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Nonhomogeneous generalized multi-term fractional heat propagation and fractional diffusion-convection equation in three-dimensional space

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Abstract: The main purpose of this article is to study non-homogeneous generalized multi-term fractional heat propagation and fractional diffusion-convection equation in three-dimensional space, where the fractional derivative is defined in the Caputo sense. The convection-diffusion equation describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection.

Keywords: Fractional partial differential equations, Fractional heat propagation, Fractional Diffusion-Convection Equation, Laplace transform; Fourier transform, Fox-Wright functions, Kelvin functions.

1. Prelude to Fractional PDEs

The partial differential equations of fractional order have been successfully used for modeling some relevant physical processes; therefore, a large body of research in the solutions of these equations has been published in the literature. Debnath [15] has discussed the solutions of the various types of partial differential equations occurring in the fluid mechanics. Nikolova and Boyadjiev [16] found solution of the time-space fractional diffusion equations by means of the fractional generalization of the Fourier transform and the classical Laplace transform. Solutions of fractional reaction-diffusion equations are investigated in a number of recent papers by Saxena *et al* [17,18]. Also, in [10,11] the authors employed integral transforms to solve certain – non homogenous heat and wave equations.

Many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine/sine transform.

The object of this paper is to present solutions of generalized multi-term fractional heat propagation and fractional diffusion-convection equation in three-dimensional space involving the Caputo time-fractional derivative and by employing the joint Laplace and Fourier transforms. In order to obtain the solutions, the definitions and notations of the well-known Laplace transform, Fourier transform, their inverses and fractional derivatives of a function $u(x,t)$ are described below.

The Laplace transform of a function $u(x,t)$ (which is supposed to be continuous or sectionally continuous, and of exponential order as $t \to +\infty$) with respect to the variable t is defined by

$$L\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt \coloneqq U(x,s),$$

where $(s) > 0$, and the inverse Laplace transform of $U(x,s)$ with respect to s is given by

$$L^{-1}\{U(x,s)\} = u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ts}U(x,s)ds,$$

where γ is a fixed real quantity.

The Fourier transform of a function $u(x,t)$ with respect to x is defined as

$$F\{u(x,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega x}u(x,t)dx \coloneqq \bar{u}(\omega,t)\ (w \in R).$$

The inverse Fourier transform of a function $\bar{u}(\omega,t)$ with respect to ω is given by

$$F^{-1}\{\bar{u}(\omega,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ix\omega}\bar{u}(\omega,t)d\omega.$$

The finite sine transform of a function $u(x,t)$ with respect to $x \in (0,L)$ (L is finite) is defined by

$$F_s\{u(x,t); \mathrm{x} \to \mathrm{n}\} = \int_0^L u(x,t)\sin\frac{\mathrm{n\pi x}}{\mathrm{L}}\mathrm{dx} \coloneqq \bar{u}(n,t)\ (n \in N \cup \{0\}).$$

The inverse finite sine transform is given by

$$F_s^{-1}\{\bar{u}(n,t)\} = u(x,t) = \frac{2}{\mathrm{L}}\sum_{\mathrm{n}=0}^{\infty} u(n,t)\sin\frac{\mathrm{n\pi x}}{\mathrm{L}}.$$

The Caputo fractional derivative of arbitrary order α is defined as

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\xi)^{n-\alpha-1}u^{(n)}(x,\xi)d\xi\ \ (t > 0)$$

where $n-1 < \alpha < n$ $(n \in N)$ and $u^{(n)}(x,t)$ is the partial derivative of order n of the function $u(x,t)$ with respect to the variable t.

The Laplace transform of Caputo's fractional derivative is given by [3]

$$L\{{}_0^C D_t^\alpha u(x,t)\} = S^\alpha U(x,s) - \sum_{\mathrm{r}=0}^{\mathrm{n}-1} S^{\alpha-r-1}u^{(r)}(x,0)\ \ (\mathrm{n}-1 < \alpha \le n).$$

The above formula play an important role in deriving the solution of differential and integral equations of fractional order governing certain physical problems of reaction and diffusion. One may refer to the monographs by Podlubny [3], Samko *et al* [4], Mathai *et al* [5] and Kilbas *et al* [1].

