NEW TRENDS IN MATHEMATICAL SCIENCES
Vol. 1, No. 1, 2013, p13-17
ISSN 2147-5520 - www.ntmsci.com

The Jacobsthal Sequences in The Groups Q_{2^n}, $Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}$ and $Q_{2^n} \times \mathbb{Z}_{2m}$

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Abstract: In [8], Deveci et.al defined the generalized order-k Jacobsthal orbit $J_A^k(G)$ of a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \cdots, a_k\}$ to be the sequence $\{x_i\}$ of the elements of G such that

$$x_i = a_{i+1} \text{ for } 0 \le i \le k-1, \quad x_{i+k} = \begin{cases} (x_i)^2 (x_{i+1}), & k = 2, \\ (x_i)\cdots(x_{i+k-2})^2 (x_{i+k-1}), & k \ge 3 \end{cases} \quad \text{for } i \ge 0.$$

The length of the period of the generalized order-k Jacobsthal orbit $J_A^k(G)$ is denoted by $LJ_A^k(G)$ and is called the generalized order-k Jacobsthal length of G [8].

In this study, we obtain the generalized order-k Jacobsthal lengths of the quarternion group Q_{2^n}, the semidirect product $Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}$ and the direct product $Q_{2^n} \times \mathbb{Z}_{2m}$ for $m, n \ge 3$.

2000 Mathematics Subject Classification: 11B50, 20F05, 20D60, 15A36
Keywords: Group, Sequence, Length.

1 Introduction and Preliminaries

The well-known Jacobsthal sequence $\{J_n\}$ is defined by the following recurrence relation:

for $n \ge 2$

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.1}$$

where $J_0 = 0$ and $J_1 = 1$.

In [13], Koken and Bozkurt showed that the Jacobsthal numbers are also generated by a matrix

$$F = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad F^n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix}.$$

Kalman [11] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding *k* terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \cdots, c_{k-1}$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

In [15], Yilmaz and Bozkurt defined the k sequences of the generalized order-*k* Jacobsthal numbers as follows:

for $n > 0$ and $1 \le \mathrm{i} \le k$

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + \ldots + J_{n-k}^i, \tag{1.2}$$

with initial conditions

$$J_n^i = \begin{cases} 1 & \text{if } n = 1\text{-}i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0,$$

where J_n^i is the nth term of the ith sequence. If $k = 2$ and $i = 1$ the generalized order-*k* Jacobsthal sequence is reduced to the conventional Jacobsthal sequence.

In [15], Yilmaz and Bozkurt showed that

$$\begin{bmatrix} J_{n+1}^i \\ J_n^i \\ J_{n-1}^i \\ \vdots \\ J_{n-k+2}^i \end{bmatrix} = C \cdot \begin{bmatrix} J_n^i \\ J_{n-1}^i \\ J_{n-2}^i \\ \vdots \\ J_{n-k+1}^i \end{bmatrix} \tag{1.3}$$

where C is called the generalized order-*k* Jacobsthal matrix and C is a *k*-square matrix as following:

$$C = \begin{bmatrix} 1 & 2 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.4}$$

Also, it was obtained that $B_n = C \cdot B_{n-1}$ where

$$B_n == \begin{bmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & & \vdots \\ J_{n-k+1}^1 & J_{n-k+1}^2 & \cdots & J_{n-k+1}^k \end{bmatrix}. \tag{1.5}$$

Lemma 1.1 (Yilmaz and Bozkurt [15]). Let C and B_n be as (1.4) and (1.5), respectively. Then, for all integers $n \geq 0$

$$B_n = C^n.$$

Reducing the generalized order-k Jacobsthal sequence $(k \geq 2)$ by a modulus m, we can get the repeating sequences, denoted by

$$\left\{J_n^{k,m}\right\} = \left\{J_{1-k}^{k,m}, J_{2-k}^{k,m}, \cdots, J_0^{k,m}, J_1^{k,m}, \cdots, J_i^{k,m}, \cdots\right\}$$

where $J_i^{k,m} \equiv J_i^k \pmod m$. It has the same recurrence relation as in (1.2) [8].

Theorem 1.1 (Deveci et al [8]). The sequence $\left\{J_n^{k,m}\right\}$ $(k \geq 2)$ is periodic.

The notation $hJ^{k,m}$ denotes the smallest period of $\left\{J_n^{k,m}\right\}$ $(k \geq 2)$ [8].

