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Isophote Curves in a Strict Walker 3-Manifold and Application in Optical Fiber

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Abstract

In this paper, we introduice and investigate the geometry of isophote curves in a strict Walker 3-manifold using the Darboux frame. The considered curve is timelike and lying in a timelike surface. We give some characterisations about isophote and its axis and we give an application about optical fiber and its polarization vector.

Keywords: *Curves, Isophote curve, Walker manifolds.*
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1. Introduction

Investigation of the submanifolds in an ambient space is a very interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here the ambient space is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For details see [1, 2]. A Walker manifold of dimension three is a Riemannian manifold admitting such a nul parallel vector field; this manifold is locally reducible. The same property remains true for a pseudo-Riemannian manifold admitting either a spacelike or a timelike parallel vector field. The metric of Walker have used as a best tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties. Much research has been done on the geometric properties of curves on Walker manifolds. See for more informations about curves in a Walker 3-manifold [6, 7].
One can look isophote curves on a surface as a good consequence of Lambert's cosine law in optics branch of physics, see [3, 4, 5] . Lambert's law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector U and the light vector d. In other words, isophote curves of a surface are curves with the property that their points have the same light intensity from a given source (a curve of constant illumination intensity).
The isophote curve method is one of the most efficient methods that can be used to analyze and visualize surfaces by lines of equal light intensity. Isophote curve whose normal vectors make a constant angle with a fixed vector (the axis) is one of the curves to characterize surfaces such as parameter, geodesics, and asymptotic curves or lines of curvature. Moreover, this curve is used in computer graphics and it is also interesting to study for geometry.
Then, to find an isophote on a surface we use the formula

$$\frac{\langle U(u,v),d\rangle}{\|U(u,v)\|}=\cos\gamma,\quad 0\leq\gamma\leq\frac{\pi}{2},$$

where is d the unit fixed vector and γ is the constant angle between the surface normal vector U and d. The isophote curve is called a silhouette curve when $\gamma=\frac{\pi}{2}$.
In the paper [3], the authors determine the axis of an isophote curve via its Darboux frame and give some characterizations about the isophote curve and its axis in Euclidean 3-space. In particularly, they obtain other characterizations for isophote curves lying on a canal surface.
In the paper [9], the authors described the isophote curve and optical descriptions of these curves. The reader can see more informations about optical fiber and their study. More informations about isophote curves the reader can see also [4, 10].
In this paper, we investigate isophote timelike curves on a timelike surfaces in a strict Walker 3-manifold and we find that its axis d verifying some geometric conditions through the Walker Darboux frame. At the end we give an application about optical fiber and its polarization

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vector.
The paper is organised as follow: in section 2, we recall some preliminaries results for Walker manifold (M, g^{ε}_{f}). In the section 3, we study the geometric properties of isophote timelike curves lying on a timelike surface. In the last section we give an application of isophote curve using the optical fiber.

2. Preliminaries

In this section we give some geometric properties of the three dimensional Walker manifold. We refer the reader to see [8, 1].
A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel q-planes, with $q \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [1]. Walker has derived adapted coordinates to a parallel plan field. The metric of Walker 3-manifold (M, g^{ε}_{f}) with coordinates (x, y, z) is given by

$$g^{\varepsilon}_{f} = dx \circ dz + \varepsilon dy^2 + f(x,y,z)dz^2 \tag{2.1}$$

and its matrix form as

$$g^{\varepsilon}_{f} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{ with inverse } (g^{\varepsilon}_{f})^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x,y,z)$, where $\varepsilon = \pm 1$ and thus $D = \text{Span}\partial_x$ as the parallel degenerate line field. Notice that when $\varepsilon = 1$ and $\varepsilon = -1$ the Walker manifold has signature $(2,1)$ and $(1,2)$ respectively, and therefore is Lorentzian in both cases.

