NEW TRENDS IN MATHEMATICAL SCIENCES
Vol. 1, No. 1, 2013, p.100-106
ISSN 2147-5520 - www.ntmsci.com

On a Special Type Nearly Quasi-Einstein Manifold

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Abstract: In the present paper, we consider a special type of nearly quasi-Einstein manifold denoted by $N(\mathrm{QE})_n$. Most of the sections are based on some properties of $N(\mathrm{QE})_n$. We give some theorems about these manifolds. In the last section, a special type nearly quasi-Einstein spacetime is investigated.

Keywords: Quasi-Einstein manifold, nearly quasi-Einstein manifold, spacetime.

1. Introduction

A non-flat n-dimensional Riemannian or a semi-Riemannian manifold (M,g) $(n>2)$ is said to be an Einstein manifold if the condition

$$S(X,Y)=\frac{r}{n}g(X,Y) \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M,g), respectively.

Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat n-dimensional Riemannian manifold (M,g) $(n>2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following condition

$$S(X,Y)=ag(X,Y)+bA(X)A(Y) \tag{1.2}$$

where $a,b\in\mathbb{R}$ and A is a non-zero 1-form such that

$$g(X,U)=A(X) \tag{1.3}$$

for all vector fields X on M, [4]. Then A is called the associated 1-form and U is called the generator of the manifold.

Also M.C. Chaki and R.K. Maity [1] studied the quasi-Einstein manifolds by considering a and b as scalars such that $b\neq 0$ and U as a unit vector field.

In 2008, U.C. De and A.K. Gazi [2] introduced the notion of nearly quasi-Einstein manifold. A non-flat n-dimensional Riemannian manifold (M,g) $(n>2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bE(X,Y) \tag{1.4}$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type (0,2).

Then E is called the associated tensor and a and b are called the associated scalars of M. An n-dimensional nearly quasi-Einstein manifold is denoted by $N(\mathrm{QE})_n$. An example of $N(QE)_4$ has been given in [2].

The nearly quasi-Einstein manifolds have also studied by A.K. Gazi, U.C. De [5], D.G. Prakasha, C.S. Bagewadi [7] and R.N. Singh, M.K. Pandey, D. Gautam [8].

In [8], R.N. Singh, M.K. Pandey, D. Gautam consider a type of nearly quasi-Einstein manifold whose associated tensor E of type (0,2) is in the form

$$E(X,Y) = A(X)B(Y) + B(X)A(Y) \tag{1.5}$$

where A and B are non-zero 1-forms associated with orthogonal unit vector fields V and U, i.e.,

$$g(U,U) = 1, \quad g(V,V) = 1 \quad \text{and} \quad g(U,V) = 0. \tag{1.6}$$

These vector fields are defined by

$$g(X,U) = A(X), \quad g(X,V) = B(X)$$

for every vector field X.

In the present paper, we consider a special type of nearly quasi-Einstein manifold, $N(\mathrm{QE})_n$, whose associated tensor E is of the form (1.5) with the condition (1.6). Some theorems about this manifold are proved and some properties are obtained.

2. A Special Type Nearly Quasi-Einstein Manifold

In this section, we consider a special type of $N(\mathrm{QE})_n$ whose Ricci tensor satisfies the conditions (1.5) and (1.6), i.e., it satisfies the following condition

$$S(X,Y) = ag(X,Y) + b[A(X)B(Y) + B(X)A(Y)] \tag{2.1}$$

where A and B are non-zero 1-forms, a and b are non-zero scalars.

Definition 1. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the following condition

$$\nabla_X \xi = \rho X + \phi(X)\xi \tag{2.2}$$

where $X \in TM$, ϕ is a linear form and ρ is a function, [10].

In the local transcription, this reads

$$\nabla_i \xi^h = \rho\, \delta_i^h + \xi^h \phi_i \tag{2.3}$$

where ξ^h and ϕ_i are the components of ξ and ϕ, and δ_i^h is the Kronecker symbol.

