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Filtration on a ring make a quasi valuation or valuation ring

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Abstract: In this paper we show that if R is a filtered ring then we can define a quasi valuation ring. And there exists a valuation ring if R is some kind of filtered ring. Then we prove some properties and relations between filtered ring and quasi valuation ring and valuation ring.

Keywords: Quasi valuation ring, Valuation ring, Filtered ring, Strongly filtered ring.

1. Introduction

In algebra valuation ring and filtered ring are two most important structures [6],[7],[8]. We know that filtered ring is also the most important structure since filtered ring is a base for graded ring especially associated graded ring and completion and some similar results, on the Andreadakis–Johnson filtration of the automorphism group of a free group [2], on the depth of the associated graded ring of a filtration [3],[4]. So, as these important structures, the relation between these structures is useful for finding some new structures, and if R is a discrete valuation ring then R has many properties that have many usage for example Decidability of the theory of modules over commutative valuation domains [8], Rees valuations and asymptotic primes of rational powers in Noetherian rings and lattices [7]. M.H. Anjom SHoa and M.H.Hosseini in [1] proved that if there exists a valuation ring then we can defined a filtration on R, consequently they prove some properties.

In this article we investigate the relation between filtered ring and valuation and quasi valuation ring. We prove that if we have filtered ring then we can find a quasi valuation on it. Continuously we show that if R be a strongly filtered then exist a valuation, Similarly we prove it for PID. At the end we explain some properties for them.

2. Preliminaries

Definition 2.1 A filtered ring R is a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying in the following conditions:

i. $R_0 = R$;

ii. $R_{n+1} \subseteq R_n$ for all $n \geq 0$;

iii. $R_n R_m \subseteq R_{n+m}$ for all $n, m \geq 0$.

Definition 2.2. Let R be a ring together with a family $\{R_n\}_{n\geq 0}$ of subgroups of R satisfying the following conditions:

i. $R_0 = R$;

ii. $R_{n+1} \subseteq R_n$ *for all* $n \geq 0$;

iii. $R_n R_m = R_{n+m}$ *for all* $n, m \geq 0$.

Then we say R has a strong filtration.

Definition 2.3. Let R be a ring and I an ideal of R. Then $R_n = I^n$ is called I-adic filtration.

Definition 2.4. A map $f : M \to N$ is called a homomorphism of filtered modules if: (i) f is R-module an homomorphism and (ii) $f(M_n) \subseteq N_n$ for all $n \geq 0$.

Definition 2.5. A subring R of a filed K is called a valuation ring of K if for every $\alpha \in K, \alpha \neq 0$, either $\alpha \in R$ or $\alpha^{-1} \in R$.

Definition 2.6. Let Δ be a totally ordered abelian group. A valuation v on R with values in Δ is a mapping $v: R^* \to \Delta$ satisfying:

i. $v(ab) = v(a) + v(b)$;

ii. $v(a + b) \geq Min\{v(a), v(b)\}$.

Definition 2.7. Let Δ be a totally ordered abelian group. A quasi valuation v on R with values in Δ is a mapping $v: R^* \to \Delta$ satisfying :

i. $v(ab) \geq v(a) + v(b)$;

ii. $v(a + b) \geq Min\{v(a), v(b)\}$.

Remark 2.1. R is said to be **vaulted ring**; $R_v = \{x \in R : v(x) \geq 0\}$ and $v^{-1}(\infty) = \{x \in R: v(x) = \infty\}$.

Definition 2.8. Let K be a filed. A discrete valuation on K is a valuation $v: K^* \to \mathbb{Z}$ which is surjective.

Theorem 2.1. If R is a UFD then R is a PID (see [3]).

Proposition 2.1. Any discrete valuation ring is a Euclidean domain(see[4]).

Remark 2.2. If R is a ring, we will denote by $Z(R)$ the set of **zero-divisors** of R and by $T(R)$ the **total ring of fractions** of R.

Definition 2.9. A ring R is said to be a **Manis valuation ring** (or simply a **Manis ring**) if there exist a valuation v on its total fractions $T(R)$, such that $R = R_v$.