The non zero components of the Levi-Civita connection of the metric in (2.1) is given by:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x + f_z)\partial_x - \frac{\varepsilon}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z \end{aligned} \tag{2.2}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. We remark that, if (M, g^{ε}_{f}) is a strict Walker manifolds then the associated Levi-Civita connection satisfies

$$\nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \; \nabla_{\partial_z}\partial z = \frac{1}{2}f_z\partial_x - \frac{\varepsilon}{2}f_y\partial_y. \tag{2.3}$$

Remark that, if the Walker 3-manifold is strict then the non-zero components of the Christoffel symbols and the curvature tensor of the metric g^{ε}_{f} as follows:

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2}f_y, \; \Gamma^1_{33} = \frac{1}{2}f_z, \; \Gamma^2_{33} = -\frac{\varepsilon}{2}f_y \tag{2.4}$$

By using the local coordinates (x,y,z) for which (2.1) holds, one can see that

$$e_1 = \partial_y, \; e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z, \; e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z$$

are local pseudo-orthonormal frame fields on (M, g^{ε}_{f}), with $g^{\varepsilon}_{f}(e_1,e_1) = 1$, $g^{\varepsilon}_{f}(e_2,e_2) = \varepsilon$ and $g^{\varepsilon}_{f}(e_3,e_3) = 1$. Thus the signature of the metric g^{ε}_{f} is $(1, \varepsilon, -1)$.
Let now u and v be two vectors in M. Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in $\mathbb{R}^3$.
The vector product of u and v in (M, g^{ε}_{f}) with respect to the metric g^{ε}_{f} is the vector denoted by $u \times v$ in M defined by

$$g^{\varepsilon}_{f}(u \times v, w) = \det(u, v, w) \tag{2.5}$$

for all vector w in M, where $\det(u,v,w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$. If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then by using (2.5), we have:

$$u \times v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \varepsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k} \tag{2.6}$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g^{\varepsilon}_{f})$ be a curve parametrized by its arc-length s.
The Frenet frame of α is the vectors T, N and B along α where T is the tangent, N the principal normal and B the binormal vector. They satisfied the Frenet formulas

$$\begin{cases} \nabla_T T &= \varepsilon_2 \kappa N \\ \nabla_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B \\ \nabla_T B &= \varepsilon_2 \tau N \end{cases} \tag{2.7}$$

where κ and τ are respectively the curvature and the torsion of the curve α, with $\varepsilon_1 = g_f(T;T)$; $\varepsilon_2 = g_f(N;N)$ and $\varepsilon_3 = g_f(B,B)$.
Now let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g^{\varepsilon}_{f})$ be a timelike curve lying in a timelike surface S in M. Let U be the unit normal of S, the Darboux frame is

given by $\{T,Y,U\}$, where T is the tangent vector of the curve $\alpha(s)$ and $Y=U\times T$.
The usual transformations between the Walker Frenet frame and the Darboux takes the form

$$Y=\cos\theta N-\sin\theta B \tag{2.8}$$
$$U=\sin\theta N+\cos\theta B, \tag{2.9}$$

where θ is an angle between the surface normal vector N and the binormal vector B of α.
Derivating Y along the curve alpha we get

$$\nabla_T Y=\cos\theta\nabla_T N-\theta'\sin\theta N-\sin\theta\nabla_T B-\theta'\cos\theta B.$$

Using the Frenet equation in (2.7) we have

$$\nabla_T Y=\cos\theta(-\varepsilon_1\kappa T-\varepsilon_3\tau B)-\theta'\sin\theta N-\sin\theta(\varepsilon_2\tau N)-\theta'\cos\theta B.$$

Now we suppose that the principal normal and the binormal have the same sign. then we get

$$\nabla_T Y=-\varepsilon_1\kappa\cos\theta T-(\theta+\varepsilon_2\tau)U \tag{2.10}$$

The same calculus gives

$$\nabla_T U=-\varepsilon_1\kappa\sin\theta T+(\theta+\varepsilon_2\tau)Y. \tag{2.11}$$

Then the Walker Darboux equation is expressed as

$$\begin{cases}\nabla_T T=-\varepsilon_2\kappa_g Y-\varepsilon_2\kappa_n U\\ \nabla_T Y=-\varepsilon_1\kappa_g T-\tau_g U\\ \nabla_T U=-\varepsilon_1\kappa_n T+\tau_g Y,\end{cases} \tag{2.12}$$

where κ_g, κ_g and τ_g are the geodesic curvature, normal curvature and geodesic torsion of $\alpha(s)$ on S, respectively. Also, (2.12) implies

$$\tau_g=\theta+\varepsilon_2\tau,\quad \kappa_g=-\kappa\cos\theta,\quad \kappa_n=-\kappa\sin\theta$$
$$\kappa^2=\kappa_g^2+\kappa_n^2,\ \text{and}\ \tau=\tau_g-\frac{\kappa_g\kappa_n'-\kappa_n\kappa_g'}{\kappa^2}. \tag{2.13}$$

We end this section with this following two lemmas.

