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Research Article

On basicity of exponential and trigonometric systems in grand Lebesgue spaces

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Abstract

Basis properties of exponential and trigonometric systems in grand Lebesgue spaces $L_{p)}(-\pi,\pi)$ are studied. Based on a shift operator, we consider the subspace $G_{p)}(-\pi,\pi)$ of the space $L_{p)}(-\pi,\pi)$, where continuous functions are dense, and the boundedness of the singular operator in this subspace is proved. We establish the basicity of exponential system $\{e^{int}\}_{n\in Z}$ for $G_{p)}(-\pi,\pi)$ and the basicity of trigonometric systems $\{\sin nt\}_{n\in N}$ and $\{\cos nt\}_{n\in N_0}$ for $G_{p)}(0,\pi)$.

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1. Introduction

Recently there arose great interest in the nonstandard spaces. Many classical facts about harmonic analysis have been extended to these spaces (see, for example, [1, 11, 12, 18, 21, 24, 27, 28], etc.). Approximation properties are also of interest in suchlike spaces. These properties have been relatively well studied in generalized Lebesgue spaces in the works [2,4,9,13,15,17,29,31–33]. These properties are different with the cases of Morrey-type and grand Lebesgue spaces: only recently the approximation matters began to be studied in these spaces, and many problems are still open. Apparently, the works [1–3,5,6,23,28,36] have been pioneers in this field.

Iwaniec and Sbordone [24] introduced so called grand Lebesgue spaces when studying the integrability properties of Jacobian in an open bounded set. These spaces are functional Banach spaces, and they have important applications in the theory of partial differential equations, interpolation theory, etc. Various properties of grand Lebesgue spaces have been studied in [14,16,19,21,30]. Properties based on harmonic analysis are of

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special interest in these spaces (see [27]). Later, the researches have appeared which studied the associated spaces of these spaces known as the small Lebesgue spaces (see [10,20]). As applications to the applications in the theory of differential equations, it is important to study the boundedness of integral operators (see, e.g., [11,22,25–27]). In [22,27], the equivalence of the boundedness of maximal operator in the weighted grand Lebesgue space to the Muckenhoupt condition has been proved. Similar result for one-dimensional singular operator has been obtained in [26,27].

Note that some direct and inverse theorems of the theory of approximations in grand Lebesgue spaces have been proved in [23]. Basicity problems in these spaces have not yet been considered because of their non-separability. Therefore, there is a need to consider a subspace required in the theory of differential equations and to study basis properties of classical systems in this subspace. Questions of solvability of Dirichlet problems for elliptic equations were considered in [7,8]. Korovkin-type theorems, as well as spectral problems with a spectral parameter in boundary conditions in grand Lebesgue spaces, were studied in [34] and [35], respectively. Note that the problem considered in [35] in Morrey-type spaces was studied in [3].

In this work, we study the basis properties of the systems of exponents, sines and cosines in the grand Lebesgue space $L_{p)}(-\pi,\pi)$. Based on shift operator, we consider the subspace $G_{p)}(-\pi,\pi)$ of $L_{p)}(-\pi,\pi)$ where continuous functions are dense. The boundedness of the singular operator in the subspace $G_{p)}(-\pi,\pi)$ is established. We prove that the classical system of exponents $\{e^{int}\}_{n\in Z}$ forms a basis for $G_{p)}(-\pi,\pi)$ and the trigonometric systems $\{\sin nt\}_{n\in N}$ and $\{\cos nt\}_{n\in N_0}$ form bases for $G_{p)}(0,\pi)$.

2. Some concepts and auxiliary results

Let us recall standard notation. Throughout this paper: N denotes the set of natural numbers; Z the set of integers; N_0 the set of non-negative integers; By $C_0^\infty[\pi,\pi]$ we will denote the space of infinitely differentiable finite functions on a segment $[-\pi,\pi]$; $L(X,Y)$ will be the space of linear bounded operators acting from Banach spaces X to Y, in particular $L(X,X)=L(X)$; $\overline{A}$ is closure of a set A in a Banach space X; δ_{nk} is the Kronecker symbol.