The simplest Wright function is defined by the series

$$W(\alpha,\beta;z) \coloneqq \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}\ (\alpha,\beta,z \in C).$$

The Fox-Wright function ${}_p\Psi_q(z)$ is defined for $z \in C$, complex numbers $a_l, b_j \in C$ and real $\alpha_l, \beta_j \in R$ $(l = 1,\dots,p\,;\ j = 1,\dots,q)$ by the series

$$ {}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_l,\alpha_l)_{1,p} \\ (b_l,\beta_l)_{1,q}\end{matrix}|z\right] := \sum_{k=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}. $$

The Laplace transform is used in a variety of applications. The most common usage of the Laplace transform is in the evaluation of certain integrals and solution to boundary value problems. In this paper we will briefly discuss applications of Laplace transform in all of the above named areas.

In the following lemma, certain integrals involving Kelvin function are evaluated by means of Laplace transform.

Lemma 1.1. The following relations hold true

1. $\int_0^\infty \frac{x bei(2\sqrt{x})}{\lambda^2 x^2+\xi^2}dx = \frac{\pi}{2\lambda^2}J_0\left(2\sqrt{\frac{\xi}{\lambda}}\right)$
2. $\int_0^\infty \frac{x bei(2\sqrt{x})}{x^2+1}dx = \frac{\pi}{2}J_0(2)$
3. $\int_0^\infty \frac{bei(2\sqrt{x})}{x}dx = \frac{\pi}{2}$

Remark. The Kelvin functions $ber(x), bei(x)$ are related to the Bessel functions in the following way:

$$ \mathrm{Re}J_0(i\sqrt{i}x) = ber(x), \qquad \mathrm{Im}J_0(i\sqrt{i}x) = bei(x). $$

Laplace transform of Kelvin functions are as following

$$ L\{J_0(2\sqrt{at}) = \frac{1}{s}\exp\left(-\frac{a}{s}\right) \Rightarrow $$

$$ L\{ber2\sqrt{at}\} = \frac{1}{s}\cos\left(\frac{a}{s}\right), \qquad L\{bei2\sqrt{at}\} = \frac{1}{s}\sin\left(\frac{a}{s}\right). $$

The Kelvin functions are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics.

Proof.

1. Let us define the following function

$$ I(\phi) = \int_0^\infty \frac{x bei(2\sqrt{\phi x})}{\lambda^2 x^2+\xi^2}dx $$

Taking Laplace transform of the above function, yields

$$ L\{I(\phi), \phi \to p\} = p^{-1}\int_0^\infty \frac{x sin(p^{-1}x)}{\lambda^2 x^2+\xi^2}dx $$

The above integral can be evaluated by means of residue theorem, that is

$$ L\{I(\phi)\} = \frac{1}{p}\left(\frac{\pi}{2\lambda^2}\right)\exp\left(-\frac{1}{p}\sqrt{\frac{\xi}{\lambda}}\right) $$

At this point, on taking inverse Laplace transform of the above relation gives

$$ I(\phi) = \frac{\pi}{2\lambda^2}J_0\left(2\sqrt{\frac{\xi\phi}{\lambda}}\right) $$

In special case $\phi = 1$, one gets the result.

2. By setting $\lambda = \xi = 1$, we get $\int_0^\infty \frac{x bei(2\sqrt{x})}{x^2+1} dx = \frac{\pi}{2} J_0(2)$.
3. By setting $\lambda = 1, \xi = 0$, we obtain $\int_0^\infty \frac{bei(2\sqrt{x})}{x} dx = \frac{\pi}{2}$.

Lemma 1.2. The following identity holds true [19]

$$L_2\{f(x,y)\} = 2\int_0^\infty K_0\left(2\sqrt{pqt}\right) f(t) dt,$$

where K_0 is modified Bessel function of zero order.

Proof. Assume that $t = xy$, then

$$L_2\{f(xy)\} = \int_0^\infty \int_0^\infty e^{-px-qy} f(xy) dy\, dx = \int_0^\infty \frac{e^{-px}}{x}\left(\int_0^\infty e^{-\frac{qt}{x}} f(t) dt\right) dx = \int_0^\infty f(t)\left(\int_0^\infty \frac{e^{-px-\frac{qt}{x}}}{x} dx\right) dt$$
$$= 2\int_0^\infty K_0\left(2\sqrt{pqt}\right) f(t) dt.$$

Lemma 1.3. The following integral relations hold true.