Lemma 2.1. *Let α be a unit speed space curve with $\kappa(s)\neq 0$. Then α is a general slant helix iff $\frac{\tau}{\kappa}(s)$ is a constant function.[3]*

Lemma 2.2. *Let α be a unit speed space curve with $\kappa(s)\neq 0$. Then α is a general slant helix iff $\sigma(s)=\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$ is a constant function.[3]*

3. The axis of an isophote curve in Walker 3-manifold

Let S be a spacelike regular surface and $\alpha: I\subset\mathbb{R}\longrightarrow M_f$ be an unit speed curve lying on the surface S and we suppose that α is an spacelike curve for some $s\in I$ where the normal unit have negative sign.
Suppose that there exist a position vector d in α such that

$$\frac{\langle U(u,v),d\rangle}{\|U(u,v)\|}=\cos\gamma, \tag{3.1}$$

where γ is the constant angle between the normal vector N of the curve and the normal vector U of the surface S.
And then we have

$$g_f^{\varepsilon}(\nabla_T U,d)=0.$$

Using (2.7) we have

$$-\varepsilon_1\kappa_n g_f^{\varepsilon}(T,d)+\tau_g g_f^{\varepsilon}(Y,d)=0.$$

If we denote by $a=g_f^{\varepsilon}(Y,d)$ we obtain $g_f^{\varepsilon}(T,d)=\frac{\varepsilon_1\tau_g}{\kappa_n}a$. Then d can be written as

$$d=\frac{\varepsilon_1\tau_g}{\kappa_n}aT+aY+\cos\gamma U. \tag{3.2}$$

Case 1. d is a spacelike vector.
Since d is spacelike, the relation $g_f^{\varepsilon}(d,d)=1$ gives $a^2(1+\frac{\tau_g^2}{\kappa_n^2})=\sin^2\gamma$. And then $a=\pm\frac{\kappa_n\sin\gamma}{\sqrt{\kappa_n^2+\tau_g^2}}$. The relation (3.2) becomes

$$d=\pm\frac{\tau_g\sin\gamma}{\sqrt{\kappa_n^2+\tau_g^2}}T\pm\frac{\kappa_n\sin\gamma}{\sqrt{\kappa_n^2+\tau_g^2}}Y+\cos\gamma U. \tag{3.3}$$

If we differentiate $\nabla_T U$ with respect to s we get

$$\nabla_T(\nabla_T U) = -(\kappa_n' + \tau_g \kappa_g)T + (\tau_g' + \kappa_n \kappa_g)Y - (\tau_g^2 + \kappa_n^2)U. \tag{3.4}$$

Since $g_f^{\varepsilon}(\nabla_T(\nabla_T U), d) = 0$ we have

$$\cot\gamma = \pm\left[\frac{\tau_g \kappa_n' + \kappa_n \tau_g'}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}} + \frac{\kappa_g}{\sqrt{\kappa_n^2+\tau_g^2}}\right]. \tag{3.5}$$

Theorem 3.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_f$ be a unit-speed spacelike curve on a regular surface S, and d is a vector axis of the curve. Then*

1. *If α is is an asymptotic curve on S, then α is isophote with axis d. And then α is a general helix with the fixed vector $d = \pm\sin\gamma T + \cos\gamma U$.*
2. *If α is geodesic and κ_n is a constant, then α is isophote with axis d.*
3. *If α is a line of curvature, then α is a plane curve and the angle $\theta = \pm\gamma$ or $\theta = \pm\gamma + \pi$.*

