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A Remark on A Fundamental System of Units of Numbers Fields of degree 2, 3, 4 and 6

M'hammed Ziane[1]

Universite Mohammed Premier, Faculte des Sciences,Departement de Math. et Info., Laboratoire "ACSA",60000 Oujda MAROC.

Abstract: Let $M_n = (D_n)^n \pm D_n > 1$ where $D_n = tv^n \pm 1 \neq 0, t, v \in N^*$ and $n \in \{4,6\}$. The integer M_n is always written as $M_n = v^n m_n$, where m_n is a non-zero positive integer; assuming m_n square-free, we exhibit a fundamental system of units for families of pure fields $K_n = \mathbb{Q}(\sqrt[n]{M_n})$, including a family already given by H.-J. Stender.

Keywords: Fundamental system of units (FSU), Parametrization, the integral basis.

1. Introduction

There is a closed link between a fundamental system of units of some number fields, the resolution of some Diophantine equations, the cycle of continued fractions, and certain protocols in cryptography, see J. Buchmann [2]. Also, the regulator of a number field K, based on knowledge of a system fundamental of units, is essential to compute the class number of K, and therefore the Hilbert class towers and the construction of a codes on this number field (see V. Guruswami [5]. This, in addition to many other applications, justifies the study of such a system.

If K is an algebraic extension of degree $n = r + 2s$ on $\mathbb{Q}$, the field of rational numbers, where r is the number of real embeddings and $2s$ is the number of complex embeddings of K, Dirichlet (1840) established that the unit group U_k of K is generated by $r + s - 1$ units. The group U_K is said to be of rank $r + s - 1$. The set $S = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{r+s-1}\}$ of all generators, form what is called a fundamental system of units of the field K. However, the explicit determination of such a system is very limited.

The methods for determining a fundamental system of units of a number field K are very varied. However, regardless to the method adopted, the way followed by several mathematicians is to find in the field K

(1) Units,
(2) an independent system of units
(3) a maximal independent system of units,
(4) a fundamental system of units.

Such a program can be illustrated as follows: L. Bernstein and H. Hasse [1] considered the field $K = \mathbb{Q}(\omega)$, where $\omega = \sqrt[n]{D^n \pm d}$, with $d|D$ and they gave a system of units. The result was generalized by F. Halter-Koch and H.-J. Stender [6] for $d|D^n$. Based on a work of G. Frei and C. Levesque [4] that ensures the maximality of this system for $n \in \{2,3,4,6\}$, H-J Stender studied:

(1) In [11] (page 211), the case $n = 4$, where he assumes that $D^4 \pm d$ is squarefree.
(2) In [13], the case $n = 4$ where he assumes that $D^4 \pm d$ is free of power fourth.
(3) In [12] (page 87), the case $n = 6$, where he assumes that $D^6 \pm d$ is squarefree.

[1] E. Mail: ziane12001@yahoo.fr

These assumptions allow him to use directly the Bernstein and H. Hasse units [1] to determine a fundamental unit of the quadratic fields $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$ and $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$ and a fundamental unit of the pur cubic field $K_3 = \mathbb{Q}(\sqrt[3]{M_6})$ hence the author determines then a fundamental system of units of the fields $K_6 = \mathbb{Q}(\sqrt[6]{M_6})$ and $K_4 = \mathbb{Q}(\sqrt[4]{M_4})$.

Question: What happens if M_n contains one nth power?

To partially answer to this question, (based on an idea of *C. Levesque, laval University, Quebec- Canada)*, we introduce the parameterizations:

$$M_n = (D_n)^n \pm D_n > 1 \ with \ D_n = tv^n \pm 1 \neq 0; t, v \in \mathbb{N}^*$$

Here the plus sign commutes with the minus sign in the expression of M_n and D_n , that is to say:

$$\begin{cases} Case \ \text{"−"} \ : M_n = (D_n)^n - D_n \ and \ D_n = tv^n + 1, \\ Case \ \text{"+"} \ : M_n = (D_n)^n + D_n \ and \ D_n = tv^n - 1 \end{cases}$$

Let

$$m_6 = ab > 1$$

where

$$(a, b) = \begin{cases} (tv^6 + 1, t^5v^{24} + 5t^4v^{18} + 10t^3v^{12} + 10t^2v^6 + 5t) \ in \ Case \ \text{" − "} \\ (t^5v^{24} - 5t^4v^{18} + 10\ t^3v^{12} - 10t^2v^6 + 5t, tv^6 - 1) \ in \ Case \ \text{" + "} \end{cases}$$

And let

$$m_4 = cd > 1$$

where

$$(c, d) = \begin{cases} (tv^4 + 1, t^3v^8 + 3t^2v^4 + 3t) \ in \ Case \ \text{" − "} \\ (t^3v^8 - 3t^2v^4 + 3t, tv^4 - 1) \ in \ Case \ \text{" + "} \end{cases}$$

In both cases, we have the form $M_n = m_n v^n, (n \in \{4,6\})$. In the following, we assume that m_n is square-free, but the M_n always, contains an nth power, $(n \in \{4,6\})$, unless $v = 1$, (the case $v = 1$ coincides with the case of Stender. In the following we always assume $v \geq 2$). Obviously, $K_n = \mathbb{Q}(\sqrt[n]{M_n}) = \mathbb{Q}(\sqrt[n]{m_n})$ but m_n no longer admits a parametrization similar to that of M_n, therefore the Bernstein units [1] are no longer valid. In this paper, we determine a fundamental systems of units of the number fields

$$K_n = \mathbb{Q}(\sqrt[n]{M_n}), \ n \in \{4,6\} \ \ and \ \ K_3 = \mathbb{Q}(\sqrt[3]{M_6})$$

and obviously those of quadratic sub-fields $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$ and $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$.

In T. Nagell [7], T. Nagell [8] and H.-J. Stender [15] we find a full theory dealing with the Diophantine equations of the form $S_C: AX^2 - BY^2 = C, (C \in \{1,2,4\})$, in connection with the fundamental unit of a quadratic field; for C = 1, we summarize (see [15], theorem 3, page 295):

Theorem 1.1 *Given a solution (x, y) of the Diophantine equation $S_1: AX^2 - BY^2 = 1, A, B \in \mathbb{N}, (A, B) = 1$ and AB is square-free, such that*

$$x < \frac{1}{4}(A + B) - \frac{1}{2} \ or \ y < \frac{1}{4}(A + B) + \frac{1}{2},$$

then

$$\eta = (x\sqrt{A} + y\sqrt{B})^2$$

is a fundamental unit>1 of positive norm of the field $K = \mathbb{Q}(\sqrt{AB})$.