1. $\int_0^\infty \frac{sinh\sqrt{t}}{\sqrt{t}} K_0\left(\sqrt{2t}\right) dt = \frac{\pi}{2}$.
2. $\int_0^\infty \frac{sin\sqrt{t}}{\sqrt{t}} K_0\left(\sqrt{t}\right) dt = \pi \ln\left(1+\sqrt{2}\right)$.

Proof.

1. By two dimensional Laplace transform table, one has

$$L_2\left\{\frac{sinh\sqrt{x\,y}}{\sqrt{x\,y}}\right\} = \frac{2}{\sqrt{4pq-1}}\left(\pi - 2Arc\tan\sqrt{4pq-1}\right) = \frac{4}{\sqrt{4pq-1}} Arc\sin\left(\frac{1}{2\sqrt{pq}}\right)$$
$$= 2\int_0^\infty \frac{sinh\sqrt{t}}{\sqrt{t}} K_0\left(2\sqrt{pqt}\right) dt.$$

In special case $p = \frac{1}{2}$ and $q = 1$ we get

$$\int_0^\infty \frac{sinh\sqrt{t}}{\sqrt{t}} K_0\left(\sqrt{2t}\right) dt = \frac{\pi}{2}.$$

2. From table we get

$$L_2\left\{\frac{sin\sqrt{x\,y}}{\sqrt{x\,y}}\right\} = \pi\left(\ln(pq) - 2\ln\left(\frac{1+\sqrt{4pq+1}}{2}\right)\right) = 2\pi \sinh^{-1}\left(\frac{1}{2\sqrt{pq}}\right) = 2\int_0^\infty \frac{sin\sqrt{t}}{t} K_0\left(2\sqrt{pqt}\right) dt.$$

If we set $p = q = \frac{1}{2}$, then

$$\int_0^\infty \frac{sin\sqrt{t}}{t} K_0\left(\sqrt{t}\right) dt = \pi \ln\left(1+\sqrt{2}\right).$$

2. Nonhomogeneous Generalized Multi-Term Fractional Heat Propagation in a Rectangle

In this section, we consider propagation of heat in a rectangular shape plate, where we used Caputo partial fractional derivatives in time of order $0 < \alpha < 1$.

Problem 2.1. We consider non-homogeneous generalized multi-term fractional heat propagation

$$ {}_0^C D_t^\alpha u(x,y,t) = a^2\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) + f(x,y,t) \tag{2.1} $$
$$ 0 < \alpha \le 1, 0 < x < b_1, 0 < x < b_2, t > 0 $$

with initial condition $u(x,y,0) = 0$ and boundary conditions

$$ u(0,y,t) = g_1(y,t), \qquad u(b_1,y,t) = g_2(y,t) \ (2.2) $$

$$ u(x,0,t) = h_1(x,t), \qquad u(x,b_2,t) = h_2(x,t). \ (2.3) $$

Solution. By using the Laplace transform with respect to t and finite sine transform with respect to x, we set

$$ L\{u(x,y,t); t \to s\} = U(x,y,s), $$

$$ F_s\{u(x,y,t); x \to n\} = \bar{u}(n,y,t). $$

By applying the joint Laplace - Fourier finite sine transforms to (2.1) and using the initial and boundary conditions (2.2), we obtain

$$ \overline{U_{yy}}(n,y,s) - \left(\frac{n^2\pi^2}{{b_1}^2} + \frac{s^\alpha}{a^2}\right)\bar{U}(n,y,s) = -\frac{n\pi}{b_1}\left(G_1(y,s) - (-1)^n G_2(y,s)\right) + \frac{1}{a^2}\bar{F}(n,y,s). $$

For the sake of simplicity, assume that

$$ \bar{K}(n,y,s) = -\frac{n\pi}{b_1}\left(G_1(y,s) - (-1)^n G_2(y,s)\right) + \frac{1}{a^2}\bar{F}(n,y,s), $$

thus

$$ \overline{U_{yy}}(n,y,s) - \left(\frac{n^2\pi^2}{{b_1}^2} + \frac{s^\alpha}{a^2}\right)\bar{U}(n,y,s) = \bar{K}(n,y,s). $$