A torse-forming vector field ξ is called

i) recurrent, if $\rho = 0$,

ii) concircular, if the form ϕ_i is a gradient covector, i.e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$,

iii) convergent, if it is concircular and $\rho = const.exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X \xi = \phi(X)\xi. \tag{2.4}$$

Also, from the Definition 1, for a concircular vector field ξ, we get

$$(\nabla_Y \xi)X = \rho g(X,Y) \tag{2.5}$$

for all $X, Y \in TM$.

Theorem 2.1. Let V_n be a $N(QE)_n$ satisfying the condition (2.1) and let U and V be the vector fields corresponding to the associated 1-forms A and B, respectively. Thus, the vector fields U and V cannot be concircular vector fields.

Proof. We consider a special type $N(QE)_n$ satisfying the condition (2.1). Let U and V corresponding to the associated 1-forms A and B be concircular vector fields, respectively. In local coordinates, thus we have

$$\nabla_i A_j = \rho g_{ij} \tag{2.6}$$

and

$$\nabla_i B_j = \sigma g_{ij} \tag{2.7}$$

where ρ and σ are non-zero scalar functions.

Taking the covariant derivative of the condition $g(U,U) = 1$, it is found that

$$(\nabla_j A_i) A^i = 0 \tag{2.8}$$

where $A^i = g^{ih} A_h$ and h is the arbitrary choice for indexing and the summation runs from 1 to n.

Multiplying (2.6) by A^j and using the equation (2.8), we get

$$\rho A_i = 0$$

which contradicts to the fact that ρ is a non-zero scalar function and A is a non-zero 1-form. Similarly, it can be shown that the generator V cannot be a concircular vector field. In this case, $N(QE)_n$ satisfying the condition (2.1) does not admit concircular vector fields U and V corresponding to the associated 1-forms A and B, respectively. Hence, the proof is completed.

Definition 2. A quadratic conformal Killing tensor is defined as a second order symmetric tensor T satisfying the condition

$$\begin{aligned}(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) \quad &+ (\nabla_Z T)(X,Y) = \alpha(X) g(Y,Z) \\ &+ \alpha(Y) g(Z,X) + \alpha(Z) g(X,Y)\end{aligned} \tag{2.9}$$

where α is a 1-form, [9].

Now, we consider a $N(QE)_n$ admitting a generator vector as a torse-forming vector field and the other be not. If we assume that the generator U is a torse-forming vector field, then we have from (1.6) and (2.3)

$$\nabla_j A_i = \rho(g_{ij} - A_i A_j) \tag{2.10}$$

where ρ is a scalar function.

Taking the covariant derivative of the condition $g(U,V)=0$ and using the equation (2.10), it can be seen that

$$A^i(\nabla_k B_i) = -\rho B_k. \tag{2.11}$$

By the aid of (2.9), (2.10) and (2.11), we prove the following theorem.

Theorem 2.2. Let V_n be a $N(QE)_n$ satisfying the condition (2.1) and admitting the Ricci tensor as a quadratic conformal Killing tensor. If the vector field U generated by the 1-form A is a torse-forming vector field and the other vector field V generated by the 1-form B is not, then the vector field V is divergence-free.

Proof. Suppose that the Ricci tensor of a $N(QE)_n$ satisfying the condition (2.1) is a quadratic conformal Killing tensor. In this case, in local coordinates, we have from (2.9)

$$\nabla_k S_{ij} + \nabla_i S_{jk} + \nabla_j S_{ki} = \alpha_k g_{ij} + \alpha_i g_{jk} + \alpha_j g_{ki} \tag{2.12}$$

where α is a 1-form.