Definition 2.10. A ring R is said to be a **Prüfer ring** if each overring of R is integrally closed in $T(R)$.

Definition 2.11. A Manis ring R_v is said to be v**-closed** if $R_v/v^{-1}(\infty)$ is a valuation domain (see Theorem 2 of [9]).

3. Filtered ring, Quasi Valuation and Valuation

Let R be a ring with unit and R a filtered ring with filtration $\{R_n\}_{n>0}$.

Lemma 3.1. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i | \alpha \in R_i \backslash R_{i+1}\}$.

Then we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

Proof. For any $\alpha, \beta \in R$ with $v(\alpha) = i$ and $v(\beta) = j$, $\alpha\beta \in R_i R_j \subseteq R_{i+j}$.

Now let $v(\alpha\beta) = k$ then we have $\alpha\beta \in R_k \backslash R_{k+1}$.

We show that $k \geq i + j$.

Let $k < i + j$ so we have $k + 1 \leq i + j$ hence $R_{K+1} \supset R_{i+j}$ then $\alpha\beta \in R_{i+j} \subseteq R_{k+1}$ it is contradiction. So $k \geq i + j$. Now we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

Lemma 3.2. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i | \alpha \in R_i \backslash R_{i+1}\}$.

Then $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$

Proof. For any $\alpha, \beta \in R$ such that $v(\alpha) = i$ and $v(\beta) = j$ and $v(\alpha + \beta) = k$ so we have $\alpha + \beta \in R_k \backslash R_{k+1}$. Without losing the generality, let $i < j$ so $R_j \subset R_i$ hence $\beta \in R_i$. Now if $k < i$ then $k + 1 \leq i$ and $R_i \subset R_{k+1}$ so $\alpha + \beta \in R_i \subset R_{k+1}$ it is contradiction. Hence $k \geq i$ and so we have $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

Theorem 3.1. Let R be a filtered ring. Then there exist a quasi valuation on R.

Proof. Let R be a filtered ring with filtration $\{R_n\}_{n>0}$. Now we define $v: R \to \mathbb{Z}$ such that for every $\alpha \in R$ and $v(\alpha) = \min\{i | \alpha \in R_i \backslash R_{i+1}\}$.

Then

i) By lemma (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$.

ii) By lemma(3.2) we have $v(\alpha + \beta) \geq min\{v(\alpha), v(\beta)\}$. So by Definition 2.7 R is quasi valuation ring.

Proposition 3.1. Let R be a strongly filtered ring. Then there exists a valuation on R.

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R.

Corollary 3.1. Let R be a strongly filtered ring, then R is a Euclidean domain.

Proof. By proposition (3.1) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

Proposition 3.2. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then there exists a valuation on R.

Proof. By theorem (3.1) we have $v(\alpha\beta) \geq v(\alpha) + v(\beta)$ and $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$. Now we show $v(\alpha\beta) = v(\alpha) + v(\beta)$. Let $v(\alpha\beta) > v(\alpha) + v(\beta)$ so $k > i + j$ then $\alpha\beta \in P^k \subsetneq P^{i+j}$ and $k \geq i + j + 1$, since P is a prime ideal hence $\alpha \in P^{i+1}$ or $\beta \in P^{j+1}$ and it is contradiction. So $v(\alpha\beta) = v(\alpha) + v(\beta)$, then there is a valuation on R.

Proposition 3.3. Let R be a PID then there is a valuation on R.

Proof. By theorem (3.1) and proposition (3.2) there is a valuation on R.

Corollary 3.2. If R is an UFD then there exists a valuation on R, then R is a Euclidean domain.

Corollary 3.3. Let R be a ring and P is a prime ideal of R. If R has a P-adic filtration and $R = \cup_{i=0}^{+\infty} P^i$, then R is a Euclidean domain.

Proof. By proposition (3.2) R is a discrete valuation and so by proposition (2.1) R is a Euclidean domain.

Corollary 3.4. Let R be a PID then R is a Euclidean domain.