Now we give the main results of this section.

Theorem 1.2 *Let t; v be two nonzero positive integers,* $D_6 = tv^6 \pm 1 \neq 0$. *Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6 v^6 > 1, \qquad \omega = \sqrt[6]{m_6}$$

Suppose that m_6 *is square-free. Then*

$$\eta_{2,6} = \frac{D_6}{(v^3 w^3 - (D_6)^3)^2}$$

is a fundamental unit of

$$K_{2,6} = \mathbb{Q}(\sqrt{M_6})$$

Proof: Consider the equation

$$S_1 : aX^2 - bY^2 = 1$$

First of all $(a, b) = 1$, indeed:

Case "-": Let d an integer such that $d|\alpha$ and $d|b = t[(tv^6+1)^4 + (tv^6+1)^3 + (tv^6+1)^2 + (tv^6+1) + 1] = t(a^4 + a^3 + a^2 + a + 1)$. Then $d|(b - t(a^4 + a^3 + a^2 + a) = t$.

an then $d|(a - tv^6) = 1$. Thus $(a, b) = 1$.

Case "+": Let d an integer such that $d|b$ and $d|a = t[(tv^6-1)^4 - (tv^6-1)^3 + (tv^6-1)^2 - (tv^6-1) + 1] = t(b^4 - b^3 + b^2 - b + 1)$. Then $d|(a - t(b^4 - b^3 - a^2 + b) = t$.

$d|(b - tv^6) = 1$. Thus $(a, b) = 1$. In addition the equation (S_1) has the solution,

$$(x, y) = \begin{cases} ((tv^6+1)^2, v^3) \ in \ (Case \text{" -"}), with: \\ \frac{1}{4}(a+b) - \frac{1}{2} > \frac{1}{4}(10t^3v^{12} + 10t^2v^6 + 5t - 1) > (tv^6+1)^2 = x \\ or \\ (v^3, (tv^6-1)^2) \ in \ (Case \text{ "+"}), \text{with:} \\ \frac{1}{4}(a+b) - \frac{1}{2} = \frac{1}{4}(t^5v^{24} - 5t^4v^{18} + 10t^3v^{12} - 10t^2v^6 + tv^6 + 5t - 1) - \frac{1}{2} > v^3 = x. \end{cases}$$

So in both cases, and by theorem 1.1,

$$\eta_{2,6} = \frac{D_6}{(v^3 w^3 - (D_6)^3)^2}$$

is the fundamental unit of the quadratic field $K_{2,6} = \mathbb{Q}(\sqrt{M_6})$.

Theorem 1.3 *Let t; v be two nonzero positive integers,* $D_4 = tv^4 \pm 1 \neq 0$. *Let*

$$M_4 = (D_4)^4 \mp D_4 = m_4 v^4 > 1, \qquad \omega = \sqrt{m_4}$$

Suppose that m_4 *is square-free.Then*

$$\eta_{2,4} = \frac{(v^2 w^2 - (D_4)^2)^2}{D_4}$$

is a fundamental unit of

$K_{2,4} = \mathbb{Q}(\sqrt{M_4})$.

Proof: Consider the equation

$$S_1: cX^2 - dY^2 = 1$$

First of all $(c, d) = 1$, indeed:

Case "-": Let l an integer such that $l|c$ and $l|d = ta^2 + ta + t$; then $l|(b - ta^2 - ta) = t$.

But $l|a$, Then $l|(a - tv^2) = 1$.

Case "+": is such. In addition the equation (S_1) has the solution,

$$(x,y) = \begin{cases} (tv^4+1, v^2) \text{ in } (Case \text{ "-"}), with: \\ 2(a+b) - 1 = 2t^3v^8 + (6t^2+2t)v^4 + 6t + 1 > tv^6 + 1 = x_1 \\ or \\ (v^2, tv^4 - 1) \text{ in } (Case \text{ "+"}), \text{ with:} \\ 2(a+b) - 1 = 2t^3v^8 + (2t - 6t^2)v^4 - 6t - 3 > v^2 = x_2. \end{cases}$$

So in both cases, and by theorem 1.1,

$$\eta_{2,4} = \frac{(v^2w^2 - (D_4)^2)^2}{D_4}$$

is the fundamental unit of $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$.

2. A Fundamental System of Units of $K_3 = \mathbb{Q}(\sqrt[3]{M})$

Let the Diophantine equation

$$(G) = Ax^3 - By^3 = 1$$

with $A, B \epsilon \mathbb{N}$, square-free, $AB > 1$. According to Stender [14], we have two possibilities for the fundamental unit of $\mathbb{Q}(\sqrt[3]{AB^2})$:

Theorem 2.4 *Let $A > 1$ and $B > 1$. Let (x, y) be a solution of the equation (G). Then*

$$\eta = (x\sqrt[3]{A} - y\sqrt[3]{B})^3$$

is either a fundamental unit, or the square of the fundamental unit of the field $K = \mathbb{Q}(\sqrt[3]{AB^2})$.

Now we give the main results of this section.

Theorem 2.5 *Let t, v be two nonzero positive integers $D_6 = tv^6 \pm 1 \neq 0$. Let*

$$M_6 = (D_6)^6 \mp D_6 = m_6v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\eta_3 = \pm\frac{((D_6)^2 - v^2w^2)^3}{D_6}$$

is either a fundamental unit, or the square of the fundamental unit of the field $K_3 = \mathbb{Q}(\sqrt[3]{M_6})$.

Proof: Case "-": Let the equation

$$(G): a^2x^3 - by^3 = 1,$$

which has the solution

$$(x, y) = (tv^6 + 1, v^2),$$

Case "+": Let the equation

$$(G): ax^3 - b^2y^3 = 1,$$

which has the solution

$$(x, y) = (v^2, tv^6 - 1).$$

In both cases and bay theorem 2.4,

$$\eta_3 = \mp \frac{(v^2w^2 - (D_6)^2)^3}{D_6}$$

is the fundamental unit, or the square of the fundamental unit of the field K_3.

