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Some Results on Important Inequalities for Univalent Functions with Positive and Negative Coefficients

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ABSTRACT
As it is known from Real Analysis, inequalities are used to give the definition of many mathematical concepts formally and to analyze them analytically. Similarly, the geometric characterizations of the range of analytic and univalent functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ can be easily analyzed with inequalities and easily classified these functions.

Keywords: *Analytic function, Convex function, Starlike function, Univalent function*

Pozitif ve Negatif Katsayılı Univalent Fonksiyonlar için Bazı Önemli Eşitsizlikler

Öz
Reel analizden bilindiği gibi, eşitsizlikler birçok matematiksel kavramın tanımını formal olarak vermek ve onları analitik olarak analiz etmek için kullanılır. Benzer olarak birim açık disk $U = \{z \in \mathbb{C} : |z| < 1\}$ da analitik ve univalent olan geometrik fonksiyonların görüntü kümeleri eşitsizliklerle kolayca analiz edilebilir ve kolayca bu fonksiyonlar sınıflandırılabilir.

Anahtar Kelimeler: *Analitik fonksiyon, Konveks fonksiyon, Starlike(yıldızıl) fonksiyon, Univalent(Yalınkat) Fonksiyon.*

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I. INTRODUCTION

Let S denoted the class of functions of the form

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + \cdots \tag{1}$$

which are both analytic and univalent in the open unit disc $U = \{z \in C : |z| < 1\}$. Functions of the form (1) under their have conditions, also satisfy the equations $f(0) = 0$ and $f'(0) = 1$ which are known as normalization conditions. It is a known fact that the concept of univalent, which counterpart in the Real analysis is one-to-one, has contributed to the development of Geometric function theory, as a much stronger property referring to both analytic and univalent function in Complex analysis. More importantly, this property is used as a basic mathematical tool in the classification of analytic functions that take the open unit disc U as the domain in the Riemann mapping theorem without loss of generality. Therefore it can be said Riemann's theorem constitutes the scientific foundations of the Geometric function theory. In this sense the inequalities given in this study are valid for univalent functions whose domain is the open unit disc U.

If the range of a function f of class S exhibits respect to the origin, then the inequality

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \tag{2}$$

is satisfied and vice versa. Accordingly, this class of functions is denoted by S^* and can be analytically given as

$$S^* = \left\{f(z) \in S\colon Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U\right\}. \tag{3}$$

The more specific subclass of starlike functions is the class of starlike functions of order α, denoted by $S^*(\alpha)$ with $0 \leq \alpha < 1$. This subclass can be given as

$$S^*(\alpha) = \left\{f(z) \in S\colon Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ \ \alpha \in [0,1), z \in U\ \ \right\}. \tag{4}$$

Another geometric characterization that a function $f(z)$ belonging to the class S can exhibit is usual convexity. A function $f \in S$ with this geometric characterization of the range provides the inequlity

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \tag{5}$$

and vice versa. The class of $f(z)$ functions that belonging to the class$S$, and additionally provide this last inequality, is denoted by C and can be analytically given by

$$C = \left\{f(z) \in S\colon Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in U\right\}. \tag{6}$$

Also, the more specific subclass of convex functions is the class of convex functions of order α, denoted $C(\alpha)$ with $0 \leq \alpha < 1$. This class can be given as

$$C(\alpha) = \left\{ f(z) \in S : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ \ \alpha \in [0,1), z \in U \ \right\}. \tag{7}$$

Based on the information given so far and the fact that a convex domain is also a starlike region with respect to every point at the same time, thus $C(\alpha) \subset S^*(\alpha) \subset S$ can be written using the subset relation. The classes $S^*(\alpha)$ and $C(\alpha)$ satisfies Alexander duality relation [2];

$$zf'(z) \in S^*(\alpha) \Leftrightarrow f(z) \in C(\alpha), 0 \leq \alpha < 1. \tag{8}$$

The basic argument provided by this theorem, also known as the Alexander duality theorem that $f(z)$ is univalent and convex if and only if $zf'(z)$ can be univalent and convex. In this case, it can be said immediately that f is convex, according to the argument of Alexander's theorem, since the function f satisfies the normalization conditions and a convex region is also a starlike region with respect to each point. This brilliant theorem, which is not difficult to prove, is used as very useful mathematical tool in obtaining many result set forth in univalent function theory. On the other hand, it is clear that the functions belonging to these classes produce a set of adequacy conditions when mapping the open unit disc to simple regions which have as cute as it is interesting geometric characterizations. The theory of geometric function basically aims to classify analytic functions that are defined in the open unit disc U and have certain conditions such as being univalent, convex and starlike by relating them to the geometric characterization that all members of the most of specific class have, but not vice versa. Moreover, the common geometric characterization belonging to a class imposes very clear limitations on the Taylor coefficient of the functions belonging to the class. The best known among this is Bieberbach conjecture states that for every function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ of class S, $|a_n| \leq n, n \geq 2$. Equality in inequality is concidered an extremal property for these functions [1]. Tere is only one funcion that satisfies this property and is therefore known as the extremal function in Univalent function theory. This is the $k(z) = z(1-z)^{-2}, |z| < 1$ function known as the Koebe function.

