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The Numerical Solution of Fractional Differential-Algebraic Equations (FDAEs)

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Abstract: In this paper, numerical solution of Fractional Differential–Algebraic Equations (FDAEs) is studied. Firstly Fractional Differential–Algebraic Equations (FDAEs) have been converted to power series and then numerical solution of Fractional Differential–Algebraic Equations (FDAEs) is obtained.

Keywords: Differential-Algebraic Equations (DAEs), Fractional Differential-Algebraic Equation (FDAEs), Power Series.

1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades because of its powerful potential applications. The applications of ordinary fractional differential equations or fractional differential-algebraic equations (FDAE) used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, chemical physics, electrochemistry, optics, polymer physics and signal processing can be successfully modelled by linear or nonlinear fractional differential equations. Meanwhile, some rich fractional dynamical motion which reflect the inherent nature of realistic physical systems are observed. In short, fractional calculus and fractional differential equations have played more and more important role in almost all the scientific fields. [1,4,5,8,12,13]

In this paper, the method is applied to solve FDAEs of the form with the initial conditions [11]

$$D_*^{\alpha_i} x_i(t) = f_i\left(t, x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n'\right), \quad i = 1,2,3,\dots,n-1, \quad t \ge 0, \quad 0 < \alpha_i \le 1$$

$$g(t, x_1, x_2, \dots, x_n) = 0 \tag{1.1}$$

$$x_i(0) = a_i, \qquad i = 1,2,3,\dots,n$$

2. Basic definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ [6], for example. Riemann-Liouville, Grunwald-Letnikow, Caputo and the generalized functions approach. The most commonly used definitions are those of Riemann-Liouville and Caputo. We give some basic definitions and properties of fractional calculus theory used in this paper.

Definition 2.1. A real function $f(x), x < 0.$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^P f_I(x)$, where $f_I(x) \in C[0,\infty)$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x), x < 0.$ is said to be in the space $C_\mu^m, m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function, $f \in C_{\mu}, \mu \geq -1$, is defined as [4].

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \alpha > 0, x > 0 \tag{2.1}$$

$$J^0 f(x) = f(x) \tag{2.2}$$

The properties of the operator f^{α} can be found in [6, 7]: we mention only the following.

For $f \in C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x) \tag{2.3}$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x) \tag{2.4}$$

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{2.5}$$

The Riemann- Liouville derivative has certain disadvantages when trying to model real-word phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator D_*^{α}r proposed by Caputo's work on the theory of viscoelasticity [10].

Definition 2.4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{2.6}$$

for $m-1 < \alpha \leq m, \quad m \in N, \quad x > 0, \quad f \in C_{-1}^m$.

Also, we give two basic properties of its in here. [4].

Lemma 2.1. If $m-1 < m, m \in N$ and $f \in C_{\mu}^m, m \geq -1$, then

$$D_*^{\alpha} J^{\alpha} f(x) = f(x) \tag{2.7}$$

$$J^{\alpha} D_*^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{2.8}$$

3. Our Method

Consider the differential-algebraic equations (DAEs)

$$F(t, x, x') = 0 \tag{3.1}$$

with the initial condition

$$x(t_0) = x_0$$

where F and x are vector functions. The solutions of (3.1) can be assumed that

$$x = x_0 + et \tag{3.2}$$

where e is a vector function. Substitute (3.2) into (3.1) and neglect bigger order term. We have the linear equation of e in the form

$$Ae = B \tag{3.3}$$

where A and B are constant matrices. Solving this (3.3), the coefficients of e in (3.2) can be found. Repeating the above procedure for bigger terms, we can obtain the arbitrary order power series of the solutions for (3.1) [1,2,3,9].

4. Power series of solution for DAEs

We determine another type of power series in the form

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m) t^n \tag{4.1}$$

where $p_1, p_2, \cdots, p_m$ are constants. $e_1, e_2, \cdots, e_m$ are bases of vector e, m is the size of vector e. x is a vector with m elements in (3.2). Every element can be written by the power series in (4.1).

$$x_i = x_{i,0} + x_{i,1} t + x_{i,2} t^2 + \cdots + e_i t^n \tag{4.2}$$

where x_i is the i th element of x. Substituting (4.2) into (3.1), we can get the following expression:

$$f_i = (f_{i,n} + p_{i,1} e_1 + \cdots + p_{i,m} e_m) t^{n-j} + O(t^{n-j+1}) \tag{4.3}$$

where f_i is the i th element of $f(t, x, x')$ in (3.1) and j is 0 if $f(t, x, x')$ have x', 1 if do not. From (4.3) and (3.3), we can determine the linear equation in (3.3) as follows:

$$A_{i,j} = P_{i,j} \tag{4.4a}$$

$$B_i = -f_{i,n} \tag{4.4b}$$

solving this linear equation, we have e_i $(i = 1, \cdots, m)$. Substituting e_i into (4.2), we have x_i $(i = 1, \cdots, m)$ polynomials of degree n. Repeating this procedure from (4.4), we can get the arbitrary order power series of the solution for FDAEs in (1.1). If we repeat the above procedure, we have numerical solution of FDAEs in (1.1).

5. Numerical Examples

To express the effectiveness of the method, we consider the following fractional differential-algebraic equations. All the results were calculated by using the Maple software.

Example 5.1. We consider the following fractional differential-algebraic equation.

$$\begin{aligned} &D_*^{\alpha} x(t) - t y'(t) + x(t) - (1+t) y(t) = 0, \qquad 0 < \alpha \le 1, \\ &y(t) - \sin t = 0 \end{aligned} \tag{5.1}$$

with initial conditions $x(0) = 1$, $y(0) = 0$ and exact solutions $x(t) = e^{-t} + t \sin t$, $y(t) = \sin t$ when $\alpha = 1$.