Let M be a positive integer cube free, then we set $M = fg^2$, with $(f, g) = 1$, $\bar{M} = f^2g, \Omega = \sqrt[3]{M}, et\ \bar{\Omega} = \sqrt[3]{\bar{M}}$

We say that

(1) $K = \mathbb{Q}(\sqrt[3]{M})$ is of first kind if

$$fg^2 \not\equiv \pm 1 (mod\ 9)$$

(2) $K = \mathbb{Q}(\sqrt[3]{M})$ is of second kind if

$$fg^2 \equiv \pm 1 (mod\ 9)$$

and by Dedekind [3], we have

Proposition 2.6 *(i) If K is of first kind, then* $\{1, \Omega, \bar{\Omega}\}$ is an integral basis of $K = \mathbb{Q}(\Omega)$.

(ii) If K is of second kind, then $\{\frac{1}{3}(1 + f\Omega + g\bar{\Omega}), \Omega, \bar{\Omega}\}$ is an integral basis of $K = \mathbb{Q}(\Omega)$. Moreover each algebraic integer of $K = \mathbb{Q}(\Omega)$ can be written in the form $\frac{1}{3}(x + y\Omega + z\bar{\Omega})$, $x, y, z \in \mathbb{Z}$.

Now, and more precisely, the fundamental unit of the field $K_3 = \mathbb{Q}(\sqrt[3]{M_6})$ is given by

Theorem 2.7 *Let t; v be two nonzero positive integers, $D_6 = tv^6 \pm 1 \neq 0$. Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\eta_3 = \pm \frac{((D_6)^2 - v^2w^2)^3}{D_6}$$

is a fundamental unit of the field $K_3 = \mathbb{Q}(\sqrt[3]{M_6}) = \mathbb{Q}(\omega^2)$.

Proof: As m_6 is square free, according to the proposition 2.6, $\{1, \omega^2, \omega^4\}$ is an integral basis of $K_3 = \mathbb{Q}(\omega^2)$ if K_3 is of first kind; and $\{\frac{1}{3}(1 + f\omega^2 + \omega^4), \omega^2, \omega^4\}$ is an integral basis of $K_3 = \mathbb{Q}(\omega^2)$ if K_3 is of second kind. In addition, according to the proposition 2.6, each algebraic integer of $K_3 = \mathbb{Q}(\omega^2)$ can be written in the form

$$\frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \quad x, y, z \in \mathbb{Z}$$

(1) Case"-": $m_6 = D_6(t^5v^{24} + 5t^4v^{18} + 10t^3v^{12} + 10t^2v^6 + 5t)\ et$

$$\eta_3 = 1 - (3(D_6)^3v^2)\omega^2 + (3D_6v^4)\omega^4 \tag{2.1}$$

Suppose that $\eta_3 = \zeta^2$, where ζ is a unit of K_3.

(a) Let K_3 is of first kind. Then

$$\zeta = x + y\omega^2 + z\omega^4, \qquad with\ x, y, z \in \mathbb{Z}$$

as $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 1 \tag{2.2}$$
$$2xy + z^2m_6 = -3(D_6)^3v^2 \tag{2.3}$$
$$2xz + y^2 = 3D_6v^4 \tag{2.4}$$

Let's show that

$$(*) \qquad xy > 0 \ \ and \ \ y \neq 0.$$

According to (2.3), $x \neq 0$ and $y \neq 0$ In addition $z \neq 0$; Indeed, suppose $z = 0$; then according to (2.2), $x = \pm 1$; according to (2.3), $2y = \pm 3(D_6)^3v^2$, i.e. $4y^2 = 9(D_6)^6v^4$; but according to (2.4), $4y^2 = 12D_6v^4$; then we have $12D_6v^4 = 9(D_6)^6v^4$, i.e. 3|4, a contradiction. According to (2.3), $xy < 0$, and according to (2.2), $yz < 0$. Then x and z have the same sign, i.e. $xz > 0$.

According to (2.2), we have

$$(**) \qquad (x, m_6) = 1.$$

Then $(x, D_6) = 1$. According to (2.3), $D_6|2xy$, i.e. $D_6|2y$. Then (2.4) becomes

$$8xz + (2y)^2 = 12D_6v^4 \tag{2.5}$$

Then $(D_6)^2|(8z^2)$. And (2.3) becomes

$$2(8)^2xy + (8z)^2m_6 = -3(8)^2(D_6)^3v^2 \tag{2.6}$$

Then $(D_6)^3|2(8)^2y$, since $D_6|m_6$. And (2.4) becomes

$$(2^2)(8^4)2xz + (2(8^2)y)^2 = (2^2)(8^4)3D_6v^4 \tag{2.7}$$

But $D_6|8z$; then, (seeing that $xz > 0$),

$$(2^2)(8^4)2xz = (8^4)xz_1D_6 > 0$$

But $(D_6)^6|(2(8^2)y)^2$; then

$$(2(8^2)y)^2 = (y_1)^2(D_6)^6 > 0$$

But

$$(2^2)(8^4)3D_6v^4 < (2^2)(8^4)3(D_6)^2$$

Then (2.7) implies

$$(8^4)xz_1D_6 + (y_1)^2(D_6)^6 < (2^2)(8^4)3D_6)^2 \tag{2.8}$$

which is impossible for $D_6 \geq 16$; but $v \geq 2$, whereby that $D_6 = tv^6 + 1 > 2^6 = 64$.

(b) Let K_3 of the second kind. Then

$$\zeta = \frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \quad x, y, z \in \mathbb{Z}$$

As $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 9 \tag{2.9}$$

$$2xy + z^2 m_6 = -27(D_6)^3 v^2, \tag{2.10}$$

$$2xz + y^2 = 27 D_6 v^4, \tag{2.11}$$

Then $(x, m_6) = 1, 3\ ou\ 9$. The 9 is excluded because m_6 is square free. whether $(x, m_6) = 1$ We have then the propriete (**) of first case, and get the equivalent of (2.8), namely

$$(8^4)xz_1 D_6 + (y_1)^2 (D_6)^6 < (2^2)(8^4)27(D_6)^2$$

which is impossible for $D_6 \geq 27$, i.e. for all $v \geq 2$.

Whether $(x, m_6) = 3$ according to (2.9), $3|y$ or $3|z$. If $3|y$, then according to (2.10), $3|z$. If $3|z$, then according to (2.11), $3|y$. Brief, $3|y$ and $3|z$. Let

$$x_1 = \left(\frac{x}{3}\right), \quad y_1 = \left(\frac{y}{3}\right) \quad et \quad z_1 = \left(\frac{z}{3}\right)$$

Then

$$x_1^2 + 2y_1 z_1 m_6 = 1. \tag{2.12}$$

$$2x_1 y_1 + z_1^2 m_6 = -3(D_6)^3 v^2 \tag{2.13}$$

$$2x_1 z_1 + y_1^2 = 3(D_6) v^4 \tag{2.14}$$

Which brings us back again to the same contradiction above.