Theorem 1.2 (Deveci et.al [8]). If p is a prime such that $p \neq 2$, then $hJ^{k,p^a} = \left|\langle C\rangle_{p^a}\right|$.

The usual notation $G_1 \times_\varphi G_2$ is used for the semidirect product of the group $G_1$by G_2, where $\varphi : G_2 \to \mathrm{Aut}(G_1)$ is a homomorphism such that $b\varphi = \varphi_b$ and $\varphi_b : G_1 \to G_1$ is an element of $\mathrm{Aut}(G_1)$.

The quaternion group Q_{2^n}, $(n \geq 3)$ are defined by presentation

$$Q_{2^n} = \left\langle x, y : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xy = x^{-1} \right\rangle.$$

Let $m, n \geq 3$ be integers. By the definitions of the direct and semidirect products, we get the following presentations:

$$Q_{2^n} \times \mathbb{Z}_{2m} = \left\langle x, y, z : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xyx = z^{2m} = [x,z] = [y,z] = e \right\rangle,$$

$$Q_{2^n} \times_\varphi \mathbb{Z}_{2m} = \left\langle x, y, z : x^{2^{n-1}} = e, y^2 = x^{2^{n-2}}, y^{-1}xyx = z^{2m} = e, z^{-1}xzx = e, z^{-1}yzy = e \right\rangle,$$

where if $\mathbb{Z}_{2m} = \langle z \rangle$, then $\varphi : \mathbb{Z}_{2m} \to \mathrm{Aut}\left(Q_{2^n}\right)$ is a homomorphism such that $z\varphi = \varphi_z$; $\varphi_z : Q_{2^n} \to Q_{2^n}$ is defined by $x\varphi_z = x$ and $y\varphi_z = y^{-1}$

For more information see [9,10].

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $a, b, c, d, e, b, c, d, e, b, c, d, e, \cdots$ is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, d, e, f, a, b, c, d, e, f, a, b, c, d, e, f, \cdots$ is simply periodic with period 6.

Many references may be given for some special linear recurrence sequences in groups and related issues; see for example, [1-7,9,12,14,16]. Deveci et.al [8] expanded the theory to the Jacobsthal sequence. In this study, we obtain the generalized order-k Jacobsthal lengths of the quarternion group Q_{2^n}, the semidirect product $Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}$ and the direct product $Q_{2^n} \times \mathbb{Z}_{2m}$ $(m, n \geq 3)$ for initial (seeds) sets y, x and y, x, z.

2 Main Results and Proofs

Definition 2.1. Let $hJ^{k,m}_{(a_1,a_2,\cdots a_k)}$ denote the smallest period of the integer-valued recurrence relation $u_n = u_{n-1} + 2u_{n-2} + \cdots + u_{n-k}$, $u_1 = a_1, u_2 = a_2, \cdots, u_k = a_k$ when each entry is reduced modulo m.

Theorem 2.1. Let $a_1, a_2, \cdots, a_k, x_1, x_2, \cdots, x_k \in \mathbb{Z}$ and let p be a prime with $p \neq 2$, $\gcd(a_1, a_2, \cdots, a_k, p) = 1$ and $\gcd(x_1, x_2, \cdots, x_k, p) = 1$. Then we have

$$hJ^{k,p}_{(a_1,a_2,\cdots a_k)} = hJ^{k,p}_{(x_1,x_2,\cdots x_k)}.$$

Proof. Let $hJ^{k,p} = \left|\langle C \rangle_p\right| = r$. From (1.3), we can write $\begin{bmatrix} u_{n+r} \\ u_{n+r-1} \\ \vdots \\ u_{n+r-k+1} \end{bmatrix} = C^r \cdot \begin{bmatrix} u_n \\ u_{n+r-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix}$. So, we get $\begin{bmatrix} u_{n+r} \\ u_{n+r-1} \\ \vdots \\ u_{n+r-k+1} \end{bmatrix} \equiv \begin{bmatrix} u_n \\ u_{n+-1} \\ \vdots \\ u_{n-k+1} \end{bmatrix}$ mod p, in the natural way. Thus the proof is completes.

Theorem 2.2. $LJ^2_{(y,x)}(Q_{2^n}) = hJ^{2,2^{n-1}}$.

Proof. The orbit $J^2_{(y,x)}(Q_{2^n})$ is

$$y, x, x^{2^{n-2}+1}, \cdots.$$

It is clear from Theorem 2.1 that this sequence has period $hJ^{2,2^{n-1}}$.

