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The semi normed space defined by χ sequences

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Abstract: In this paper we introduce the sequence spaces $\chi(p,\sigma,q,s), \Lambda(p,\sigma,q,s)$ and define a semi normed space (X,q) semi normed by q. We study some properties of these sequence spaces and obtain some inclusion relations.

Keywords: Chi sequence, Analytic sequence, Invariant mean, Semi norm.

1. Introduction

A complex sequence, whose kth term is x_k , is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be anlaytic $\sup_k (|x_k|)^{\frac{1}{k}} < \infty$f. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called chi sequence if $\lim_{k\to\infty}(k!\,|x_k|)^{\frac{1}{k}} = 0$.

The vector space of all chi sequences will be denoted by χ. Let σ be a one-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n)), m = 1,2,3,\ldots$

A continuous linear functional ϕ on Λ is said to be an invariant mean or a σ-mean if and only if (1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$for all n (2) $\phi(e) = 1$ where $e = (1,1,1,\ldots)$ and (3) $\phi(\{x_{\sigma}(n)\}) = \phi(\{x_n\})$ for all $x \in \Lambda$. For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in C$. Consequently $C \subset V_\sigma$, where V_σ is the set of analytic sequences all of those σ-means are equal.

If $x = (x_n)$, set $Tx = (Tx)^{1/n} = (x_\sigma(n))$. It can be shown that

$$V_\sigma = \left\{x = (x_n): \lim_{m\to\infty} t_{mn}(x_n)^{1/n} = L \text{ uniformly in } n, L = \sigma - \lim_{n\to\infty}(x_n)^{\frac{1}{n}}\right\} \text{ where}$$

$$t_{mn}(x) = \frac{(x_n + Tx_n + \cdots + T^m x_n)^{1/n}}{m+1} \qquad (1)$$

Given a sequence $x = \{x_k\}$ its nth section is the sequence $x^{(n)} = \{x_1, x_2, \ldots, x_n, 0,0,\ldots\}, \delta^{(n)} = (0,0,\ldots,1,0,0,\ldots)$, 1 in the nth place and zeros elsewhere. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1,2,\ldots)$ are continuous.

2. Definitions and Preliminaries

Definition 2.1. The space consisting of all those sequences x in w such that $(k!\,|x_k|)^{\frac{1}{k}} \to 0$ as $k \to \infty$ is denoted by χ. In other words $(k!\,|x_k|)^{1/k}$ is a null sequence χ is called the space of chi sequences. The space χ is a metric space with the metric $d(x,y) = \left\{\sup_k (k!\,|x_k - y_k|)^{\frac{1}{k}}, k = 1,2,3,\ldots\right\}$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in χ.

Definition 2.2. The space consisting of all those sequence x in w such that $\left(\sup_k (|x_k|)^{\frac{1}{k}}\right) < \infty$ is denoted by Λ. In other words $\left(\sup_k (|x_k|)^{\frac{1}{k}}\right)$ is a bounded sequence.

Definition 2.3. Let p, q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_n) is a sequence such that $p(x_n) \to 0$, then also $q(x_n) \to 0$. If each is stronger than the other, then p and q are said to be equivalent.

Lemma 2.4. Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$for all $x \in X$.

Definition 2.5. A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$, for all $k \in N$.

Definition 2.6. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.7. From the above two definitions, it is clear that a sequence space E is solid implies that E is monotone.

Definition 2.8. A sequence E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies that $y_k = 0$.

Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k < \sup p_k = G$. Let $D = \max(1, 2^{G-1})$. Then for $a_k, b_k \in C$, the set of complex numbers for all $k \in N$ we have.

$$|a_k + b_k|^{1/k} \leq D\{|a_k|^{1/k} + |b_k|^{1/k}\} \tag{2}$$

Let (X, q) be a semi normed space over the field C of complex numbers with the semi norm q. The symbol $\Lambda(X)$ denotes the space of all analytic sequences defined over X. We define the following sequence spaces:

$$\Lambda(p, \sigma, q, s) = \left\{x \in \Lambda(X) \colon \sup_{n,k} k^{-s} \left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} < \infty \text{ uniformly in } n \geq 0, s \geq 0\right\}$$

$$\chi(p, \sigma, q, s) = \left\{x \in \chi(X) \colon k^{-s} \left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right]^{p_k} \to 0, \text{as } k \to \infty \text{ uniformly in } n \geq 0, s \geq 0\right\}$$

3. Main Results

Theorem 3.1. $\chi(p, \sigma, q, s)$ is a linear space over the set of complex numbers..

Proof. It is routine verification. Therefore the proof is omitted.

Theorem 3.2. $\chi(p, \sigma, q, s)$is paranormed space with

$$g^*(x) = \left\{\sup_{k \geq 1} k^{-s} \left[q\left(\sigma^k(n)! \left|x_{\sigma^k(n)}\right|\right)^{\frac{1}{k}}\right], \text{uniformly in } n > 0\right\}$$

where $H = \max\left(1, \sup_k p_k\right)$.

Proof. Clearly $g(x) = g(-x)$ and $g(\theta) = 0$, where θ is the zero sequence. It can be easily verified that $g(x + y) \leq g(x) + g(y)$. Next $x \to \theta, \lambda$ fixed implies $g(\lambda x) \to 0$. Also $x \to \theta$ and $\lambda \to 0$ imply $g(\lambda x) \to 0$. The case $\lambda \to 0$ and x fixed implies that $g(\lambda x) \to 0$ follows from the following expressions.

$$g(\lambda x) = \left\{\sup_{k \geq 1} k^{-s} \left[q\left(\left|x_{\sigma^k(n)}\right|^{1/k}\right)\right] \text{ uniformly in } n, m \in N\right\}$$

$$g(\lambda x) = \left\{\left(|\lambda|^{1/k} r\right)^{p_m/H} \colon \sup_{k \geq 1} k^{-s} \left[q\left(\sigma^k(n)! \left|x_{\sigma^k(n)}\right|\right)^{1/k}\right], r > 0, \text{uniformly in } n, m \in N\right\}.$$

where $r = \frac{1}{|\lambda|^{1/k}}$. Hence $\chi(p,\sigma,q,s)$ is a paranormed space. This completes the proof.