Let $L_{p)}(-\pi,\pi)$ be the grand Lebesgue space of measurable functions f on $[-\pi,\pi]$ with the norm

$$\|f\|_{L_{p)}(-\pi,\pi)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}.$$

$L_{p)}(-\pi,\pi)$ is a non-separable Banach space (see [11]).

Let's take an arbitrary function $f\in L_{p)}(-\pi,\pi)$. Let us extend the function f by zero to the entire axis R , i.e. $f(t)=0,\ t\in R\setminus[-\pi,\pi]$. Consider the set $\tilde{G}_{p)}(-\pi,\pi)$ of functions $f\in L_{p)}(-\pi,\pi)$ satisfying the condition

$$||f(\cdot+\delta)-f(\cdot)||_{L_{p)}(-\pi,\pi)}\to 0, \quad \delta\to 0.$$

It is clear that $\tilde{G}_{p)}(-\pi,\pi)$ is a linear manifold in $L_{p)}(-\pi,\pi)$. Let $\overline{\tilde{G}_{p)}(-\pi,\pi)}=G_{p)}(-\pi,\pi)$. There is a strict continuous embedding $L_p(-\pi,\pi)\subset G_{p)}(-\pi,\pi)$([35], Lemma 2.2):

$$||f(\cdot)||_{L_{p)}(-\pi,\pi)} \le (p-1)||f(\cdot)||_{L_p(-\pi,\pi)}, \qquad \forall f\in L_p(-\pi,\pi). \tag{2.1}$$

Moreover, the set $C_0^\infty[-\pi,\pi]$ is dense in $G_{p)}(-\pi,\pi)$ (see [34,35]).

Next, we need the fact that the singular integral is bounded in grand Lebesgue spaces.

Theorem 2.1 ([26]). *Let Γ be a simple rectifiable curve. In order for the singular operator S_Γ defined by the formula*

$$S_\Gamma(f)(t)=\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi-t}d\xi, \quad t\in\Gamma,$$

is bounded in $L_{p)}(\Gamma)$, $1 < p < +\infty$, *it is necessary and sufficient than the curve* Γ *be Carleson, i.e. when*

$$\exists c_0 > 0 : |D(t, r)| \leq c_0 r,$$

where $D(t, r) = \Gamma \cap B(t, r)$, $\quad B(t, r) = \{z \in C : |z - t| < r\}, t \in \Gamma$.

We present the notion of a double basis associated with a system of exponentials.

Definition 2.2. A system $\{x_n\}_{n\in Z}$ of elements from X is called a basis in X if for $\forall x \in X$ there is a unique sequence of scalars $\{a_n\}_{n\in Z}$ such that

$$x = \sum_{n\in Z} a_n x_n,$$

i.e.

$$x = \lim_{n,m\to\infty} \sum_{k=-m}^{n} a_k x_k.$$

Suppose that the system $\{x_n\}_{n\in Z}$ forms a basis in a Banach space X and K is the space of sequences of coefficients in the expansion of elements in terms of the basis, i.e. the space of sequences of scalars $\{a_n\}_{n\in Z}$ for which the series $\sum_{n\in Z} a_n x_n$ converges. Let us show that K is a Banach space with the norm

$$\|\{a_n\}_{n\in Z}\|_K = \sup_{n,m\in N_0} \left\| \sum_{k=-m}^{n} a_k x_k \right\|_X .$$

The validity of the axioms of the norm is obvious. Let $a^{(i)} = \{a_n^{(i)}\}_{n\in Z}$, $\quad i \in N$, be a fundamental sequence in K. Then

$$\forall \varepsilon > 0 \quad \exists i_0(\varepsilon) : \forall i, j > i_0 \quad \left\|a^{(i)} - a^{(j)}\right\|_K = \sup_{n,m\in N_0} \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X < \varepsilon. \quad (2.2)$$