It should also be noted that all Taylor coefficients of $f(z) \in S$ functions are positive, including the coefficient. Further, another subclass can be created by imposing a more specific condition on $f(z)$ function. For example, let's take the function

$$w = f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n = z - a_2z^2 - \cdots \tag{9}$$

where all nonzero Taylor coefficients are negative starting from the second Taylor coefficient. Let the class of functions belonging to class S, which have the form (9) and are both analytic and univalent, be denoted by λ [6], [7]. According to the information given, class λ is a subclass of class$S$. In this case, the classes of α order starlike and α order convex functions denoted $\lambda^*(\alpha)$ and $\mathcal{K}(\alpha)$ with $\alpha \in [0,1)$, respectively [4], [5]. A function of the form (9) belongs to class λ if and only if the inequality $\sum_{n=2}^{\infty} na_n \leq 1$ is satisfied [8]. Similarly, the condition $\sum_{n=2}^{\infty} n|a_n| \leq 1$ is sufficient for all functions $f(z)$ of the form (1) to be in S. Further, such functions are also starlike, since in this case$\left|\frac{zf'(z)}{f(z)}\right| < 1(z \in U)$ is proved. The coefficient restriction and thus geometric characterization make class λ more guidable then class$S$. Therefore, the solution of problems in class S is completely based on the solution of related problems in classλ.

II. MAIN RESULTS

Theorem 2.1. Let the function $f(z)$ have the form (1). In this case, the condition

$$\sum_{n=2}^{\infty}(n-2^{-r})\,|a_n| < 1-2^{-r}, r \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} \tag{10}$$

is a sufficient conditions for $f(z) \in S^*(2^{-r})$.

Proof. Based on the hypothesis of the theorem, to prove its conclusion, it is sufficient to show that $zf'(z)/f(z)$ is in a circle with center $w = 1$ and radius$1-2^{-r}$. Then,

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{zf'(z)-f(z)}{f(z)}\right| = \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z+\sum_{n=2}^{\infty}a_n z^n}\right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n||z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}.$$

The resulting last inequality is bounded above by $1-2^{-r}$ if

$$\sum_{n=2}^{\infty}(n-2^{-r})\,|a_n| \leq (1-2^{-r})\left(1-\sum_{n=2}^{\infty}|a_n|\right).$$

which is equivalent to

$$\sum_{n=2}^{\infty}(n-2^{-r})\,|a_n| < 1-2^{-r}. \tag{11}$$

But (11) is accurate by hypothesis. Hence$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-2^{-r}$, and the theorem is proved.

A more general case of Theorem 2.1 can be found in [3]. In the light of the information given first chapter, the following corollary it can be easily deduced from Theorem 2.1.

Corollary 2.1. Let the function $f(z)$ have the form (1). In this case, the condition

$$\sum_{n=2}^{\infty}n(n-2^{-r})\,|a_n| < 1-2^{-r}, r \in \mathbb{N} \tag{12}$$

then $f(z) \in C(2^{-r})$.

Proof. According to the Alexander's duality theorem (8), which points to the relations between $C(2^{-r})$ and $S^*(2^{-r})$, that is $f(z) \in C(2^{-r}) \Leftrightarrow zf'(z) \in S^*(2^{-r})$. On the other hand, since $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, may be replace a_n with na_n in Theorem 2.1. At this stage, the desired result is obtained with a direct calculation.

Theorem 2.2. Let the function $f(z)$ have the form (9). Then, $f(z) \in \lambda^*(2^{-r})$ if and only if

$$\sum_{n=2}^{\infty} (n - 2^{-r})\,|a_n| < 1 - 2^{-r}, r \in \mathbb{N}. \tag{13}$$

Proof. From Theorem 2.1, it is sufficient to prove only the if part. Since,

$$Re\left(\frac{zf'(z)}{f(z)}\right) = Re\left(\frac{z - \sum_{n=2}^{\infty} n|a_n|z^n}{z - \sum_{n=2}^{\infty} |a_n|z^n}\right) > 2^{-r}, r \in \mathbb{N}, z \in \mathbb{N}. \tag{14}$$

Here, when the z values choose on the real axis. If $z \to 1$ is taken after necessary simplifications in (14), we have

$$1 - \sum_{n=2}^{\infty} n\,|a_n| \geq 2^{-r}\left(1 - \sum_{n=2}^{\infty} |a_n|\right).$$

Thus $\sum_{n=2}^{\infty} (n - 2^{-r})|a_n| \leq 1 - 2^{-r}$, and the proof is completed.

Corollary 2.2. Let the function $f(z)$ have the form (9). Then, $f(z) \in C(2^{-r})$ if and only if

$$\sum_{n=2}^{\infty} n(n - 2^{-r})\,|a_n| < 1 - 2^{-r}, r \in \mathbb{N}. \tag{15}$$

Its proof is a natural consequence of Theorem 2.1.

Corollary 2.3. Letting $r \to \infty$ in Theorem 2.2., then, $\lambda^*(2^{-r}) = \lambda^*(0) = \lambda^*$.

The proof is obtained directly from the proof of Theorem 2.1.

III.CONCLUSION

The main purpose of this study is to bring a different perspective to the inequalities used in the classification of analytical and univalent complex functions in the open unit disc U according to the geometric characterizations of the image regions. In this sense, general information and related resources are given in the first chapter. In the second part, our main results and their proofs are given.

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