(2) Case “+”: As

$$v^6 m_6 = (D_6)^6 + D_6 \tag{2.15}$$

We derive

$$m_6 > \frac{(D_6)^6}{v^6} \tag{2.16}$$

Furthermore,

$$\eta_3 = 1 + (3(D_6)^3 v^2)\omega^2 - (3D_6 v^4)\omega^4 \tag{2.17}$$

Suppose that $\eta_3 = \zeta^2$, We distinguish two cases.

(a) Let K_3 be of the first kind. Then

$$\zeta = x + y\omega^2 + z\omega^4, \quad with \;\; x, y, z \in \mathbb{Z}$$

Then

$$x^2 + 2yzm_6 = 1 \tag{2.18}$$

$$2xy + z^2m_6 = 3(D_6)^3v^2 \tag{2.19}$$

$$2xz + y^2 = -3D_6v^4 \tag{2.20}$$

Let's show that

$$(*') \qquad (x, m_6) = 1 \quad et \; D_6|4z$$

According to (2.18), $(x, m_6) = 1$. According to (2.19), $D_6|2y$. Then (2.20) becomes

$$4xz + 2y^2 = -6D_6v^4 \tag{2.21}$$

and $D_6|4z$

Let's show that

$$(**') \qquad xy > 0 \quad et \;\; z \neq 0$$

We have $x \neq 0$ and $z \neq 0$ otherwise according to (2.20), $y^2 = -3v^4D_6 < 0$. In addition $y \neq 0$; suppose the contrary; according to (2.19), $(4z)^2m_6 = (4^2)(3)v^2(D_6)^3$; according to $(*')$, $4z = z_1D_6$; according to (2.16), $m_6 > \frac{(D_6)^6}{v^6}$, Then

$$(4^2)(3)v^2(D_6)^3 = (4z)^2m_6 > z_1^2\left(\frac{(D_6)^8}{v^6}\right)$$

which is impossible for $v \geq 2$. According to (2.18), $x^2 = 1 - 2yzm_6 > 0$ then $yz < 0$. According to (2.20), $2xz = -3v^4D_6 - y^2 < 0$ then $xz < 0$. Then $xy > 0$ The equation (2.19) becomes

$$(4^2)2xy + (4z)^2m_6 = (4^2)3(D_6)^3v^2 \tag{2.22}$$

Then

$$(4^2)3(D_6)^3v^2 = (4^2)2xy + z_1^2D^2m_6 > z_1^2\left(\frac{(D_6)^8}{v^6}\right)$$

which is impossible for $v \geq 2$.

(b) Let K_3 be of the second kind. Then

$$\zeta = \frac{1}{3}(x + y\omega^2 + z\omega^4), \quad with \;\; x, y, z \in \mathbb{Z}$$

As $\eta_3 = \zeta^2$, we have

$$x^2 + 2yzm_6 = 9 \tag{2.23}$$

$$2xy + z^2m_6 = 27(D_6)^3v^2 \tag{2.24}$$

$$2xy + y^2 = -27D_6v^4 \tag{2.25}$$

According to (2.23), $(x, m_6) = 1{,}3$ ou 9. 9 is excluded. Whether $(x, m_6) = 1$. We have then the property $(*\,')$ and we deduce a contradiction as above. Let $(x, m_6) = 3$. According to (2.23), $3|yz$, If $3|y$, then according to (2.24), $3|z$. If $3|z$, then according to (2.25), $3|y$. Let

$$x_1 = \left(\frac{x}{3}\right),\quad y_1 = \left(\frac{y}{3}\right)\quad et\ z_1 = \left(\frac{z}{3}\right);$$

and we deduce a contradiction as above.

3. A Fundamental System of Units of $K = \mathbb{Q}(\sqrt[6]{M_6})$

We have m_6 is square-free, the field $K_6 = \mathbb{Q}(\omega), \omega = \sqrt[6]{m_6}$, is of degree 6 over $\mathbb{Q}$, in addition it admits a quadratic sub-field $K_{2,6} = \mathbb{Q}(\omega^3)$ with fundamental unit $\eta_{2,6}$ (theorem 1.2), and a cubic sub-field $K_3 = \mathbb{Q}(\omega^2)$ with fundamental unit η_3 (theorem 2.7). For the determination of a fundamental system of units of the field $K_6 = \mathbb{Q}(\sqrt[6]{M_6})$, we use the Stender theorem [12]:

Theorem 3.8 *Let $\eta_{2,6}$ be the fundamental unit of $K_{2,6}$, and let η_3 be the fundamental unit of K_3. Let $\xi_2, \xi_3 \epsilon K_6$ such that*

$$N_{K_6/K_{2,6}}(\xi_2) = \eta_{2,6},\quad N_{K_6/K_3}(\xi_2) = \pm 1$$

and

$$N_{K_6/K_{2,6}}(\xi_3) = 1,\quad N_{K_6/K_3}(\xi_2) = \pm \eta_{3,6}$$

Let $\epsilon_1 \in K_6$ be the smallest unit>1, satisfying:

$$N_{K_6/K_{2,6}}(\epsilon_1) = 1,\quad N_{K_6/K_3}(\epsilon_1) = \pm 1$$

Then

$$\{\xi_2, \xi_3, \epsilon_1\}$$

is a fundamental system of d'units of K_6.

Let ϱ be a third root of unity, $(\varrho^2 + \varrho + 1 = 0)$; the conjugates $\alpha^{(j)}$ of an algebraic integer α of field $K_6, 0 \le j \le 5$, are given by:

$$\begin{cases} \alpha^{(0)} = \alpha \\ \alpha^{(1)} = -\alpha \\ \alpha^{(2)} = \varrho\alpha \\ \alpha^{(3)} = \varrho^2\alpha \\ \alpha^{(4)} = -\varrho\alpha \\ \alpha^{(5)} = -\varrho^2\alpha \end{cases} \tag{3.26}$$

And according to Stender [12], the product $\alpha\omega$ can be written in the form:

$$\alpha\omega = \frac{1}{6}(x_0 + x_1\omega + x_2\frac{\omega^2}{h} + x_3\frac{\omega^3}{h}x_4\frac{\omega^4}{gh^2} + x_5\frac{\omega^5}{gh^2}) \tag{3.27}$$

with $x_i \epsilon \mathbb{Z}, 0 \le i \le 5$.