Taking the covariant derivative of (2.1), we get

$$\begin{aligned} \nabla_k S_{ij} &= a_k g_{ij} + b_k(A_i B_j + A_j B_i) \\ &+ b((\nabla_k A_i)B_j + A_i(\nabla_k B_j) + (\nabla_k A_j)B_i + A_j(\nabla_k B_i)) \end{aligned} \tag{2.13}$$

where a and b are the associated scalars of this manifold and $a_k = \partial_k a$, $b_k = \partial_k b$.

If the vector field U generated by the 1-form A is a torse-forming vector field, then we have the relation (2.10). Changing the indices by cyclic in (2.13), using (2.10) and (2.12), it can be obtained that

$$\begin{aligned} &(a_k + 2b\rho B_k - \alpha_k)g_{ij} + (a_i + 2b\rho B_i - \alpha_i)g_{jk} + (a_j + 2b\rho B_j - \alpha_j)g_{ik} \\ &+ b_k(A_i B_j + A_j B_i) + b_i(A_j B_k + A_k B_j) + b_j(A_k B_i + A_i B_k) \\ &+ b(A_i(\nabla_k B_j) + A_j(\nabla_k B_i) + A_j(\nabla_i B_k) + A_k(\nabla_j B_i) + A_k(\nabla_i B_j) + A_i(\nabla_j B_k)) \\ &- 2b\rho(A_i A_k B_j + A_j A_k B_i + A_i A_j B_k) = 0. \end{aligned} \tag{2.14}$$

Multiplying (2.14) by g^{ij} and considering (2.11), we get

$$\begin{aligned} &(n+2)(a_k + 2b\rho B_k - \alpha_k) + 2b_i(A^i B_k + A_k B^i) \\ &- 4b\rho B_k + 2b(A^i(\nabla_i B_k) + A_k(\nabla_i B^i)) = 0. \end{aligned} \tag{2.15}$$

Moreover, multiplying (2.15) by A^k and B^k, respectively, and using the condition (1.6), we obtain the following equations

$$(n+2)(a_k - \alpha_k)A^k + 2b_k B^k + 2b\nabla_k B^k = 0 \tag{2.16}$$

$$(n+2)(a_k - \alpha_k)B^k + 2nb\rho + 2b_k A^k = 0. \tag{2.17}$$

On the other hand, multiplying (2.14) by $A^i A^j A^k$ and using (2.11), it is found that

$$(a_k - \alpha_k)A^k = 0. \tag{2.18}$$

Multiplying (2.14) by $B^i B^j A^k$, we find

$$(a_k - \alpha_k)A^k + 2b_k B^k = 0. \tag{2.19}$$

Since b is a non-zero scalar function, from (2.16), (2.18) and (2.19), it can be seen that

$$\nabla_k B^k = 0.$$

Thus, the vector field V generated by the 1-form B is divergence-free. This completes the proof.

Definition 3. A non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$ is called a generalized Ricci-recurrent manifold if its Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y, Z) = \gamma(X)S(Y, Z) + \delta(X)g(Y, Z) \tag{2.20}$$

where γ and δ are non-zero 1-forms, [3]. If $\delta = 0$, then the manifold reduces to a Ricci-recurrent manifold, [6].

Theorem 2.3. Let $N(QE)_n$ be a generalized Ricci-recurrent manifold. Thus, the vector fields U and V generated by the 1-forms A and B cannot be torse-forming vector fields.

Proof. We consider that V_n is a $N(QE)_n$ satisfying the condition (2.1). In this case, in local coordinates, we have the equation (2.13) by Theorem 2.2. Let the vector field U generated by the 1-form A be a torse-forming vector field and the other be not. Then the relation (2.10) is satisfied. If we suppose that V_n is a generalized Ricci-recurrent manifold, by the aid of (2.10), (2.13) and (2.20), we obtain

$$\begin{aligned}&(a_k - \delta_k - a\gamma_k)g_{ij} + (b_k - b\gamma_k)(A_i B_j + A_j B_i) + b[\rho(g_{ik} - A_i A_k)B_j \\ &+ A_i(\nabla_k B_j) + \rho(g_{jk} - A_j A_k)B_i + A_j(\nabla_k B_i)] = 0\end{aligned} \tag{2.21}$$

where γ_k and δ_k denote the components of the 1-forms γ and δ.