Proof. By proposition (3.3) and proposition (2.1) we have R is a Euclidean domain.

Corollary 3.5. Let R be a UFD then R is a Euclidean domain.

Corollary 3.6. Let R be a strongly filtered ring. Then R is Manis ring.

Corollary 3.7. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is Manis ring.

Proposition 3.4. Let R_v be a Manis ring. If R_v is v-closed, then R_v is Prüfer.

Proof. See proposition 1 of [10]

Proposition 3.5. Let R be a strongly filtered ring. Then R is v-closed.

Proof. By proposition (3.1) and definition (2.9) we have R is Manis ring and $R = R_v$.

Now let $\alpha, \beta \in R$ and

$$v(\alpha) = i \text{ and } v(\beta) = j$$

Consequently if

$$\big(\alpha + v^{-1}(\infty)\big)\big(\beta + v^{-1}(\infty)\big) \in v^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in v^{-1}(\infty)$ or $\beta \in v^{-1}(\infty)$ .Hence by definition (2.11) R is v-closed.

Corollary 3.8. Let R be a strongly filtered ring. Then R is Prfer.

Proof. By proposition (3.7) R is v-closed so by proposition (3.4) R is Prüfer.

Theorem 3.2. Let R be a domain. Then the following conditions are equivalent.

i) R is a Prüfer domain;

ii) Every tow-generated ideal of R is invertible;

iii) R_P is a valuation for every prime ideal P of R;

iv) R_m is a valuation for every maximal ideal m of R;

v) Each finitely-generated non-zero ideal I of R is a cancelation ideal, that is $IJ = IJ$ for ideals J, K, implies $J = K$;

vi) R is integrally closed and there exists integer $n > 0$ such that for every two elements $a, b \in R, (a, b)^n = (a^n, b^n)$;

Proof. See [3], [4], [5].

Proposition 3.6. Let R be a strongly filtered domain. Then we have following condition for R:

i) Every tow-generated ideal of R is invertible;

ii) R_P is a valuation for every prime ideal P of R;

iii) R_m is a valuation for every maximal ideal m of R;

iv) Each finitely-generated non-zero ideal I of R is a cancelation ideal, that is $IJ = IJ$ for ideals J, K, implies $J = K$;

v) R is integrally closed and there exists integer $n > 0$ such that for every two elements $a, b \in R, (a, b)^n = (a^n, b^n)$;

Proof. By Corollary(3.8) R is a Prüfer domain so by Theorem(3.2) we have the conditions.

Proposition 3.7. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is v-closed.

Proof. By proposition (3.2) and definition (2.9) we have R is Manis ring and $R = R_v$.

Now let $\alpha, \beta \in R$ and

$$v(\alpha) = i \text{ and } v(\beta) = j$$

Consequently if

$$\left(\alpha + v^{-1}(\infty)\right)\left(\beta + v^{-1}(\infty)\right) \in v^{-1}(\infty)$$

Then $i + j \geq \infty$ so $\alpha \in v^{-1}(\infty)$ or $\beta \in v^{-1}(\infty)$ .Hence by definition (2.11) R is v-closed.

Corollary 3.9. Let P is a prime ideal of R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then R is Prüfer.

Proof. By proposition (3.7) R is v-closed so by proposition (3.4) R is Prüfer.

Corollary 3.10. Let P is a prime ideal of domain R and $\{P^n\}_{n\geq 0}$ be P-adic filtration. Then there exists following condition for R.

i) Every tow-generated ideal of R is invertible;

ii) R_P is a valuation for every prime ideal P of R;

iii) R_m is a valuation for every maximal ideal m of R;

iv) Each finitely-generated non-zero ideal I of R is a cancelation ideal, that is $IJ = IJ$ for ideals J, K, implies $J = K$;

v) R is integrally closed and there exists integer $n > 0$ such that for every two elements $a, b \in R$, $(a, b)^n = (a^n, b^n)$;

Proof. Since R by corollary(3.8) is a Prüfer domain. Then by theorem (3.2) we have the above conditions.

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