Proof. (1) First we suppose that $\kappa_n = 0$. By (3.3) we get

$$d = \pm\sin\gamma T + \cos\gamma U. \tag{3.6}$$

Differentiating (3.6) with respect to s, we have

$$\nabla_T d = \pm\varepsilon_2 \sin\gamma \kappa_g Y + \cos\gamma \tau_g Y. \tag{3.7}$$

In the other hand if we use the equation in (3.5), we have

$$\cos\gamma = \pm\frac{\kappa_g}{\tau_g}\sin\gamma. \tag{3.8}$$

And then the equation in (3.7) becomes

$$\nabla_T d = \pm\Big(\varepsilon_2 \sin\gamma\kappa_g + \sin\gamma\kappa_g\Big)Y. \tag{3.9}$$

Since the normal of the curve have negative sign, then $\varepsilon_2 = -1$ and we have $\nabla_T d = 0$. And then d is constant vector.
In the other we have $\kappa_g = -\kappa$ and $\tau_g = \tau$. Using (3.8) we have $\frac{\tau}{\kappa} = constant$, and then α is slant curve by lemma 2.1.
(2) Since α is geodesic we have $\kappa_g = 0$. Then differentiating (3.3) we have

$$\begin{aligned} \nabla_T d &= \pm\left(\frac{\tau_g \sin\gamma}{\sqrt{\kappa_n^2+\tau_g^2}}\right)' T \pm \left(\frac{\kappa_g \sin\gamma}{\sqrt{\kappa_n^2+\tau_g^2}}\right)' Y + \cos\gamma(\kappa_g T + \tau_g Y) \\ &= \left(\pm\sin\gamma\frac{\kappa_n(\tau_g'\kappa_n - \tau_g\kappa_n')}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}} - \cos\gamma\kappa_n\right)T + \left(\pm\sin\gamma\frac{\tau_g(\tau_g\kappa_n' - \tau_g'\kappa_n)}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}} + \cos\gamma\tau_g\right)Y. \end{aligned} \tag{3.10}$$

Since $\kappa_g = 0$, the equation in (3.5) gives

$$\cos\gamma = \pm\frac{\kappa_n'\tau_g + \kappa_n\tau_g'}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}}\sin\gamma. \tag{3.11}$$

And then the equation in (3.10) becomes

$$\begin{aligned} \nabla_T d &= \pm\left[\sin\gamma\frac{\kappa_n(\tau_g'\kappa_n - \tau_g\kappa_n')}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}} - \sin\gamma\frac{\kappa_n(\tau_g\kappa_n' + \tau_g'\kappa_n)}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}}\right]T \\ &= \pm\left[\sin\gamma\frac{\tau_g(\tau_g\kappa_n' - \tau_g'\kappa_n)}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}} + \sin\gamma\frac{\tau_g(\tau_g\kappa_n' + \tau_g'\kappa_n)}{(\kappa_n^2+\tau_g^2)^{\frac{3}{2}}}\right]Y. \end{aligned} \tag{3.12}$$

Since κ_n is constant ie $\kappa_n' = 0$, we have $\nabla_T d = 0$ and then we get d is a constant vector.
(3) Since α is a line of curvature, the equation (3.5) becomes

$$\tan\gamma = \pm\frac{\kappa_n}{\kappa_g} = \pm\frac{-\kappa\sin\theta}{-\kappa\cos\theta} = \pm\tan\theta.$$

That gives $\gamma = \pm\theta$ or $\gamma = \pm\theta + \pi$.
By the equation in (2.13) we have $\tau_g = \theta' + \varepsilon_2\tau = 0$, and this imply $\tau = 0$. So the curve α is a plane curve. □

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_f$ be an isophote unit-speed spacelike geodesic curve on a regular surface S, with axis d. Then α is slant helix with the fixed vector $d = \pm\frac{\tau\sin\gamma}{\sqrt{\kappa^2+\tau^2}}T \pm \frac{\kappa\sin\gamma}{\sqrt{\kappa^2+\tau^2}}Y + \cos\gamma U$ if the function $\frac{2\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}$ is a constant.*

Proof. Since the curve is geodesic we have $\kappa_n = -\kappa$ and $\tau_g = \tau$. And using the equation (3.11) we get

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)' = \cot\gamma - \frac{2\kappa'\tau}{(\kappa^2+\tau^2)^{\frac{3}{2}}}.$$

Using the hypothesis in the corollary we get $\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'$ is a constant. Then α is a slant helix by the lemma 2.2. □

Case 2. d is a timelike vector.
Since d is timelike, the relation $g_f^{\varepsilon}(d,d) = -1$ gives $a^2(1+\frac{\tau_g^2}{\kappa_n^2}) = -1-\cos^2\gamma$. That gives $a^2 = -\frac{(1+\cos^2\gamma)\kappa_n^2}{\kappa_n^2+\tau_g^2}$ which is not possible.