From (2.2) it follows that for $\forall n, m \in N_0$

$$\forall \varepsilon > 0 \quad \exists i_0(\varepsilon) : \forall i, j > i_0 \quad \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X < \varepsilon. \quad (2.3)$$

Then, taking into account (2.3), we obtain

$$\begin{aligned} \left\|(a_n^{(i)} - a_n^{(j)}) x_n\right\|_X &= \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k^{(j)}) x_k - \sum_{k=-m}^{n-1} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X \\ &\leq \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X + \left\| \sum_{k=-m}^{n-1} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X < 2\varepsilon, \end{aligned}$$

as well as

$$\left\|(a_{-m}^{(i)} - a_{-m}^{(j)}) x_{-m}\right\|_X = \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k^{(j)}) x_k - \sum_{k=-m+1}^{n} (a_k^{(i)} - a_k^{(j)}) x_k \right\|_X < 2\varepsilon.$$

Thus, for $\forall k \in Z$ the sequence $\{a_k^{(i)}\}_{i\in N}$ is a fundamental sequence of numbers. So for $\forall k \in Z$ the sequence of numbers $\{a_k^{(i)}\}_{i\in N}$ converges. Let $a_k = \lim_{i\to\infty} a_k^{(i)}$. It remains to check $\{a_k\}_{k\in Z} \in K$ and the convergence of the sequence $\{a_k^{(i)}\}_{k\in Z}$ with the limit $\{a_k\}_{k\in Z}$.

Let $S_n^+ = \sum_{k=0}^{n} a_k x_k$ and $S_m^- = \sum_{k=-m}^{0} a_k x_k, \quad n, m \in N$. Let $\forall \delta > 0$ and $\forall p \in N$. Let $\varepsilon > 0$ so that $4\varepsilon < \delta$. Passing to the limit in (2.3) for $j \to \infty$, we obtain

$$\forall i > i_0(\varepsilon), \quad \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k) x_k \right\|_X \leq \varepsilon \quad \forall n, m \in N_0. \tag{2.4}$$

Since the series $\sum_{k \in Z} a_k^{(i)} x_k, \quad i > i_0(\varepsilon)$ converges, we obtain that for $\forall n \in N_0$

$$\exists n_0 = n_0(i), \quad \forall n > n_0, \quad \left\| \sum_{k=0}^{n} a_k^{(i)} x_k - \sum_{k=0}^{n+p} a_k^{(i)} x_k \right\|_X < \frac{\delta}{2}. \tag{2.5}$$

So, using (2.4) and (2.5), for $\forall n > n_0$ we get

$$\begin{aligned} \left\| S_n^+ - S_{n+p}^+ \right\|_X &= \left\| \sum_{k=0}^{n} (a_k - a_k^{(i)}) x_k + \sum_{k=0}^{n+p} (a_k^{(i)} - a_k) x_k + \sum_{k=0}^{n} a_k^{(i)} x_k - \sum_{k=0}^{n+p} a_k^{(i)} x_k \right\|_X \\ &\leq \left\| \sum_{k=0}^{n} (a_k^{(i)} - a_k) x_k + \sum_{k=0}^{n+p} (a_k^{(i)} - a_k) x_k \right\|_X + \left\| \sum_{k=0}^{n} a_k^{(i)} x_k - \sum_{k=0}^{n+p} a_k^{(i)} x_k \right\|_X \\ &\leq \left\| \sum_{k=0}^{n} (a_k^{(i)} - a_k) x_k \right\|_X + \left\| \sum_{k=0}^{n+p} (a_k^{(i)} - a_k) x_k \right\|_X + \frac{\delta}{2} < \varepsilon + \varepsilon + \frac{\delta}{2} < \delta. \end{aligned}$$

Therefore, the sequence S_n^+ is fundamental. Then exists $\lim_{n \to \infty} S_n^+$. The existence of the limit $\lim_{m \to \infty} S_m^-$ can be proven similarly. So the series $\sum_{n \in Z} a_n x_n$ converges, and so $\{a_k\}_{k \in Z} \in K$. Next, from (2.4) we obtain

$$\forall i > i_0(\varepsilon) \quad \left\| a^{(i)} - a \right\|_K = \sup_{n, m \in N_0} \left\| \sum_{k=-m}^{n} (a_k^{(i)} - a_k) x_k \right\|_X \leq \varepsilon,$$

i.e. the sequence $\{a_k^{(i)}\}_{k \in Z}$ converges to $\{a_k\}_{k \in Z}$ in space K.