Remark 3.9 *Since m_6 is square free, then we can take $g = h = 1$ in (3.27), (see [12] page 80 and page 87).*

In addition, we have:

Proposition 3.10 *Let α be an algebraic integer of the field $K_6 = \mathbb{Q}(\omega)$. Let β be a unit >1 such that*

$$N_{K_6/K_{2,6}}(\beta) = 1, \quad N_{K_6/K_3}(\beta) = \pm 1$$

Suppose that $\beta = \alpha^n$. Then

$$|x_i| < \frac{k_i}{\omega^{i-1}}\left(\sqrt[n]{\beta} + 2\sqrt[n]{|\beta^{(4)}|} + 3\right), \qquad 0 \le i \le 5; \tag{3.28}$$

where $k_0 = k_1 = 1, k_2 = k_3 = h, k_4 = k_5 = gh^2$. In addition $x_4 \neq 0$ and $x_5 \neq 0$.

Now we give the main results of this section.

Theorem 3.11 *Let t, v be two nonzero integers, $D_6 = tv^6 \mp 1 > 0$. Let*

$$M_6 = (D_6)^6 \pm D_6 = m_6 v^6 > 1, and \quad \omega = \sqrt[6]{m_6}.$$

Suppose that m_6 is square-free. Then

$$\{\xi_2 = \pm\frac{v\omega + D_6}{v\omega - D_6}, \xi_3 = \frac{v^3\omega^3 - (D_6)^3}{(v\omega - D_6)^3}, \qquad \epsilon_1 = \xi_2^3 {\eta_{2,6}}^{-1}\}$$

is a fundamental system of units of $K_6 = \mathbb{Q}\left(\sqrt[6]{M_6}\right)$

Proof: ξ_2 and ξ_3 satisfies theorem 3.8, namely:

$$N_{K_6/K_{2,6}}(\xi_2) = \eta_{2,6}, \quad N_{K_6/K_3}(\xi_2) = \pm 1$$

And

$$N_{K_6/K_{2,6}}(\xi_3) = 1, \quad N_{K_6/K_3}(\xi_3) = \pm\eta_3$$

For

$$\epsilon_1 = \xi_2^3 {\eta_{2,6}}^{-1} = \xi_3^2 {\eta_3}^{-1} = \xi_6 {\eta_3}^{-1} {\eta_{2,6}}^{-1}$$

where

$$\xi_6 = \frac{D_6}{(v\omega - D_6)^6}$$

we have

$$\epsilon_1 > 1, \qquad N_{K_6/K_{2,6}}(\epsilon_1) = 1, \quad et \quad N_{K_6/K_3}(\epsilon_1) = \pm 1 \tag{3.29}$$

Let's show that ϵ_1 is the smallest unit that verifies (3.29):

Lemma 3.12 *(i) In Case"-", we have*

$$\begin{cases} \dfrac{4v^6\omega^6}{D_6} < \eta_{2,6} < 4(D_6)^5, \\ \dfrac{12v^6\omega^6}{D_6} < \xi_2 < 12(D_6)^5 \end{cases}$$

(ii) In Case "+", we have

$$\begin{cases} 4(D_6)^5 < \eta_{2,6} < \dfrac{4v^6\omega^6}{D_6}, \\ 12(D_6)^5 < \xi_2 < \dfrac{12v^6\omega^6}{D_6} \end{cases}$$

Proof: (i) Case “-“:

$$D_6 - 1 < v\omega < D_6.$$

Since

$$D_6 = (D_6)^6 - (v\omega)^6,$$

we deduce

$$\frac{D_6}{(D_6)^3 - (v\omega)^3} = (D_6)^3 + (v\omega)^3$$

and

$$\frac{D_6}{D_6 - v\omega} = ((D_6)^5 + v\omega(D_6)^4 + v^2\omega^2(D_6)^3 + v^3\omega^3(D_6)^2 + v^4\omega^4 D_6 + v^5\omega^5).$$

We have

$$\eta_{2,6} = \frac{D_6}{(v^3\omega^3 - (D_6)^3)^2} = \frac{1}{D_6}(\frac{D_6}{v^3\omega^3 - (D_6)^3})^2 = \frac{1}{D_6}((D_6)^3 + v^3\omega^3)^2$$

then

$$\frac{4v^6\omega^6}{D_6} < \eta_2 < 4(D_6)^5.$$

Similarly

$$\xi_2 = -\frac{v\omega + D_6}{v\omega - D_6} = \frac{v\omega + D_6}{D_6}((D_6)^5 + v\omega(D_6)^4 + v^2\omega^2(D_6)^3 + v^3\omega^3(D_6)^2 + v^4\omega^4 D_6 + v^5\omega^5)$$

Then

$$\frac{12v^6\omega^6}{D_6} < \xi_2 < 12(D_6)^5.$$

(ii) Case “+”:$v\omega > D_6$, and just swap D_6 and $v\omega$.

Lemma 3.13

$$1 < \epsilon_1 < \begin{cases} 3^3 2^4 \dfrac{(D_6)^{16}}{v^6\omega^6} \ \ Case\ "-", \\ 3^3 2^4 \dfrac{v^{18}\omega^{18}}{(D_6)^8} \ \ Case\ "+", \end{cases}$$

Proof: Case “-“

$$\epsilon_1 = \xi_2^3{\eta_2}^{-1} < (12(D_6)^5)^3\left(\frac{D_6}{4(v\omega)^6}\right) = 3^3 2^6\left(\frac{(D_6)^{16}}{4(v\omega)^6}\right) = 3^3 2^4\left(\frac{(D_6)^{16}}{(v\omega)^6}\right)$$

On the other hand, according to lemma 3.12,

$$\epsilon_1 = \xi_2^3 \eta_2^{-1} > \left(\frac{12v^6\omega^6}{D_6}\right)^3 (4(D_6)^5)^{-1} > 1$$

In Case"+", we use lemma 3.12 and the fact that $v\omega > D_6$.