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M_f$ be a unit-speed spacelike isophote curve on a regular surface S with position vector axis d. Then d cannot be timelike.*

Now we suppose that the curve α lying in the surface S is timelike and the vector d also is timelike. Then the relation $g_f^{\varepsilon}(d,d) = -1$ gives $a^2(1-\frac{\tau_g^2}{\kappa_n^2}) = -(1-\cos^2\gamma)$. This relation gives

$$a^2 = \frac{(1+\cos^2\gamma)\kappa_n^2}{\tau_g^2-\kappa_n^2}. \tag{3.13}$$

The equation in (3.2) becomes

$$d = -\tau_g\sqrt{\frac{1+\cos^2\gamma}{\tau_g^2-\kappa_n^2}}T + \kappa_n\sqrt{\frac{1+\cos^2\gamma}{\tau_g^2-\kappa_n^2}}Y + \cos\gamma U, \tag{3.14}$$

with the condition:

$$\tau_g^2 > \kappa_n^2 \quad \textbf{(C1)}$$

and

$$\tau_g^2 > 2\kappa_g^2 \quad \textbf{(C2)}$$

If we differentiate $\nabla_T U$ with respect to s we get

$$\nabla_T(\nabla_T U) = (\kappa_n' + \tau_g\kappa_g)T + (\tau_g' + \kappa_n\kappa_g)Y - (\tau_g^2 - \kappa_n^2)U, \tag{3.15}$$

Theorem 3.4. *Let α be a unit speed timelike curve lying in the spacelike surface S in M_f. We suppose α satisfy **(C1)** and **(C2)**. If α is asymptotic then α is isophote with axis d.*

Proof. Since $g_f^{\varepsilon}(\nabla_T(\nabla_T U), d) = 0$ with the hypothesis $\kappa_n = 0$, we have

$$\tan\gamma = \pm\sqrt{\frac{\tau_g^2}{\kappa_g^2} - 2}. \tag{3.16}$$

If we suppose that α is an asymptotic curve then the relation (3.14) becomes

$$d = -\sqrt{1+\cos^2\gamma}T + \cos\gamma U. \tag{3.17}$$

Now if we differentiate the relation (3.17) with respect to s we get

$$\nabla_T d = -\sqrt{1+\cos^2\gamma}\kappa_g + \tau_g\cos\gamma. \tag{3.18}$$

Using the equation (3.16) in the equation (3.18), we get $\nabla_T d = 0$; that is, d is a constant vector. Then the result. □

4. Applications to the Optical fiber

Let $\alpha : I \subset \mathbb{R} \to M_f$ be a curve lying in a surface S (spacelike or timelike) in M. Let U be the unit normal of S, the Darboux frame is given by $\{T,Y,U\}$, where T is the tangent vector of the curve $\alpha(s)$ and $Y = U \times T$. We suppose that α is iphote curve.
In this section we suppose that α describe the optical fiber and the axis σ of this optical fiber is the direction of the polarization vector. Then the angle between the polarization vector σ and the Darboux frame fields are constant noted ϕ. Then using the theorem 3.1 we can stay the following:

- If α is asymptotic on S, then the polarization vector is constant and the optical fiber is a general helix and we have

$$\Sigma = \pm\sin\phi T + \cos\phi U. \tag{4.1}$$

- If the optical fiber is geodesic and κ_n constant, then the polarization vector of the optical fiber is constant.
- If the optical fiber is a line of curvature, then it is a plane curve and we have the angle θ between the normal vector of α and U is expressed by $\theta = \pm\phi$ or $\theta = \pm\phi + \pi$.

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