Let the operator $F : K \to X$ be defined by the formula $Fa = \sum_{n \in Z} a_n x_n$. It is clear that $\|F\| \leq 1$ and $F : K \to X$ forms an isomorphism.

We will need the following criterion for the basis property of a double system.

Theorem 2.3. *Let X be a Banach space. The system of elements $\{x_n\}_{n \in Z}$ from X forms a basis in X if and only if the following conditions hold:*

(1) *system $\{x_n\}_{n \in Z}$ is complete in X;*
(2) *system $\{x_n\}_{n \in Z}$ is minimal X;*
(3) $\exists M > 0 \quad \forall x \in X : \left\| \sum_{k=-m}^{n} f_k(x) x_k \right\|_X \leq M \|x\|_X, \quad \forall n, m \in N_0,$

where $\{f_n\}_{n \in Z}$ is the biorthogonal system to $\{x_n\}_{n \in Z}$.

Proof. Necessary. Condition 1) is obvious. Let's take an arbitrary $x \in X$. Let $x = \sum_{n \in Z} a_n x_n$. For $\forall k \in Z$ we define a linear functional f_k by the formula $f_k(x) = a_k$. It's clear that $f_k(x_n) = \delta_{kn}, \forall k, n \in Z$. For $\forall n, m \in N_0$ we have

$$\begin{aligned}
|f_n(x)| &= \frac{1}{\|x_n\|}\|a_n x_n\| = \frac{1}{\|x_n\|_X}\left\|\sum_{k=-m}^{n} a_k x_k - \sum_{k=-m}^{n-1} a_k x_k\right\|_X \\
&\le \frac{1}{\|x_n\|_X}\left\|\sum_{k=-m}^{n} a_k x_k\right\|_X + \frac{1}{\|x_n\|_X}\left\|\sum_{k=-m}^{n-1} a_k x_k\right\|_X \\
&\le \frac{2\|a\|_K}{\|x_n\|_X} \le \frac{2\|F^{-1}\|_{L(X,K)}}{\|x_n\|_X}\|x\|_X, \\
|f_{-m}(x)| &= \frac{1}{\|x_{-m}\|}\|a_{-m}x_{-m}\|_X = \frac{1}{\|x_{-m}\|_X}\left\|\sum_{k=-m}^{n} a_k x_k - \sum_{k=-m+1}^{n} a_k x_k\right\|_X \\
&\le \frac{2\|a\|_K}{\|x_{-m}\|_X} \le \frac{2\|F^{-1}\|_{L(X,K)}}{\|x_{-m}\|_X}\|x\|_X,
\end{aligned}$$

i.e. f_k is bounded in X. So $\{f_n\}_{n\in Z}$ and $\{x_n\}_{n\in Z}$ are biorthogonal systems. Therefore, the system $\{x_n\}_{n\in Z}$ is minimal in X.
Finally, for $\forall x \in X$ and $\forall n, m \in N_0$ we get

$$\left\|\sum_{k=-m}^{n} f_k(x)x_k\right\|_X \le \sup_{n,m\in N_0}\left\|\sum_{k=-m}^{n} f_k(x)x_k\right\|_X = \left\|F^{-1}x\right\| \le \left\|F^{-1}\right\|_{L(X,K)}\|x\|_X.$$