Using the boundary conditions (2.3), the solution of the above equation is as

$$ \bar{U}(n,y,s) = \overline{H_1}(n,s)\frac{\sinh(b_2 - y)\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}} + \overline{H_2}(n,s)\frac{\sinh y\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}} $$
$$ - \int_0^y \frac{\bar{K}(n,w,s)}{\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}\frac{\sinh w\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}\sinh(b_2 - y)\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}\,dw $$
$$ - \int_y^{b_2} \frac{\bar{K}(n,w,s)}{\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}\frac{\sinh(b_2 - w)\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}}\sinh y\sqrt{\frac{s^\alpha}{a^2} + \frac{n^2\pi^2}{{b_1}^2}}\,dw. $$

Applying the inverse Laplace transform, one gets

$$ \bar{u}(n,y,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{U}(n,y,s)e^{ts}ds. $$

We assume that $\alpha = 1$. For evaluation of the inverse Laplace transform of

$$L^{-1}\left\{\frac{\sinh(b_2-y)\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}=?$$

Since, $\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}$ has simple zeroes in

$$b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}=m\pi i\ (m\in Z\{0\})\ or\ s_m=-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2.$$

Thus

$$\lim_{s\to s_m}\left(\frac{s-s_m}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}e^{ts}\sinh(b_2-y)\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\right)=\frac{2a^2\pi}{{b_2}^2}(-1)^{m+1}me^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t}\sin\left(\frac{b_2-y}{b_2}\right)m\pi.$$

Thus

$$L^{-1}\left\{\frac{\sinh(b_2-y)\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}=\frac{2a^2\pi}{{b_2}^2}(-1)^{m+1}me^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t}\sin\left(\frac{b_2-y}{b_2}\right)m\pi.$$

For the general case $0<\alpha<1$,

$$\begin{aligned}L^{-1}&\left\{\frac{\sinh(b_2-y)\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}\\&=\frac{4a^2\pi}{{b_2}^2}\sum_{m=1}^{\infty}(-1)^{m+1}m\sin\left(\frac{b_2-y}{b_2}\right)m\pi\times\int_0^t e^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2\tau}\frac{1}{t}W(-\alpha,0;-\tau t^{-\alpha})d\tau\\&=\frac{4}{{b_2}^2\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}m}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{b_2-y}{b_2}\right)m\pi\left(\frac{1}{t}\,{}_1\Psi_1\left[\begin{matrix}(1,1)\\(1,-\alpha)\end{matrix}\middle|-\frac{1}{\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t^\alpha}\right]\right)\\&\coloneqq\overline{p_1}(n,y,t)\end{aligned}$$

By the same procedure, for $0<\alpha<1$, we have

$$\begin{aligned}L^{-1}\left\{\frac{\sinh(y)\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}&=\frac{4a^2}{{b_2}^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}m}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{y}{b_2}\right)m\pi\left(\frac{1}{t}\,{}_1\Psi_1\left[\begin{matrix}(1,1)\\(1,-\alpha)\end{matrix}\middle|-\frac{1}{\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t^\alpha}\right]\right)\\&\coloneqq\overline{p_2}(n,y,t)\end{aligned}$$

$$L^{-1}\left\{\frac{\sinh w\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\sinh(b_2-y)\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\sinh b_2\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}$$

$$=\frac{4a^2}{b_2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{w}{b_2}\right)m\pi\sin\left(\frac{b_2-y}{b_2}\right)m\pi$$

$$\times\left(\frac{1}{t}\,{}_1\Psi_1\left[\begin{matrix}(1,1)\\(1,-\alpha)\end{matrix}\middle|-\frac{1}{\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t^\alpha}\right]\right):=\overline{p_3}(n,w,y,t),$$

$$L^{-1}\left\{\frac{\sinh(b_2-w)\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\sinh y\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\sinh b_2\sqrt{\frac{s^\alpha}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\right\}$$

$$=\frac{4a^2}{b_2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{b_2-w}{b_2}\right)m\pi\sin\left(\frac{y}{b_2}\right)m\pi$$

$$\times\left(\frac{1}{t}\,{}_1\Psi_1\left[\begin{matrix}(1,1)\\(1,-\alpha)\end{matrix}\middle|-\frac{1}{\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2t^\alpha}\right]\right):=\overline{p_4}(n,w,y,t).$$