Multiplying (2.21) by g^{ij} and using the condition (2.11), it can be seen that

$$a_k = \delta_k + a\gamma_k. \tag{2.22}$$

Moreover, multiplying (2.21) by $A^i A^j$ and using (1.6), we get

$$a_k - \delta_k - a\gamma_k + 2bA^i(\nabla_k B_i) = 0. \tag{2.23}$$

By the aid of (2.11), (2.22) and (2.23), it is found that

$$b\rho B_k = 0$$

which contradicts to the fact that b and ρ are non-zero scalar functions and B is a non-zero 1-form. Therefore, the vector field U of this manifold cannot be a torse-forming vector field. By similar calculations it can be easily obtained that the vector field V of this manifold also cannot be a torse-forming vector field. Thus, the proof is completed.

3. A Special Type $N(QE)_n$ Spacetime

In this section, we will examine $N(QE)_4$ spacetime which will be denoted by $N(QES)_4$ satisfying the condition (2.1).

The Einstein field equations (EFE) without cosmological constant is written as the following form

$$kT(X,Y) = S(X,Y) - \frac{r}{2} g(X,Y) \tag{3.1}$$

where S is the Ricci tensor, r is the scalar curvature, g is the metric tensor, k is a constant and T is the energy-momentum tensor.

Theorem 3.1. In a $N(QES)_4$ satisfying the condition (2.1), the trace of the energy-momentum tensor is constant if and only if the associated scalar a is constant.

Proof. Let us consider a $N(QES)_4$ satisfying the condition (2.1). From (3.1) and (2.1), it is obtained that

$$kT(X,Y) = (a - \frac{r}{2}) g(X,Y) + b(A(X)B(Y) + A(Y)B(X)). \tag{3.2}$$

Moreover, using (2.1), the scalar curvature of a $N(QES)_4$ is found as

$$r = 4a. \tag{3.3}$$

From (3.2) and (3.3), we have

$$kT(X,Y) = -ag(X,Y) + b(A(X)B(Y) + A(Y)B(X)). \tag{3.4}$$

Contracting (3.4) over X and Y, we obtain

$$\tilde{T} = -\frac{4}{k} a \tag{3.5}$$

where $\tilde{T}$ denotes the trace of the energy-momentum tensor.

It follows from (3.5) that if the associated scalar a is constant, then the trace of the energy-momentum tensor is constant. The converse is also true. Hence, the proof is completed.

Theorem 3.2. In a perfect fluid $N(QES)_4$ spacetime satisfying the condition (2.1) with the constant associated scalar a, the change of the isotropic pressure is proportional to the change of the energy density.

Proof. In a perfect fluid spacetime, the energy-momentum tensor is in the form

$$T(X,Y) = (\sigma + p)\lambda(X)\lambda(Y) + pg(X,Y) \tag{3.6}$$

where σ is the energy density, p is the isotropic pressure and λ is a non-zero 1-form such that $g(X,V) = \lambda(X)$ for all X, V being the velocity vector field of the flow, that is, $g(V,V) = -1$. Also, $\sigma + p \neq 0$.

Using (3.6) in (3.1) and contracting the resulting equation over X and Y, and considering the condition $g(V,V) = -1$ and (3.3), it can be seen that

$$3p - \sigma = -\frac{4}{k} a \tag{3.7}$$

where a is the associated scalar of the manifold and k is a constant.

If the associated scalar a of $N(QES)_4$ is constant, then taking the covariant derivative of the equation (3.7) yields

$$3\nabla_Z p = \nabla_Z \sigma \tag{3.8}$$

for all vector fields Z .

Thus, the change of the isotropic pressure is proportional to the change of the energy density. This completes the proof.

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