Sufficient. Take arbitrary $\varepsilon > 0$ and $x \in X$. Due to the completeness $\{x_n\}_{n\in Z}$ in X there exists $y = \sum_{k=-m}^{n} f_k(y)x_k, \quad n, m \in N_0$, such that $\|x - y\|_X < \varepsilon$. For $\forall n_1, m_1 \in N_0$ such that $n_1 > n$ and $m_1 > m$ it is clear that $y = \sum_{k=-m_1}^{n_1} f_k(y)x_k$. Hence

$$\begin{aligned}
\left\|x - \sum_{k=-m_1}^{n_1} f_k(x)x_k\right\|_X &= \left\|x - y + y - \sum_{k=-m_1}^{n_1} f_k(x)x_k\right\|_X \\
&\le \|x - y\|_X + \left\|\sum_{k=-m_1}^{n_1} f_k(x - y)x_k\right\|_X \\
&\le \|x - y\|_X + M\|x - y\|_X < \varepsilon(1 + M).
\end{aligned}$$

Thus $x = \sum_{n\in Z} f_n(x)x_n$.

□

3. On the basis property of the system of exponents and trigonometric systems in grand-Lebesgue spaces

Let us study the basic properties of the system of exponentials $\{e^{int}\}_{n\in Z}$ in the subspace $G_{p)}(-\pi, \pi)$ of the space $L_{p)}(-\pi, \pi), \quad 1 < p < +\infty$. We first calculate the norms of the functions belonging to this system. We have

$$\|e^{int}\|_{G_{p)}(-\pi,\pi)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|e^{int}\right|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1}\varepsilon^{\frac{1}{p-\varepsilon}} = p - 1.$$

The following theorem studies the minimality of the system of exponentials $\{e^{int}\}_{n\in Z}$ in the space $L_{p)}(-\pi, \pi)$.

Theorem 3.1. *The system $\{e^{int}\}_{n\in Z}$ is minimal in $L_{p)}(-\pi, \pi), \quad 1 < p < +\infty$.*

Proof. Consider a system of functionals according to the formula

$$\nu_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, \quad f \in L_{p)}(-\pi, \pi), \quad n \in Z.$$

Let us show the boundedness of the functional $\nu_n(\cdot)$. For any $\varepsilon \in (0, p-1)$ we have

$$
\begin{aligned}
|\nu_n(f)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(t)e^{-int}dt\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|dt \\
&\le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}(2\pi)^{\frac{1}{(p-\varepsilon)'}} \\
&= \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\varepsilon^{-\frac{1}{p-\varepsilon}} \le \varepsilon^{-\frac{1}{p-\varepsilon}}\|f\|_{L_{p)}(-\pi,\pi)},
\end{aligned}
$$

i.e. functional $\nu_n(\cdot)$ is bounded. On the other hand, for $\forall n, m \in Z$ we have

$$
\nu_n(e^{int}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-m)t}dt = \delta_{nm},
$$

i.e. the systems $\{e^{int}\}_{n\in Z}$ and $\nu_n(\cdot)$ are biorthogonal. Thus, the system $\{e^{int}\}_{n\in Z}$ is minimal in $L_{p)}(-\pi,\pi)$. □

The following theorem shows the completeness of the system of exponents $\{e^{int}\}_{n\in Z}$ in the space $G_{p)}(-\pi,\pi)$.

Theorem 3.2. *The exponential system $\{e^{int}\}_{n\in Z}$ is complete in space $G_{p)}(-\pi,\pi)$, $1 < p < +\infty$.*

Proof. Let's take an arbitrary $\eta > 0$ and an arbitrary function $f \in G_{p)}(-\pi,\pi)$. Then, due to the density $C_0^\infty[-\pi,\pi]$ in $G_{p)}(-\pi,\pi)$ (see [34], Lemma 3.1) there exists a function $g_\eta \in C_0^\infty[-\pi,\pi]$ such that

$$
\|f - g_\eta\|_{G_{p)}(-\pi,\pi)} < \eta.
$$

It is known that the sequence of partial sums of the Fourier series of the function g_η converges uniformly to g_η. Hence

$$
\exists m_0 \quad \forall m > m_0 \quad \sup_{t\in[-\pi,\pi]}|g_\eta(t) - P_m(t)| < \eta,
$$