Theorem 3.3. $\chi(p,\sigma,q,s) \cap \Lambda(p,\sigma,q,s) \subseteq \chi(p,\sigma,q,s)$.

Proof. It is routine verification. Therefore the proof is omitted.

Theorem 3.4. $\chi(p,\sigma,q,s) \subset \Lambda(p,\sigma,q,s)$.

Proof. It is routine verification. Therefore the proof is omitted.

Remark 3.5. Let q_1 and q_2 be two semi norms on X, we have

(i) $\chi(p,\sigma,q_1,s) \cap \chi(p,\sigma,q_2,s) \subseteq \chi(p,\sigma,q_1+q_2,s)$;

(ii) If q_1 is stronger than q_2, then $\chi(p,\sigma,q_1,s) \subseteq \chi(p,\sigma,q_2,s)$;

(iii) If q_1 is equivalent to q_2, then $\chi(p,\sigma,q_1,s) = \chi(p,\sigma,q_2,s)$.

Theorem 3.6. *(i)* Let $0 \le p_k \le r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\chi(r,\sigma,q,s) \subset \chi(p,\sigma,q,s)$;

(ii) $s_1 \le s_2$ implies $\chi(p,\sigma,q,s_1) \subset \chi(p,\sigma,q,s_2)$.

Proof of *(i)*.

$$\text{Let } x \in \chi(r,\sigma,q,s) \tag{3}$$

$$k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{\frac{1}{k}}\right]^{r_k} \to 0 \text{ as } k \to \infty \tag{4}$$

Let $t_k = k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{\frac{1}{k}}\right]^{r_k} \to 0$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda > \lambda_k$. Define $u_t = t_k\ (t_k \ge 1)$; $u_k = 0\ (t_k < 1)$; and $v_k = 0\ (t_k \ge 1)$; $v_k = t_k\ (t_k < 1)$; $t_k = u_k + v_k t_k^{\lambda_k} + v_k^{\lambda_k}$. Now it follows that

$$u_k^{\lambda_k} \le t_k \text{ and } v_k^{\lambda_k} \le v_k^{\lambda} \tag{5}$$

(i.e.) $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$ by (5)

$$k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{\lambda_k} \le k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{r_k}$$

$$k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k/r_k} \le k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{r_k}$$

$$k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k} \le k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{r_k}.$$

But $k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{r_k} \to 0$ as $k \to \infty$ by (4).

$$k^{-s}\left[q\left(\sigma^k(n)!\,\left|x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k} \to 0 \text{ as } k \to \infty.$$

Hence

$$x \in \chi(r,\sigma,q,s) \tag{6}$$

From (3) and (6) we get $\chi(r,\sigma,q,s) \subset \chi(p,\sigma,q,s)$. Hence the proof.

Proof of *(ii)*. It is routine verification. Therefore the proof is omitted.

Theorem 3.7. The space $\chi(p,\sigma,q,s)$ is solid and as such is monotone.

Proof. Let $(x_k) \in \chi(p,\sigma,q,s)$ and (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then

$k^{-s}\left[q\left(\sigma^k(n)!\left|\alpha_k x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k} \leq k^{-s}\left[q\left(\sigma^k(n)!\left|\alpha_k x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k}$ for all $k \in N$.

$\left[q\left(\sigma^k(n)!\left|\alpha_k x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k} \leq \left[q\left(\sigma^k(n)!\left|\alpha_k x_{\sigma^k(n)}\right|\right)^{1/k}\right]^{p_k}$ for all $k \in N$. This completes the proof.

Theorem 3.8. The space $\chi(p,\sigma,q,s)$ are not convergence free in general.

Proof. The proof follows from the following example.

Example 3.9. Let $s = 0$; $p_k = 1$ for k even and $p_k = 2$ for k odd. Let $X = C$, $q(x) = |x|$ and $\sigma(n) = n+1$ for all $n \in N$. Then we have $\sigma^2(n) = \sigma(\sigma(n)) = \sigma(n+1) = (n+1)+1 = n+2$ and $\sigma^3(n) = \sigma(\sigma^2(n)) = \sigma(n+2) = (n+2)+1 = n+3$. Therefore, $\sigma^k(n) = (n+k)$ for all $n,k \in N$. Consider the sequences (x_k) and (y_k) defined as $x_k = \left(\frac{1}{k}\right)^k \times \frac{1}{k!}$ and $(y_k) = k^k \times \frac{1}{k!}$ for all $k \in N$. (i.e.) $|x_k|^{1/k} = \frac{1}{k} \times \frac{1}{k!}$ and $|y_k|^{1/k} = \frac{1}{k} \times \frac{1}{k!}$ for all $k \in N$.

Hence $\left|\left(\frac{1}{(n+k)}\right)^{n+k}\right|^{p_k} \to 0$ as $k \to \infty$. Therefore $(x_k) \in \chi(p,\sigma)$. But $\left|\left(\frac{1}{(n+k)}\right)^{n+k}\right|^{p_k} \to 0$ as $k \to \infty$. Hence $(y_k) \notin \chi(p,\sigma)$. Hence the space $\chi(p,\sigma,q,s)$ are not convergence free in general. This completes the proof.

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