s dt \right)^{\frac{1}{s}} ,$$

where $\frac{1}{r} + \frac{1}{s} = 1$, using Hölder's inequality

$$= O((n+1)^{\delta}) \left( \int_{\frac{1}{n+1}}^{\pi} \left( \frac{\xi(t)}{t^{\beta+1-\delta}} \right)^s dt \right)^{\frac{1}{s}} ,$$

using Lemma 1 and (22)

$$= O((n+1)^{\delta}) \left( \int_{\frac{1}{\pi}}^{n+1} \left( \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-\beta-1}} \right)^s \frac{dy}{y^2} \right)^{\frac{1}{s}} ,$$

since $\xi(t)$ is a positive increasing function, so is $\xi(1/y)/(1/y)$. Using second mean value theorem we get

$$= O((n+1)^{1+\delta} \xi\left( \frac{1}{n+1} \right)) \left( \int_{\varepsilon}^{n+1} \frac{dy}{y^{s(\delta-\beta-1)+2}} \right)^{\frac{1}{s}} , \text{ for some } \frac{1}{\pi} \le \varepsilon \le n+1$$

$$= O\left( (n+1)^{1+\delta} \xi\left( \frac{1}{n+1} \right) \right) O\left( (n+1)^{\beta+1-\delta-\frac{1}{s}} \right)$$

$$= O\left( (n+1)^{\beta+\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right) \tag{25}$$

Then from (24) and (25) , we have

$$|\tau_n - f(x)| = O\left( (n+1)^{\beta+\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right) , \text{ for } r \ge 1 .$$

$$\|\tau_n - f(x)\|_r = \left( \int_0^{2\pi} O\left( (n+1)^{\beta+\frac{1}{r}} \xi\left( \frac{1}{n+1} \right) \right)^r dx \right)^{\frac{1}{r}} , r \ge 1.$$

$$= O\left( (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right)\left(\int_0^{2\pi} dx\right)^{\frac{1}{r}}$$

$$= O\left( (n+1)^{\beta+\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right).$$

This completes the proof of the theorem.

**5 Corollaries**

Following corollaries can be derived from the main theorem.

**Corollary 1.** The degree of approximation of a function $f$ belonging to the class $Lip(\alpha, r), 0 < \alpha \leq 1, r \geq 1$ is given by

$$\|\tau_n - f\|_r = O\left( (n+1)^{-\alpha+\frac{1}{r}} \right) .$$

**Proof:** The corollary follows by putting $\beta = 0$ and $\xi(t) = t^\alpha$ in the main theorem.

**Corollary 2.** The degree of approximation of a function $f$ belonging to the class $Lip(\alpha), 0 < \alpha \leq 1$ is given by

$$\|\tau_n - f\|_\infty = O\left( (n+1)^{-\alpha} \right) .$$

**Proof:** The corollary follows by letting $r \to \infty$ in corollary 6.1.

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