where

$$
P_m(t) = \sum_{n=-m}^{m}\nu_n(g_\eta)e^{int}, \quad m \in N_0.
$$

Then

$$
\begin{aligned}
\|g_\eta - P_m\|_{G_{p)}(-\pi,\pi)} &= \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|g_\eta(t) - P_m(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \\
&< \eta\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{1}{p-\varepsilon}} = \eta(p-1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|f - P_m\|_{G_{p)}(-\pi,\pi)} &\le \|f - g_\eta\|_{G_{p)}(-\pi,\pi)} + \|g_\eta - P_m\|_{G_{p)}(-\pi,\pi)} \\
&< \eta + \eta(p-1) = p\eta.
\end{aligned}
$$

It follows that the system $\{e^{int}\}_{n\in Z}$ is complete in space $G_{p)}(-\pi,\pi)$.

□

Let $\gamma = \{\tau : |\tau| = 1\}$ be a unit circle. Consider the identification operator $T : L_{p)}(\gamma) \to L_{p)}(-\pi,\pi)$ defined by the formula $Tf(t) = f(e^{it})$, $t \in [-\pi,\pi]$. Denote by $G_{p)}(\gamma)$ the image of $G_{p)}(-\pi,\pi)$ under the mapping T^{-1}.

The theorem below asserts the invariance of $G_{p)}(\gamma)$ with respect to the singular operator S_γ.

Lemma 3.3. *The singular operator S_γ acts boundedly in $G_{p)}(\gamma), 1 < p < +\infty$.*

Proof. By Theorem 2.1, the operator S_γ is bounded in $L_{p)}(\gamma)$. Let $M = \|S_\gamma\|_{L(L_{p)}(-\pi,\pi))}$. Let us show that for $\forall f \in G_{p)}(\gamma) \quad S_\gamma f \in G_{p)}(\gamma)$. Take arbitrary $\varepsilon > 0$ and $f \in G_{p)}(\gamma)$. Due to the density $L_p(\gamma)$ in $G_{p)}(\gamma)$ there exists $g \in L_p(\gamma)$ such that

$$\|f - g\|_{G_{p)}(-\pi,\pi)} \leq \frac{\varepsilon}{M}.$$

Then

$$\|S_\gamma f - S_\gamma g\|_{L_{p)}(-\pi,\pi)} \leq \|S_\gamma\|_{L(L_{p)}(-\pi,\pi))} \|f - g\|_{G_{p)}(-\pi,\pi)} < \varepsilon.$$

Since $S_\gamma g \in L_p(\gamma)$, it follows from the last relation that $S_\gamma f$ belongs to the closure $L_p(\gamma)$ in $L_{p)}(\gamma)$, i.e. $S_\gamma f \in G_p(\gamma)$.

□

We now prove the following main theorem about the basis property of the system of exponentials $\{e^{int}\}_{n\in Z}$ in the space $G_{p)}(-\pi,\pi)$.

Theorem 3.4. *The exponential system $\{e^{int}\}_{n\in Z}$ forms a basis in the space $G_{p)}(-\pi,\pi)$, $1 < p < +\infty$.*

Proof. For an arbitrary $\eta > 0$ and an arbitrary function $f \in G_{p)}(-\pi,\pi)$. By virtue of Theorems 3.1 and 3.2, the exponential system $\{e^{int}\}_{n\in Z}$ is complete and minimal in $G_{p)}(-\pi,\pi)$. To prove the theorem according to the basis property criterion (Theorem 2.3) of the system, it suffices to show that the projectors are uniformly bounded in $G_{p)}(-\pi,\pi)$

$$S_{n,m}(f)(x) = \sum_{n=-m}^{n} \nu_n(f)e^{inx}, \quad n, m \in N_0.$$

It is easy to show that this is equivalent to the uniform boundedness of the system of projectors

$$P_m(f)(x) = \sum_{n=-m}^{m} \nu_n(f)e^{inx}, \quad f \in G_{p)}(-\pi,\pi), \quad m \in N_0.$$

.