Now we get

$$\bar{u}(n,y,t)=\int_0^t\overline{h_1}(n,t-z)\overline{p_1}(n,y,z)dz+\int_0^t\overline{h_2}(n,t-z)\overline{p_2}(n,y,z)dz-\int_0^y\int_0^t\bar{k}(n,w,t-z)\overline{p_3}(n,w,y,z)dz\,dw$$
$$-\int_y^{b_2}\int_0^t\bar{k}(n,w,t-z)\overline{p_4}(n,w,y,z)dz\,dw.$$

Lastly, by using the finite Fourier sine inversion formula, we get the exact solution as follows

$$u(x,y,t)=\frac{2}{b_1}\sum_{n=1}^{\infty}\left(\int_0^t\overline{h_1}(n,t-z)\overline{p_1}(n,y,z)dz+\int_0^t\overline{h_2}(n,t-z)\overline{p_2}(n,y,z)dz\right.$$
$$\left.-\int_0^y\int_0^t\bar{k}(n,w,t-z)\overline{p_3}(n,w,y,z)dz\,dw-\int_y^{b_2}\int_0^t\bar{k}(n,w,t-z)\overline{p_4}(n,w,y,z)dz\,dw\right)\sin\frac{n\pi x}{b_1}.$$

When $\alpha=1$, one has

$$\bar{U}(n,y,s)=\overline{H_1}(n,s)\frac{\sinh(b_2-y)\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}+\overline{H_2}(n,s)\frac{\sinh y\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}$$

$$-\int_0^y\frac{\bar{K}(n,w,s)}{\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\frac{\sinh w\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\sinh(b_2-y)\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\,dw$$

$$-\int_y^{b_2}\frac{\bar{K}(n,w,s)}{\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\frac{\sinh(b_2-w)\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}{\sinh b_2\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}}\sinh y\sqrt{\frac{s}{a^2}+\frac{n^2\pi^2}{{b_1}^2}}\,dw.$$

By applying the inverse Laplace transform, we can find

$$\bar{u}(n,y,t)=\frac{2a^2\pi}{{b_2}^2}(-1)^{m+1}m\sin\left(\frac{b_2-y}{b_2}\right)m\pi\int_0^t\overline{h_1}(n,t-z)e^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2z}dz$$

$$+\frac{4a^2}{{b_2}^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}m}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{y}{b_2}\right)m\pi\int_0^t\overline{h_2}(n,t-z)e^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2z}dz$$

$$-\frac{4a^2}{b_2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{b_2-y}{b_2}\right)m\pi\times\int_0^y\int_0^t\bar{k}(n,w,t-z)\sin\left(\frac{w}{b_2}\right)m\pi\,e^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2z}dz$$

$$-\frac{4a^2}{b_2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}}\sin\left(\frac{y}{b_2}\right)m\pi$$

$$\times\int_y^{b_2}\int_0^t\bar{k}(n,w,t-z)\sin\left(\frac{b_2-w}{b_2}\right)m\pi\,e^{-\left(\frac{m^2}{{b_2}^2}+\frac{n^2}{{b_1}^2}\right)\pi^2a^2z}dz\,dw.$$

Now, by using the finite Fourier sine inversion formula, we obtain the exact solution for $\alpha=1$ as

$$u(x,y,t)=\frac{2}{b_1}\sum_{n=1}^{\infty}\bar{u}(n,y,t)\sin\frac{n\pi x}{b_1}.$$