From initial condition, the solutions of (5.1) can be supposed as

$$\begin{aligned} x(t) &= x_0 + e_1 t \quad \rightarrow \quad x(t) = 1 + e_1 t \\ y(t) &= y_0 + e_2 t \quad \rightarrow \quad y(t) = e_2 t \end{aligned} \tag{5.2}$$

Substituting (5.2) into (5.1) and neglecting higher order terms, we have

$$\begin{aligned} &1+e_1+O(t)=0 \\ &(-1+e_2)t+O(t^2)=0 \end{aligned} \tag{5.3}$$

These formulae correspond to (4.3). The linear equation that corresponds (4.4) can be given in the following:

$$Ae=B, \tag{5.4}$$

Where;

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad B=\begin{pmatrix}-1\\ 1\end{pmatrix} \qquad e=\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

From Eq. (5.4) we have linear equation

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

Solving this linear equation, we have

$$e=\begin{pmatrix}-1\\ 1\end{pmatrix}$$

and

$$\begin{aligned} x(t)&=1-t \\ y(t)&=t \end{aligned} \tag{5.5}$$

from (5.5) the solutions of (5.1) can be supposed as

$$\begin{aligned} x(t)&=1-t+e_1t^2 \\ y(t)&=t+e_2t^2 \end{aligned} \tag{5.6}$$

In like manner substituting (5.6) into (5.1) and neglecting higher order terms, we have

$$\begin{aligned} &(-3+2e_1)t+O(t^2)=0 \\ &-e_2t^2+O(t^3)=0 \end{aligned} \tag{5.7}$$

where

$$A=\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix} \qquad B=\begin{pmatrix}3\\ 0\end{pmatrix} \qquad e=\begin{pmatrix}e_1\\ e_2\end{pmatrix}$$

From Eq. (5.7) we have linear equation

$$\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}3\\ 0\end{pmatrix}$$

By solving this linear equation, we have

$$e=\begin{pmatrix}3/2\\ 0\end{pmatrix}$$

Therefore

$$x(t) = 1 - t + 3/2t^2$$
$$y(t) = t$$

Repeating the above procedure, we have

$$x^*(t) = 1 - t + 1.500000000\,t^2 - 0.1666666667\,t^3 - 0.1250000000\,t^4 - 0.00833333333333\,t^5$$
$$+ 0.009722222222\,t^6 - 0.00001984126984\,t^7$$
$$- 0.0001736111111\,t^8 - 0.2755731922\ \ 10^{-5}\ t^9$$

$$y^*(t) = t - 0.1666666667\,t^3 + 0.00833333333333\,t^5 - 0.00001984126984\,t^7 + 0.2755731922\ \ 10^{-5}\ t^9$$

Table 1. Numerical results of the solution in Example 5.1

	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
t	$x^*(t)$	$x^*(t)$	$x^*(t)$	$x_{exact}(t)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.76429238	0.84929941	0.91482085	0.91482076
0.2	0.75450959	0.80166956	0.85846473	0.85846462
0.3	0.79031612	0.79789989	0.82947437	0.82947428
0.4	0.85249500	0.82508727	0.82608746	0.82608739
0.5	0.93232467	0.87601449	0.84624350	0.84624343
0.6	1.02420517	0.94545816	0.88759718	0.88759712
0.7	1.12379061	1.02907565	0.94753775	0.94753768
0.8	1.22732913	1.12295936	1.02321382	1.02321384
0.9	1.33139163	1.22343656	1.11156381	1.11156388
1.0	1.43275528	1.32697596	1.20935035	1.20935043

Table 1 shows the approximate solutions for Eq. (5.1) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

Example 5.2: Consider the following fractional differential-algebraic equation.

$$\begin{aligned} & x(t) + y(t) = e^{-t} + \sin t \\ & D_*^{\alpha} x(t) + x(t) - y(t) = -\sin t, \qquad 0 < \alpha \le 1, \end{aligned} \tag{5.8}$$

with initial conditions $x(0) = 1$, $y(0) = 0$ and exact solutions in this case $x(t) = e^{-t}$, $y(t) = \sin t$ when $\alpha = 1$.

Repeating the above procedure, we have obtained the numerical results shown in Table 2 by using Maple 15 software.

Table 2. Numerical results of the solution in Example 5.2

	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	
t	$x^*(t)$	$x^*(t)$	$x^*(t)$	$x_{exact}(t)$
0.0	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.76089099	0.83739311	0.90483738	0.90483741
0.2	0.69092614	0.74943903	0.81873062	0.81873075
0.3	0.63965013	0.68161285	0.74081815	0.74081822
0.4	0.59708770	0.62503221	0.67031998	0.67032004
0.5	0.55999258	0.57601215	0.60653064	0.60653065
0.6	0.52688938	0.53262381	0.54881712	0.54881163
0.7	0.49696401	0.49371280	0.49658769	0.49658530
0.8	0.46970221	0.45851976	0.44932904	0.44932896
0.9	0.44474480	0.42650762	0.40656968	0.40656965
1.0	0.42182078	0.39727365	0.36787945	0.36787944

Table 2 shows the approximate solutions for Eq. (5.2) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

6. Conclusion

In this study, the present method has been extended to solve fractional differential-algebraic equations (FDAEs). Two examples are given to demonstrate to powerfulness of the method. The results obtained by the method are in good-agreement with the exact solutions. The study shows that the method is a reliable technique to solve fractional differential–algebraic equations, and offer notable advantages from the points of applicability, computational costs, and accuracy.

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