Transform $P_m(f)$ as follows:

$$\begin{aligned} P_m(f)(x) &= \sum_{n=-m}^{m} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt\right) e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \sum_{n=-m}^{n} e^{in(x-t)}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)dt, \end{aligned}$$

where

$$K_m(t) = \frac{e^{-imt} - e^{i(m+1)t}}{1 - e^{it}}.$$

We have

$$K_m(x-t) = \frac{e^{-im(x-t)} - e^{i(m+1)(x-t)}}{1 - e^{i(x-t)}} = \frac{e^{i(m+1)t}}{e^{it} - e^{ix}}e^{-imx} - \frac{e^{imt}}{e^{it} - e^{ix}}e^{i(m+1)x}.$$

So

$$
\begin{aligned}
P_m(f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)K_m(x-t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)e^{i(m+1)t}}{e^{it}-e^{ix}}dte^{-imx} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(t)e^{-imt}}{e^{it}-e^{ix}}dte^{i(m+1)x} \\
&= \frac{1}{2\pi i}\int_{\gamma} \frac{T^{-1}(fe_m)(\tau)}{\tau - e^{ix}}d\tau e^{-imx} - \frac{1}{2\pi i}\int_{\gamma} \frac{T^{-1}(fe_{-(m+1)})(\tau)}{\tau - e^{ix}}d\tau e^{i(m+1)x} \\
&= e^{-imx}S_\gamma T^{-1}(fe_m)(e^{ix}) - e^{i(m+1)x}S_\gamma T^{-1}(fe_{-(m+1)})(e^{ix}) \\
&= e^{-imx}TS_\gamma T^{-1}(fe_m)(x) - e^{i(m+1)x}TS_\gamma T^{-1}(fe_{-(m+1)})(x),
\end{aligned}
$$

where $e_m(t) = e^{imt}, m \in Z$. Then, due to the boundedness of the singular operator S_γ in $G_{p)}(-\pi,\pi)$ we obtain

$$
\begin{aligned}
\|P_m(f)\|_{G_{p)}(-\pi,\pi)} &= \left\|e^{-imx}TS_\gamma T^{-1}(fe_m)(x) - e^{i(m+1)x}TS_\gamma T^{-1}(fe_{-(m+1)})(x)\right\|_{G_{p)}(-\pi,\pi)} \\
&\le \left\|S_\gamma T^{-1}(fe_m)\right\|_{G_{p)}(\gamma)} + \left\|S_\gamma T^{-1}(fe_{-(m+1)})\right\|_{G_{p)}(\gamma)} \\
&\le M\left\|fe_m\right\|_{G_{p)}(-\pi,\pi)} + M\left\|fe_{-(m+1)}\right\|_{G_{p)}(-\pi,\pi)} \\
&= 2M\left\|f\right\|_{G_{p)}(-\pi,\pi)}.
\end{aligned}
$$

□

Now consider the question of the basis property of trigonometric systems $\{\sin nt\}_{n\in N}$ and $\{\cos nt\}_{n\in N}$ in space $L_{p)}(0,\pi)$. The following theorem is true.

Theorem 3.5. *Systems of sines $\{\sin nt\}_{n\in N}$ and cosines $\{\cos nt\}_{n\in N}$ form bases in space $G_{p)}(0,\pi), 1 < p < +\infty$.*

Proof. Lets first consider the system of sines $\{\sin nt\}_{n\in N}$. Let us show the minimality of the system $\{\sin nt\}_{n\in N}$ in the space $L_{p)}(0,\pi), 1 < p < +\infty$. We define a system of linear functionals

$$
g_n(f) = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt dt, \quad f \in L_{p)}(0,\pi), \quad n \in N.
$$

