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Traveling wave solutions to the $K(m,n)$ equation with generalized evolution using the first integral method

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Abstract: In this paper, we investigate the first integral method for solving the $K(m,n)$ equation with generalized evolution.

$$(u^n)_t + a(u^m)_{u_x} + b(u^n)_{xxx} = 0$$

A class of traveling wave solutions for the considered equations are obtained where $4n = 3(m+1)$. This idea can obtain some exact solutions of this equations based on the theory of Commutative algebra.

Keywords: Traveling wave solutions, First integral method $K(m,n)$ equation.

1. Introduction

In 1993, Rosenau and Hyman [1] introduced and studied a genuinely nonlinear dispersive equation, a special type of KdV equation, of the form

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0,\ n > 1, \tag{1}$$

where a is a constant and both the convection term $(u^n)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. Many powerful methods were applied to construct the exact solutions for Eq.(1), such as Adomain method [1], homotopy perturbation method [2], Exp-function method [3], variational iteration method [4], variational method [5, 6]. In [7], Wazwaz studied a generalized forms of the Eq.(1), that is $mK(n,n)$equations and defined by

$$u^{n-1}u_t + a(u^n)_x + b(u^n)_{xxx} = 0,\ n > 1, \tag{2}$$

where $a,\ b$ are constants. He showed how to construct compact and non-compact solutions for Eq.(2) and discussed it in higher dimensional spaces in [8]. Chen et al. [9] showed how to construct the general solutions and some special exact solutions for Eq.(2) in higher dimensional spatial domains. He et al. [10] considered the bifurcation behavior of travelling wave solutions for Eq.(2). Under different parametric conditions, smooth and non-smooth periodic wave solutions, solitary wave solutions and kink and anti-kink wave solutions were obtained. Yan [11] further extended Eq.(2) to be a more general form,

$$u^{m-1}u_t + a(u^n)_x + bk_{xxx} = 0,\ nk \neq 0, \tag{3}$$

and using some direct ansatz, some abundant new compacton solutions, solitary wave solutions and periodic wave solutions of Eq.(3) were obtained. By using some transformations, Yan [12] obtained some Jacobi elliptic function

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solutions for Eq.(3) Biswas [13] obtained 1-soliton solution of equation with the generalized evolution term,

$$u_t^l + a(u^m)_{u_x} + b(u^n)_{xxx} = 0 \tag{4}$$

where a, b are constants, while l, m and n are positive integers. Zhu et al. [14] applied the decomposition method and symbolic computation system to develop some new exact solitary wave solutions for the $K(2,2,1)$ equation,

$$u_t + a\left(u^2\right)_x - b\left(u^2\right)_{xxx} + u_{xxxxx} = 0, \tag{5}$$

and the $K(3,3,1)$ equation

$$u_t + \left(u^3\right)_x - \left(u^3\right)_{xxx} + u_{xxxxx} = 0, \tag{6}$$

In [15], Xu and Tian introduced the osmosis $K(2,2)$ equation

$$u_t + \left(u^2\right)_x - \left(u^2\right)_{xxx} = 0, \tag{7}$$

where the negative coefficient of dispersive term denotes the contracting dispersion. They obtained the peaked solitary wave solution and the periodic cusp wave solution for Eq.(7).

As is well known that searching for solitary solutions of nonlinear equations in mathematical physics has become more and more attractive in solitary theory. In order to obtain the exact solutions, a number of methods have been proposed, such as the Bäcklund transformation method [16, 17, 18], Hirotas direct method [19], tanh-sech method [20, 21], extended tanh method [22], the exp- function method [23], sine-cosine method [24, 25, 26], Jacobi elliptic function expansion method [27], F-expansion method [28].

The first integral method was first proposed in [29] in solving Burgers-KdV equation which is based on the ring theory of commutative algebra. This method was further developed by the same author [30, 31] and some other mathematicians. In this work, we use the first integral method to find the exact solutions of the $K(m,n)$ equation with generalized evolution [32, 33],

$$(u^n)_t + a(u^m)_{u_x} + b(u^n)_{xxx} = 0,$$

where m , n are positive integrs and a, b are free constants.