Lemma 3.14 *(i) Case "-":*

$$1 < \left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| < 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

(ii) Case "+":

$$1 < \left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| < 12\sqrt{3}\left(\frac{v^9\omega^9}{(D_6)^3}\right)$$

Proof: According to (3.26), $\epsilon_1^{(4)} = \overline{\epsilon_1^{(5)}}$. Then $\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right|$. On the other hand,

$$\left|\epsilon_1^{(4)}\right| = \left|\left(\xi_2^{(4)}\right)^3\left(\eta_{2,6}^{(4)}\right)^{-1}\right| = \left|\xi_2^{(4)}\right|^3\left|\eta_{2,6}^{(4)}\right|^{-1}$$

and

$$\left|\xi_2^{(4)}\right|^2 = \xi_2^{(4)}\left(\overline{\xi_2^{(4)}}\right) = \xi_2^{(4)}\xi_2^{(5)} = \frac{(D_6)^2 + D_6 v\omega + v^2\omega^2}{(D_6)^2 - D_6 v\omega + v^2\omega^2} > 1.$$

Then

$$\left|\xi_2^{(4)}\right| > 1 \quad and \quad \left|\xi_2^{(4)}\right|^3 > 1$$

We have

$$1 < \eta_{2,6} = \left|\eta_{2,6}{}^{(1)}\right|^{-1} = \left|\eta_{2,6}{}^{(4)}\right|^{-1} = \left|\eta_{2,6}{}^{(5)}\right|^{-1}$$

Then

$$\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| = \left|\xi_2^{(4)}\right|^3\left|\eta_{2,6}{}^{(4)}\right|^{-1} > 1$$

On the other hand,

$$\left|\epsilon_1^{(4)}\right| = \left|\epsilon_1^{(5)}\right| = \left(\xi_2^{(4)}\xi_2^{(5)}\right)^{3/2}\left(\eta_{2,6}{}^{(4)}\eta_{2,6}{}^{(5)}\right)^{-1/2}$$

$$= \left(\frac{(D_6)^2 + D_6 v\omega + v^2\omega^2}{(D_6)^2 - D_6 v\omega + v^2\omega^2}\right)^{3/2} \frac{(v^3\omega^3 + (D_6)^3)^2}{D_6}$$

Then

Case "-": We have $v\omega < D_6$; then

$$\left|\epsilon_1^{(4)}\right| < \left(\frac{3(D_6)^2}{v^2\omega^2}\right)^{3/2}\left(\frac{4(D_6)^6}{D_6}\right) = 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

Case "+":We have $v\omega > D_6$; and

$$\left|\epsilon_1^{(4)}\right| < \left(\frac{3v^2\omega^2}{(D_6)^2}\right)^{3/2}\left(\frac{4v^6\omega^6}{D_6}\right) = 12\sqrt{3}\left(\frac{v^9\omega^9}{(D_6)^3}\right)$$

Lemma 3.15 *ϵ_1 is the smallest unit > 1 of field $K_6 = \mathbb{Q}(\omega)$ such that*

$$N_{K_6/K_{2,6}}(\epsilon_1) = 1, \quad et \quad N_{K_6/K_3}(\epsilon_1) = \pm 1$$

Proof: Argue by contradiction and assume that

$$\epsilon_1 = \alpha^n \quad with \ n > 1$$

There $n \notin \{2,3\}$, because $\sqrt{\eta_3} \notin K_6$ and $\sqrt[3]{\eta_{2,6}} \notin K_6$. Let $n \geq 5$. In specializing $\beta = \epsilon_1$, in the proposition 3.10, then for $i \in \{4.5\}$ we have

$$|x_i| < \left(\frac{1}{\omega^{i-1}}\right)\left(\sqrt[5]{\epsilon_1} + 2\sqrt[5]{\left|\epsilon_1^{(4)}\right|} + 3\right)$$

Case "-":

$$\epsilon_1 = 3^3 2^4 \left(\frac{(D_6)^{16}}{v^6 \omega^6}\right)$$

such that

$$\left|\epsilon_1^{(4)}\right| < 12\sqrt{3}\left(\frac{(D_6)^8}{v^3\omega^3}\right)$$

we obtain

$$|x_5| < \left(\frac{(D_6)^3}{v^4\omega^4}\right)\left(\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{20}}{v^6\omega^6}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{20}}{v^3\omega^3(D_6)^7}\right)} + \left(\frac{3v^4}{(D_6)^3}\right)\right). \tag{3.30}$$

But

$$G_1 = \sqrt[5]{3^3 2^4\left(\frac{D_6 v^{20}}{v^6\omega^6}\right)} < \sqrt[5]{3^3 2^4\left(\frac{(D_6)^5}{(D_6)^6 - D_6}\right)}$$

$$= \sqrt[5]{3^3 2^4\left(\frac{1}{D_6 - (D_6)^{-4}}\right)}$$

$$G_2 = 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{20}}{v^3\omega^3(D_6)^7}\right)} < 2\sqrt[5]{12\sqrt{3}\left(\frac{(D_6)^4}{v^3\omega^3(D_6)^7}\right)}$$

$$< 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{(D_6)^3}\right)}$$

$$G_3 = \frac{3v^4}{(D_6)^3} < \frac{3}{(D_6)^2},$$

$$\frac{(D_6)^3}{v^4\omega^4} < \frac{(D_6)^5}{(D_6)^6 - D_6} = \frac{1}{D_6 - (D_6)^{-4}}$$

Then $|x_5| < 1$, because $D_6 > 2^6 = 64$, and $x_5 = 0$ because x_5 is an integer. In addition,

$$|x_4| = \left(\frac{(D_6)^3}{v^3\omega^3}\right)\left(\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{15}}{v^6\omega^6}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{v^{15}}{v^3\omega^3(D_6)^7}\right)} + \left(\frac{3v^3}{(D_6)^3}\right)\right) \tag{3.31}$$

But

$$\sqrt[5]{3^3 2^4\left(\frac{D_6 v^{15}}{v^6\omega^6}\right)} < G_1$$

$$2\sqrt[5]{12\sqrt{3}\left(\frac{v^{15}}{v^3\omega^3(D_6)^7}\right)} < G_2$$

$$\left(\frac{3v^3}{(D_6)^3}\right) < G_3$$

$$\left(\frac{(D_6)^3}{v^3\omega^3}\right) < \frac{1}{1-(D_6)^{-5}}$$

Then $x_4 = 0$.