For $\forall f \in L_{p)}(0,\pi)$ we have

$$
\begin{aligned}
|g_n(f)| &= \frac{2}{\pi}\left|\int_0^{\pi} f(t)\sin nt dt\right| \le \frac{2}{\pi}\int_0^{\pi}|f(t)|\,dt \\
&= \frac{2}{\pi}\left(\int_0^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\pi^{\frac{1}{(p-\varepsilon)'}} = 2\left(\frac{\varepsilon}{\pi}\int_0^{\pi}|f(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}\varepsilon^{-\frac{1}{p-\varepsilon}} \\
&\le 2\varepsilon^{\frac{1}{p-\varepsilon}}\left\|f\right\|_{p)},
\end{aligned}
$$

where $\varepsilon \in (0, p-1)$. Thus, g_n is a linear continuous functional in $L_{p)}(0,\pi)$. On the other hand, it is easy to show that

$$
g_n(\sin mt) = \delta_{mn},
$$

i.e. the systems $\{\sin nt\}_{n\in N}$ and $\{g_n\}_{n\in N}$ are biorthogonal, and, consequently, the system $\{\sin nt\}_{n\in N}$ is minimal in the space $L_{p)}(0,\pi)$.

Consider an arbitrary $f \in G_{p)}(0,\pi)$. Extend the function f as an odd one to $[-\pi,\pi]$ and denote this extension by $F(t)$. We have

$$
F(t) = \begin{cases} f(t), & t \in [0,\pi] \\ -f(-t), & t \in [-\pi,0) \end{cases}
$$

It is clear that $F \in G_{p)}(-\pi,\pi)$ and

$$||F||_{L_{p)}(-\pi,\pi)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |F(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}}$$
$$= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{\pi} \int_{0}^{\pi} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} = ||f||_{L_{p)}(0,\pi)}.$$

By Theorem 3.4, the system $\{e^{int}\}_{n\in Z}$ forms a basis for $G_{p)}(-\pi,\pi)$. Then we have an expansion

$$F(t) = \sum_{n=-\infty}^{+\infty} c_n e^{int},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{-int} dt, \quad n \in Z.$$

For the coefficients c_n we have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) e^{-int} dt = \frac{1}{2\pi} \int_{0}^{\pi} f(t) e^{-int} dt - \frac{1}{2\pi} \int_{0}^{\pi} f(t) e^{int} dt$$
$$= -\frac{1}{2\pi} \int_{0}^{\pi} f(t)(e^{int} - e^{-int}) dt = \frac{1}{\pi i} \int_{0}^{\pi} f(t) \sin nt dt = \frac{1}{2i} g_n(f), \quad n \in N.$$

On the other hand, it is clear that $c_{-n} = -c_n$. Therefore, for $\forall m \in N$ we obtain

$$\sum_{n=-m}^{m} c_n e^{int} = \sum_{n=1}^{m} c_n e^{int} - \sum_{n=1}^{m} c_n e^{-int}$$
$$= \sum_{n=1}^{m} c_n (e^{int} - e^{-int}) = 2i \sum_{n=1}^{m} c_n \sin nt = \sum_{n=1}^{m} g_n(f) \sin nt.$$

It is easy to show that $F(t) - \sum_{n=-m}^{m} c_n e^{int}$ is an odd extension of $f(t) - \sum_{n=1}^{m} g_n(f) \sin nt$ to $[-\pi,\pi]$. Therefore, for $\forall m \in N$ we obtain

$$\left\| f - \sum_{n=1}^{m} g_n(f) \sin nt \right\|_{G_{p)}(0,\pi)} = \left\| F - \sum_{n=-m}^{m} c_n e^{int} \right\|_{G_{p)}(-\pi,\pi)}.$$

Hence, passing to the limit as $m \to \infty$, we obtain

$$f(t) = \sum_{n=1}^{+\infty} g_n(f) \sin nt,$$

i.e. the system $\{\sin nt\}_{n\in N}$ forms a basis for the space $G_{p)}(0,\pi)$, $\quad 1 < p < +\infty$.

□

Similarly, we can prove the basicity of the system of cosines $\{\cos nt\}_{n\in N_0}$ for the space $G_{p)}(0,\pi)$, $\quad 1 < p < +\infty$.

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