3. Non-homogenous Fractional Diffusion-Convection Equation

Problem 3.1. We consider fractional diffusion-convection equation

$$\begin{gathered}{}_0^CD_t^{\alpha}u=a^2\Delta^2u+2\beta_1u_x+2\beta_2u_y-ku+f(x,y,t)\\ 0,<\alpha\le 1,-\infty<x,y<\infty\end{gathered}\tag{3.1}$$

with initial condition $u(x,y,0)=g(x,y)$ and boundary conditions

$$\lim_{|x|\to\infty}u=\lim_{|y|\to\infty}u=0.$$

Solution. We use the joint Laplace-Fourier transform and assume that

$$L\{u(x,y,t);t\to s\}=U(x,y,s),$$

$$F_2\{u(x,y,t);x\to\omega_1,y\to\omega_2\}=u(\omega_1,\omega_2,t).$$

Taking the joint Laplace-Fourier transform of equation (3.1), we find

$$S^{\alpha}\overline{\overline{U}}(\omega_1,\omega_2,s) - S^{\alpha-1}\overline{\overline{g}}(\omega_1,\omega_2) = -a^2(\omega_1^2+\omega_2^2)\overline{\overline{U}}(\omega_1,\omega_2,s) - 2i(\beta_1\omega_1+\beta_2\omega_2)\overline{\overline{U}}(\omega_1,\omega_2,s) - k\overline{\overline{U}}(\omega_1,\omega_2,s) + \overline{\overline{F}}(\omega_1,\omega_2,s),$$

or

$$\overline{\overline{U}}(\omega_1,\omega_2,s) = \frac{1}{a^2}\frac{S^{\alpha-1}\overline{\overline{g}}(\omega_1,\omega_2)+\overline{\overline{F}}(\omega_1,\omega_2,s)}{\left(\omega_1+\frac{\beta_1 i}{a}\right)^2+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\frac{S^\alpha}{a^2}+\frac{\beta_1^2+\beta_2^2+k}{a^2}}.$$

Let $\gamma = \frac{\beta_1^2+\beta_2^2+k}{a^2}$, then applying the Fourier inversion formula with respect to ω_1 and convolution theorem in Fourier integrals gives

$$\overline{\overline{U}}(x,\omega_2,s) = \frac{S^{\alpha-1}}{2a^2}\int_{-\infty}^{\infty}\overline{g}(x-z,\omega_2)e^{\frac{\beta_1 z}{a}}\frac{\exp\left(-|z|\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}\right)}{\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}}dz$$
$$+\frac{1}{2a^2}\int_{-\infty}^{\infty}\overline{F}(x-z,\omega_2,s)e^{\frac{\beta_1 z}{a}}\frac{\exp\left(-|z|\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}\right)}{\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}}dz.$$

For $0<\alpha<1$ we use the integral representation

$$\frac{\exp\left(-|z|\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}\right)}{\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)}e^{-\frac{|z|^2}{4a^2\eta^2}}e^{-\eta^2 S^\alpha}d\eta.$$

Then

$$L^{-1}\left\{\frac{\exp\left(-|z|\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}\right)}{\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}}\right\} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)}e^{-\frac{|z|^2}{4a^2\eta^2}}\frac{1}{t}W(-\alpha,0;-\eta^2 t^{-\alpha})d\eta,$$

Similarly, for $0<\alpha<1$

$$L^{-1}\left\{\frac{\exp\left(-|z|\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}\right)}{S^{1-\alpha}\sqrt{\frac{S^\alpha}{a^2}+\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma}}\right\}$$
$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\eta^2\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)}e^{-\frac{|z|^2}{4a^2\eta^2}}\frac{1}{t^\alpha}W(-\alpha,1-\alpha;-\eta^2 t^{-\alpha})d\eta.$$

So for $0 < \alpha < 1$,

$$\bar{u}(x,\omega_2,t) = \frac{1}{2a^2}\int_{-\infty}^{\infty} \bar{g}(x-z,\omega_2)e^{\frac{\beta_1 z}{a}} \times\left(\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\eta^2\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)}e^{-\frac{|z|^2}{4a^2\eta^2}}\frac{1}{t^{\alpha}}W(-\alpha,1-\alpha;-\eta^2 t^{-\alpha})d\eta\right)dz$$
$$+\frac{1}{2a^2}\int_{-\infty}^{\infty} e^{\frac{\beta_1 z}{a}}\int_0^t \bar{f}(x-z,\omega_2,t-u) \times\left(\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\eta^2\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)}e^{-\frac{|z|^2}{4a^2\eta^2}}\frac{1}{u}W(-\alpha,0;-\eta^2 u^{-\alpha})d\eta\right)du\,dz.$$