Case "+":

$$|x_5| < \sqrt[5]{3^3 2^4\left(\frac{1}{v^{22}\omega^2}\right)} + 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{v^6\omega^{11}}\right)} + \left(\frac{3}{\omega^3}\right) \tag{3.32}$$

Then in a analogous manner $x_5 = 0$. In addition,

$$|x_5| < \sqrt[5]{3^3 2^4\left(\frac{1}{v^5}\right)\sqrt{\frac{(D_6)^6 + D_6}{(D_6)^8}}} + 2\sqrt[5]{12\sqrt{3}\left(\frac{1}{v^6\omega^6}\right)} + \left(\frac{3}{\omega^3}\right) \tag{3.33}$$

Then $x_4 = 0$. This completes the prof of theorem 3.11.

4. A Fundamental System of Units of $K = \mathbb{Q}(\sqrt[4]{M_4})$

We assume that m_4 is square-free, the field $K_4 = \mathbb{Q}(\omega), (\omega = \sqrt[4]{m_4})$, is of degree 4 over $\mathbb{Q}$, in addition it admits a sub-quadratic field $K_{2,4} = \mathbb{Q}(\omega^2)$ with fundamental unit $\eta_{2,4}$ (theorem 1.3). We introduce here the proprieties of fields of degree 4 taken follows [9] and [11].

Every algebraic integer α of K_4 can be written as form

$$\alpha = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \quad with\ x_0, \qquad x_1, x_2, x_3 \in \mathbb{Z} \tag{4.34}$$

We denote by

$$\begin{cases} \omega = \sqrt[4]{m_4} \\ \omega^{(1)} = -\omega \\ \omega^{(2)} = i\omega \\ \omega^{(3)} = -i\omega \end{cases} \tag{4.35}$$

the four conjugates ω. replacing ω respectively by $\omega^{(1)}, \omega^{(2)}, \omega^{(3)}$ in (4.42), we get

$$\alpha^{(1)} = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \tag{4.36}$$

$$\alpha^{(2)} = \frac{1}{4}(x_0 + x_1 i\omega - x_2\omega^2 - x_3 i\omega^3) \tag{4.37}$$

$$\alpha^{(3)} = \frac{1}{4}(x_0 - x_1 i\omega - x_2\omega^2 + x_3 i\omega^3) \tag{4.38}$$

If in addition β is an algebraic integer such that $\beta = \pm\alpha^n, n \geq 1$, then

$$|x_3| \leq \left(\frac{1}{\omega^3}\right)\sum_{j=0}^{3}\sqrt{|\beta^{(j)}|} \tag{4.39}$$

Denote by ε_0 the smallest unit >1 of K_4 satisfying the property

$$\varepsilon_0\varepsilon_0^{(1)} = 1; \tag{4.40}$$

then any other unit ε of K_4 which satisfies the properties (4.40), is of the form

$$\varepsilon = \varepsilon_0^n, \;\; n \geq 1 \tag{4.41}$$

writing

$$\varepsilon_0 = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \quad \text{with } x_0, x_1, x_2, x_3 \in \mathbb{Z} \tag{4.42}$$

then according to (4.40) and (4.41) we have in addition

$$0 \neq |x_3| < \frac{1}{\omega^3}(3 + \sqrt{\varepsilon}) \tag{4.43}$$

Theorem 4.16 *Let $\eta_{2,4}$ be the fundamental unit of the quadratic field $K_{2,4} = \mathbb{Q}(\sqrt{M_4})$, and let ε_0 be the smallest unit of K_4 satisfying $\varepsilon_0\varepsilon_0^{(1)} = 1$. If $\sqrt{(\eta_{2,4})^{-1}\varepsilon_0} \notin K_4$, then*

$$\{\eta_{2,4}, \varepsilon_0\}$$

is a fundamental system of unit of K_4.

Now we give the main results of this section.

Theorem 4.17 *Let t, v be two nonzero positive integers, $D_4 = tv^4 \mp 1$. Let*

$$M_4 = D_4^4 \pm D_4, m_4 = \frac{M_4}{v^4}, \omega = \sqrt[4]{m_4}.$$

Suppose that m_4 is square-free. Then

$$\left\{\eta_{2,4} = \frac{(v^2\omega^2 + D_4^2)^2}{D_4}, \varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4}\right\}$$

is a fundamental system of units of the quartique real field $K_4 = \mathbb{Q}(\sqrt[4]{M_4})$

Proof: Remains to verify that the unit ε_1 satis_es the property (4.40), and that K_4 is of the first kind (i.e.: $\sqrt{(\eta_{2,4})^{-1}\varepsilon_1} \notin K_4$). In fact,

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4}$$

is a unit of K_4 of norm 1 because

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4} = 2(D_4)^3 \mp 1 + 2(D_4)^2 v\omega + 2D_4 v^2\omega^2 + 2v^3\omega^3$$

is an algebraic integer such that

$$N_{K/\mathbb{Q}}(\varepsilon_1) = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right)\left(\frac{iv\omega + D_4}{iv\omega - D_4}\right)\left(\frac{-iv\omega + D_4}{-iv\omega - D_4}\right) = 1$$

Lemma 4.18 *Let* $\varepsilon_1 = \pm\left(\frac{v\omega+D_4}{v\omega-D_4}\right)$ *et* $\varepsilon_0 = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3)$ *with* $x_i \epsilon \mathbb{Z}$. *Assume that* $\varepsilon_1 = \varepsilon_0^n$ *with* $n \geq 2$. *Then*

1. *Case "+":* $|x_3| < \frac{3}{\omega^3} + \frac{(v^3)\sqrt{8}}{D_4}$
2. *Case "-":* $|x_3| < \frac{3}{\omega^3} + \frac{(v^3 D_4)\sqrt{8D_4}}{(tv^4)^3}$

Proof: Since $v^4\omega^4 = (D_4)^4 \pm D_4$, then

$$(*_0)\begin{cases} Case\ "+": we\ have\ D_4 < v\omega < D_4 + 1, \\ Case\ "-": we\ have\ D_4 - 1 < v\omega < D_4 \end{cases}$$

But

$$\varepsilon_1 = \pm\frac{v\omega + D_4}{v\omega - D_4} = 2(D_4)^3 \mp 1 + 2(D_4)^2 v\omega + 2D_4 v^2\omega^2 + 2v^3\omega^3 \tag{4.44}$$

which gives us

$$(*_1)\begin{cases} Case\ "+": 8(D_4)^3 < \varepsilon_1 < 8\frac{(v\omega)^4}{D_4}, \\ Case\ "-": 8\frac{(v\omega)^4}{D_4} < \varepsilon_1 < 8(D_4)^3 \end{cases}$$

then

$$(*_2)\begin{cases} Case\ "+": \frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{\sqrt{8}}{D_4}, \\ Case\ "-": \frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{(D_4)\sqrt{8D_4}}{(tv^4)^3} \end{cases}$$

in fact

Case "+": According to $(*_1)$, $\sqrt{\varepsilon_1} < (\sqrt{8})\left(\frac{(v\omega)^2}{\sqrt{D_4}}\right)$, and then one applies $(*_0)$.