Theorem 2.3. $LJ^3_{(y,x,z)}(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m}) = \mathrm{lcm}(2^{n-2} - 7, hJ^{3,2m})$.

Proof. The orbit $J^3_{(y,x,z)}(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m})$ is

$$y, x, z, yx^2z, yxz^3, x^{-1}y^{-1}z^6, x^{2^{n-2}-1}z^{13}, yx^2z^{28}, xz^{60}, x^{-2}z^{129},$$
$$yx^2z^{277}, yxz^{595}, x^{-3}y^{-1}z^{1278}, x^{-2^{n-2}+1}z^{2745}, yz^{5896}, x^{2^{n-1}-3}z^{12664}, \cdots.$$

Using the above information, the orbit $J^3_{(y,x,z)}(Q_{2^n} \times_{\varphi} \mathbb{Z}_{2m})$ becomes:

$$x_0 = y, x_1 = x, x_2 = z, \cdots,$$
$$x_{13} = x^{-2^{n-2}+1}z^{2745}, \ x_{14} = yz^{5896}, x_{15} = x^{2^{n-1}-3}z^{12664}, x_{15} = z^{27201}, \cdots$$
$$x_{14i-1} = x^{-2^{n-2}+1}z^{J^3_{14,i-3}}, x_{14i} = z^{J^3_{14i-2}}y, x_{14i+1} = x^{2^{n-1}-4i+1}z^{J^3_{14i-1}}, x_{14i+2} = z^{J^3_{14i}}, \cdots.$$

So we need an i such that $x_{14i} = y, x_{14i+1} = x, x_{14i+2} = z$. if we choose $i = 2^{n-3}$, then we obtain

$$x_{2^{n-2}\cdot 7} = z^{J^3_{2^{n-2}\cdot 7-2}} y,\ x_{2^{n-2}\cdot 7+1} = xz^{J^3_{2^{n-2}\cdot 7-1}},\ x_{2^{n-2}\cdot 7+2} = z^{J^3_{2^{n-2}\cdot 7}},\ \cdots,$$

where $J^3_{2^{n-2}\cdot 7-k+1}$ and $J^3_{2^{n-2}\cdot 7-k+2}$ are even integers and $J^3_{2^{n-2}\cdot 7-k+3}$ is an odd integer. So, the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times_\varphi \mathbb{Z}_{2m}\right)$ can be said to form layers of length $2^{n-2}\cdot 7$. It is easy to see that the orbit has period $\text{lcm}\left(2^{n-2}-7, hJ^{3,2m}\right)$.

Theorem 2.4. $LJ^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right) = \text{lcm}\left(7, hJ^{3,2m}\right)$.

Proof. The orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ is

$$y, x, z, yx^2z, yxz^3, yx^{2^{n-2}+1}z^6, x^{2^{n-1}}z^{13}, yz^{28}, xz^{60}, z^{129},$$
$$yx^2z^{277}, yxz^{595}, yx^{2^{n-2}+1}z^{1278}, x^{2^{n-1}}z^{2745}, yz^{5896}, xz^{12664}, \cdots.$$

Using the above information, the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ becomes:

$$x_0 = yz^{J^3_{-1}}, x_1 = xz^{J^3_0}, x_2 = z^{J^3_1}, \cdots,$$
$$x_7 = yz^{J^3_6},\ \ x_8 = yz^{J^3_7}, x_9 = z^{J^3_8},$$
$$x_{14} = yz^{J^3_{13}}, x_{15} = xz^{J^3_{14}}, x_{15} = z^{J^3_{15}}, \cdots$$
$$x_{7\cdot i} = yz^{J^3_{7\cdot i-1}}, x_{7\cdot i+1} = xz^{J^3_{7\cdot i}}y, x_{7i+2} = z^{J^3_{7i+1}}, \cdots.$$

The sequence can be said to form layers of length 42. So we need an i such that $x_{7\cdot i} = y, x_{7\cdot i+1} = x, x_{7\cdot i+2} = z$. It is easy to see that the orbit $J^3_{(y,x,z)}\left(Q_{2^n} \times \mathbb{Z}_{2m}\right)$ has period $\text{lcm}\left(7, hJ^{3,2m}\right)$.

Acknowledgment

The authors thank the referees for their valuable suggestions which improved the presentation of the paper. This Project was supported by the Commission for the Scientific Research Projects of Kafkas University. The Project number. 2011-FEF-26.

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