This paper is organized as follows: Section 2 is a brief introduction to the first integral method. In section 3, we apply the first integral method to find exact solutions of the $K(m,n)$ equation with generalized evolution.

2. The First Integral Method

Consider a general nonlinear partial differential equation (PDF) in the form,

$$P\left(u, u_t, u_x, u_{xx}, u_{xxx}, u_{xt}, \right), \tag{8}$$

where $u(x,t)$is the solution of nonlinear partial differential equation (8). By means of the transformation,

$$u\left(x,t\right) = u\left(\xi\right), \;\; \xi = \left(x - ct\right), \tag{9}$$

noindent where c is arbitrary constant, we reduce eq (14) to an ordinary differential equation (ODE) of the form,

$$P\left(u,u^{'},u^{''},u^{'''},\right), \tag{10}$$

where $u = u(\xi)$and the primes denote ordinary derivatives with respect to ξ. Next, we introduce a new independent variable,

$$v(\xi) = u(\xi), \;\; w(\xi) = u^{'}(\xi) \tag{11}$$

which leads to a system of ODEs of the form,

$$\begin{cases} v^{'}(\xi) = w(\xi), \\ w^{'}(\xi) = H(v(\xi), w(\xi)). \end{cases} \tag{12}$$

According to the qualitative theory of differential equations [34], if we can find two first integrals to system (11) under the same conditions, then analytic solutions to (11) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how it find it's first integrals in a systematic way. A key idea of our approach here to find first integral is to utilize the division theorem.

Theorem 1 (Division theorem) Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$ and that $P(w,z)$ is irreducible $C[w,z]$. If $Q(w,z)$ vanishes at any zero point of $P(w,z)$, then there exists a polynomial $G(w,z)$in $C[w,z]$ such that,

$$Q(w,z) = P(w,z).G(w,z). \tag{13}$$

The Divisor Theorem follows immediately from the Hilbert-Nullstellensatz Theorem [35].

Theorem 2 (Hilbert-Nullstellensatz Theorem) Let k be a field and L an algebraic closure of k. Then (i) Every ideal γ of $k[X_1,,X_n]$ not containing 1 admits at least one zero in L^n (ii) Let $x = (x_1,x_2,,x_n)$ and $y = (y_1,y_2,,y_n)$ be two elements of L^n. For the set of polynomials of $k[X_1,,X_n]$ zero at x to be identical with the set of polynomials of $k[X_1,,X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphisms s of L such that $y_i = s_i \; for \; 1 \leq i \leq n$. (iii) For an ideal $\alpha$of $k[X_1,,X_n]$to be maximal, it is necessary and sufficient that there exists $x$in Lsuch that $\alpha$is the set of polynomials of $k[X_1,,X_n]$ zero at x. (iv) For a polynomial Q of $k[X_1,,X_n]$ to be zero on the set of zeros in L^n of an ideal of $k[X_1,,X_n]$, it is necessary and sufficient that there exists an integer $m > 0$ such that $Q^m \in \gamma$.

3. The $K(m,n)$ Equation with Generalized Evolution

Let us consider the $K(m,n)$ equation with generalized evolution

$$(u^n)_t + a(u^m)_{u_x} + b(u^n)_{xxx} = 0 \tag{14}$$

where m , n are positive integrs and a, b are free constants.

Assume that equation (14) has the solution of the form:

$$u(x,t) = u(\xi), \;\; \xi = (x - ct) \tag{15}$$

where $c$is arbitrary constant. Substituting (15) into (14) we obtain,

$$-c(u^n)' + a(u^m)u' + b(u^n)''' = 0, \tag{16}$$

where prime denotes derivative with respect to ξ. Integrating the equation (16) with respect to $\xi$and taking the integration constants to zero yields:

$$-cu^n + \frac{a}{m+1}\left(u^{m+1}\right) + b(u^n)'' = 0, \tag{17}$$

Making the following transformation:

$$u = v^{\frac{1}{(m-n+1)}}, ; m-n+1 \neq 0 \tag{18}$$

then (17) becomes

$$Mvv'' - cPv^2 + N\left(v'\right)^2 + Rv^3 = 0, \tag{19}$$

where

$$M = bn(m+1)(m-n+1), \ \ bn(m+1)(2n-m-1),$$

$$P = (m+1)(m-n+1)^2, \ R = a.(m-n+1)^2,$$

and v' and v'' denote $\frac{dv}{d\xi}$ and $\frac{d^2v}{d\xi^2}$ respectively. Equation (19) is a nonlinear ODE, and we can rewrite it as

$$v'' - dv + e\frac{\left(v'\right)^2}{v} + fv^2 = 0, \tag{20}$$

where

$$d = \frac{c(m-n+1)}{bn}, \ e = -\frac{(2n-m-1)}{(m-n+1)}, \ f = \frac{a(m-n+1)}{bn(m+1)}. \tag{21}$$

We introduce new independent variables $v = z, \ \ \frac{dv}{d\xi} = w$. Then equation (20) can be rewritten as the two-dimensional autonomous system

$$\begin{cases} \frac{dz}{d\xi} = w, \\ \frac{dw}{d\xi} = dz - e\frac{w^2}{z} - fz^2. \end{cases} \tag{22}$$

Assume that

$$\frac{d\xi}{z} = d\tau \tag{23}$$

thus system becomes

$$\begin{gathered} \frac{dz}{d\tau} = zw, \\ \frac{dw}{d\tau} = dz^2 - ew^2 - fz^3. \end{gathered} \tag{24}$$

Now, we apply the Division Theorem to seek the first integral to (24). Suppose that $z = z(\tau)$ and $w = w(\tau)$ are the nontrivial solutions to (24), and $p(w,z) = \sum_{i=0}^{e} a_i(z)w^i$, is irreducible polynomial in $C[w,z]$ such that

$$p(w(\tau), z(\xi)) = \sum_{i=0}^{r} a_i(z(\tau))w^i(\tau) = 0, \tag{25}$$

where $a_i(z)$ $(i = 0, 1, , r)$ are polynomials in z and all relatively prime in $C[w,z]$, $a_r(z) \neq 0$. Equation (25) is also called the first integral to (24). We start our study by assuming $r = 1$ in (25). Note that $\frac{dp}{d\tau}$ is polynomial in z and w and $p(w(\tau), z(\tau))$

implies $\left.\frac{dp}{d\tau}\right|_{(24)} = 0$. By the Division Theorem, there exists a polynomial $H(z,w) = h(z) + g(z)w$ in $C[w,z]$ such that

$$\left.\frac{dp}{d\tau}\right|_{(24)} = \left.\left(\frac{dp}{dz}\frac{dz}{d\tau} + \frac{dp}{dw}\frac{dw}{d\tau}\right|_{(24)} = \sum_{i=0}^{1} a_i^{'}(z)w^{i+1}z + \sum_{i=0}^{1} ia_i(z)w^{i-1}\left(dz^2 - ew^2 - fz^3\right) = (h(z) + g(z)w)\left(\sum_{i=0}^{1} a_i(z)w^i\right)\right. \tag{26}$$

where prime denotes differentiating with respect to the variable z. On equating the coefficients of w^i $(i = 0;\ 1;\ 2)$ on both sides of (26), we have

$$za_1^{'}(z) - ea_1(z) = g(z)a_1(z), \tag{27}$$

$$za_0^{'}(z) = g(z)a_0(z) + h(z)a_1(z), \tag{28}$$

$$h(z)a_0(z) = a_1(z)\left[dz^2 - fz^3\right], \tag{29}$$

since, $a_1(z)$ and $g(z)$ are polynomials, from (27) we conclude that $a_1(z)$ is a constant and $g(z) = -e$. for simplicity, we take $a_1(z) = 1$, and balancing the degrees of $a_0(z)$, and $h(z)$, we conclude that deg $h(z) = 1$ and deg $a_0(z) = 2$, only. Now suppose that