Case "-": According to $(*_1)$, $\frac{\varepsilon_1}{(v\omega)^6} < \frac{8(D_4)^3}{(v\omega)^6}$, according to $(*_0)$, $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{(D_4)\sqrt{8D_4}}{(D_4-1)^3}$, finally $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3} < \frac{(D_4)\sqrt{8D_4}}{(tv^4)^3}$

Then we have

$$\varepsilon_1\varepsilon_1^{(1)} = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right) = 1$$

According to (4.43)

$$0 < |x_3| < \frac{1}{\omega^3}(3 + \sqrt{\varepsilon_1})$$

Then

$$0 < |x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{\varepsilon_1}}{(v\omega)^3} \tag{4.45}$$

replacing $\frac{\sqrt{\varepsilon_1}}{(v\omega)^3}$, by $\frac{\sqrt{8}}{D_4}$ in Case "+", and by $\frac{(D_4)\sqrt{8D_4}}{(tv^4)^3}$ in Case "-", conclude using $(*_2)$

Lemma 4.19 *Suppose that m_4 is square free. Then*

$$\varepsilon_1 = \pm\left(\frac{v\omega + D_4}{v\omega - D_4}\right)$$

is the smallest unit of K_4 which satisfies

$$\varepsilon_1\varepsilon_1^{(1)} = 1$$

Proof: recall that $v \geq 2$. We have then

$$\varepsilon_1\varepsilon_1^{(1)} = \left(\frac{v\omega + D_4}{v\omega - D_4}\right)\left(\frac{-v\omega + D_4}{-v\omega - D_4}\right) = 1$$

Argue by contradiction. Then according to (4.41) we have,

$$\varepsilon_1 = \varepsilon_0^n, \ \ ou \ \ \varepsilon_0 = \tfrac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \text{ with } x_{\ i}\epsilon\mathbb{Z}.$$

According to lemma 4.18, we have

(i) Case "+",

$$|x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{D_4} \tag{4.46}$$

Since $\omega > 3,$ then $\frac{3}{\omega^3} < \frac{1}{9}$. Then

$$|x_3| < \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{D_4} = \frac{3}{\omega^3} + \frac{v^3\sqrt{8}}{tv^4 - 1} < \frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1}$$

But

$$\frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1} < 1 \Leftrightarrow \ 4v^4 - (9\sqrt{2})v^3 - 4 > 0$$

This is true for $v > 3$. Then for $v > 3$

$$|x_3| < \frac{1}{9} + \frac{v^3\sqrt{8}}{v^4 - 1} < 1$$

Then $x_3 = 0$, contradiction with (4.43). the result remains true for $v \in \{2,3\}$, because for $v = 3$, just directly replace in (4.46). The same for $v = 2$ and $t \geq 2$, just replace directly in (4.46). For $(v, t) = (2,1)$, just replace directly in (4.45).

In the following, we do not treat the first two values of $v,(v=2,3)$, because the result is the same by the same argument.

(ii) Case "-",

$$|x_3| < \frac{3}{\omega^3} + \frac{v^3 D_4\sqrt{8D_4}}{(tv^4)^3} < \frac{1}{9} + \left(\frac{v^3(2tv^4)(\sqrt{2}\sqrt{t}\sqrt{8})v^2}{t^3v^{12}}\right) < \frac{1}{9} + \frac{\sqrt{8}}{v^3} < 1$$

Then we have the same contradiction.

We show that $\xi = \sqrt{(\eta_{2,4})^{-1}\varepsilon_1} \notin K_4$. But

$$\xi = \sqrt{(\eta_{2,4})^{-1}\varepsilon_1} = \sqrt{\frac{\varepsilon_1 D_4}{(v^2\omega^2 + (D_4)^2)^2}} = \frac{1}{(v^2\omega^2 + (D_4)^2)^2}\sqrt{\varepsilon_1 D_4}$$

If $\xi \in K_4$ then $\sqrt{\varepsilon_1 D_4} \in K_4$. According to (4.42), we have

$$\sqrt{\varepsilon_1 D_4} = \frac{1}{4}(x_0 + x_1\omega + x_2\omega^2 + x_3\omega^3) \text{ with } x_0, x_1, x_2, x_3 \epsilon \mathbb{Z}.$$

Using (4.39), we have

$$|x_3| \leq \left(\frac{1}{\omega^3}\right)\sum_{j=0}^{3}\sqrt{|(D_4\varepsilon_1)^{(j)}|}$$

Then

$$|x_3| < \frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1} + 1 + 2\sqrt{D_4}\right)$$

According to $(*_0)$ we have

Case “+”,

$$|x_3| < \frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1} + 1 + 2\sqrt{D_4}\right) < \left(\frac{1}{v - \frac{1}{v^3}} + \frac{1}{(v - \frac{1}{v^3})^3} + \frac{1}{(v^4 - \frac{1}{v^2})^2}\right) < 1$$

Case “-“,

$$|x_3| < \frac{1}{\omega^3}\left(\sqrt{D_4\varepsilon_1} + 1 + 2\sqrt{D_4}\right)$$

$$< \frac{v^3(D_4)^2\sqrt{8}}{(D_4)^3 - 3(D_4)^2 + 3D_4 - 1} + \frac{1}{\omega^3} + \frac{2v^3\sqrt{D_4}}{(D_4)^3 - 3(D_4)^2 + 3D_4 - 1}$$

$$< \frac{\sqrt{8}}{v - \frac{3}{v^3} + \frac{3}{D_4 v^3} - \frac{1}{(D_4)^2 v^3}} + \frac{1}{\omega^3} + \frac{2}{v + \frac{3}{v^3} - \frac{1}{D_4 v^3}} < 1$$

Then $x_3 = 0$; then the same contradiction arises. This completes the demonstration of theorem 4.17.

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