Now, the Fourier inversion formula with respect to ω_2 and convolution theorem in Fourier integrals yields analytical solution

$$u(x,y,t) = \frac{1}{2\pi a^3 t^{\alpha}}\int_{-\infty}^{\infty} e^{\frac{\beta_1 z}{a}}\left(\int_0^{\infty}\frac{1}{\eta}e^{-a^2\gamma\eta^2-\frac{|z|^2}{4a^2\eta^2}}W(-\alpha,1-\alpha;-\eta^2 t^{-\alpha}) \times\left\{\int_{-\infty}^{\infty} g(x-z,y-r)e^{\frac{\beta_2 r}{a}}e^{-\frac{r^2}{4a^2\eta^2}}dr\right\}d\eta\right)dz$$
$$+\frac{1}{2\pi a^3}\int_{-\infty}^{\infty} e^{\frac{\beta_1 z}{a}}\int_0^t\left(\int_0^{\infty}\frac{1}{\eta}e^{-a^2\gamma\eta^2-\frac{|z|^2}{4a^2\eta^2}}\frac{1}{u}W(-\alpha,0;-\eta^2 u^{-\alpha}) \times\left\{\int_{-\infty}^{\infty} f(x-z,y-r,t-u)e^{\frac{\beta_2 r}{a}}e^{-\frac{r^2}{4a^2\eta^2}}dr\right\}d\eta\right)du\,dz,$$

where $0 < \alpha < 1$.

If $\alpha = 1$, then

$$\bar{U}(x,\omega_2,t) = \frac{1}{2a}\int_{-\infty}^{\infty}\bar{g}(x-z,\omega_2)e^{\frac{\beta_1 z}{a}}\frac{\exp\left(-\frac{|z|}{a}\sqrt{s+a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2}\right)}{\sqrt{s+a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2}}dz$$
$$+\frac{1}{2a}\int_{-\infty}^{\infty}\bar{F}(x-z,\omega_2,s)e^{\frac{\beta_1 z}{a}}\frac{\exp\left(-\frac{|z|}{a}\sqrt{s+a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2}\right)}{\sqrt{s+a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2}}dz.$$

The inverse Laplace transform gives

$$\bar{u}(x,\omega_2,t) = \frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty}\bar{g}(x-z,\omega_2)e^{\frac{\beta_1 z}{a}}\left(\frac{1}{\sqrt{t}}e^{-\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)t}e^{-\frac{|z|^2}{4a^2 t}}\right)dz$$
$$+\frac{1}{2a\sqrt{\pi}}\int_{-\infty}^{\infty}e^{\frac{\beta_1 z}{a}}\int_0^t\bar{f}(x-z,\omega_2,t-u)\left(\frac{1}{\sqrt{u}}e^{-\left(a^2\left(\omega_2+\frac{\beta_2 i}{a}\right)^2+\gamma a^2\right)u}e^{-\frac{|z|^2}{4a^2 u}}\right)du\,dz.$$

And lastly, the Fourier transform inversion with respect to ω_2 yields exact solution where $\alpha = 1$,

$$u(x,y,t)=\frac{1}{a^2t\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{\frac{\beta_1 z}{a}}e^{-\frac{|z|^2}{4a^2t}-\gamma a^2 t}\left(\int_{-\infty}^{\infty}g(x-z,y-r)e^{\frac{\beta_2 z}{a}}e^{-\frac{r}{4a^2t}}dr\right)dz$$
$$+\frac{1}{a^2\sqrt{8\pi}}\int_{-\infty}^{\infty}e^{\frac{\beta_1 z}{a}}\int_{0}^{t}\frac{1}{u}e^{-\gamma a^2 u}e^{-\frac{|z|^2}{4a^2u}}$$

5. Conclusion

The joint transform method is a popular method for solving linear wave and diffusion equations in an infinite or semi-infinite spatial domain and with specified initial conditions. The general procedure is as follows: We use the Laplace transform to eliminate the temporal dependence while we apply a Fourier transform in the spatial dimension. These results in an algebraic or ordinary and partial differential equation which we solve to obtain the joint transform. We then compute the inverses. Whether we invert the Laplace or the spatial transform first is usually dictated by the nature of the joint transform [8,9]. The Laplace and Fourier transforms are very useful for solving differential or integral equations for the following reasons. First, these equations are replaced by simple algebraic equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems.

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