$$h(z) = Az + B, a_0(z) = Cz^2 + Dz + E\,(A \neq 0, C \neq 0), \tag{30}$$

where $A,\ B\ and\ C$are all constants to be determined. Using (30) into (28) we obtain

$$a_0(z) = Cz^2 - Az - \frac{B}{2}, 4n = 3(m+1)\,(A \neq 0, C \neq 0), \tag{31}$$

substituting $a_0(z),\ a_1(z)$ and $h(z)$in (29) and setting all the coefficients of powers of $z$to be zero, we obtain a system of nonlinear algebraic equations, and by solving it, we obtain the following solutions:

$$A = \sqrt{-d},\ B = 0,\ C = \frac{f}{d}\sqrt{-d}, \tag{32}$$

$$A = \sqrt{-d},\ B = 0,\ C = -\frac{f}{d}\sqrt{-d}. \tag{33}$$

Using the conditions (32) in (25), we obtain

$$w = -\frac{f}{d}\sqrt{-d}z^2 + \sqrt{-d}z. \tag{34}$$

Combining this first integral with (25), the second-order differential equation (20) can be reduced to

$$\frac{dv}{d\xi} = -\frac{f}{d}\sqrt{-d}v^2 + \sqrt{-d}v. \tag{35}$$

Solving (35) directly and changing to the original variables, we obtain the complex exponential function solution to equation (14):

$$u(x,t) = \left(\frac{d}{f + c_1 dexp^{-i\sqrt{d}t}}\right)^{\frac{1}{m-n+1}} \tag{36}$$

where c_1 is an arbitrary constant and $i^2 = -1$. Similarly, for the cases of (33), we have anther complex exponential function solutions:

$$u(x,t) = \left(\frac{d}{f + c_1 d exp^{i\sqrt{d}(x-ct)}}\right)^{\frac{1}{m-n+1}} \tag{37}$$

where c_1 is an arbitrary constant and $i^2 = -1$. Consequently, we have following theorem.

Theorem 3 Suppose that $4n = 3(m+1)$, then Eq. (14) admits exact solutions

$$u_1(x,t) = \left(\frac{d}{f + c_1 d exp^{-i\sqrt{d}(x-ct)}}\right)^{\frac{1}{m-n+1}}$$

$$u_2(x,t) = \left(\frac{d}{f + c_1 d exp^{i\sqrt{d}(x-ct)}}\right)^{\frac{1}{m-n+1}}$$

where c_1 is an arbitrary constant and $i^2 = -1$. In particular, for $m = 3$ and $n = 3$, we have the following solutions

$$u_1(x,t) = \left(\frac{\frac{c}{3b}}{\frac{a}{12b} + c_1 \frac{c}{12b} exp^{i\sqrt{\frac{c}{3b}}(x-ct)}}\right),$$

$$u_2(x,t) = \left(\frac{\frac{c}{3b}}{\frac{a}{12b} + c_1 \frac{c}{12b} exp^{-i\sqrt{\frac{c}{3b}}(x-ct)}}\right),$$

where c_1, c, a and $b \neq 0$ are constants.

These solutions are all new exact solutions.

Notice that the results in this paper are based on the assumption of $r = 1$, for the $K(m,n)$equation with generalized evolution . For the cases of $r \geq 2$for these equations, the discussions become more complicated and involves the irregular singular point theory and the elliptic integrals of the second kind and the hyperelliptic integrals. Some solutions in the functional form cannot be expressed explicitly. One does not need to consider the cases $r \geq 2$ because it is well known that an algebraic equation with the degree greater than or equal to 5 is generally not solvable.

4. Conlusion

In this work, we are concerned with the $K(m,n)$ equation with generalized evolution for seeking their traveling wave solutions. We first transform each equation into an equivalent two-dimensional planar autonomous system then use the first integral method to find one first integral which enables us to reduce the $K(m,n)$ equation with generalized evolution to a first-order integrable ordinary differential equations. Finally, a class of traveling wave solutions for the considered equations are obtained where $4n = 3(m+1)$. These solutions include complex exponential function solutions. We believe that this method can be applied widely to many other nonlinear evolution equations, and this will be done in a future work.

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