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A HOMOLOGICAL CHARACTERIZATION OF Q0-PRÜFER V-MULTIPLICATION RINGS

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Abstract. Let R be a commutative ring. An R-module M is called a semi-regular w-flat module if $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion for any finitely generated semi-regular ideal I. In this article, we show that the class of semi-regular w-flat modules is a covering class. Moreover, we introduce the semi-regular w-flat dimensions of R-modules and the sr-w-weak global dimensions of the commutative ring R. Utilizing these notions, we give some homological characterizations of WQ-rings and Q_0-PvMRs.

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1. Introduction

Throughout this paper, we always assume R is a commutative ring with identity and $T(R)$ is the total quotient ring of R. Following from [17], an ideal I of R is said to be *dense* if $(0:_R I) := \{r \in R \mid Ir = 0\} = 0$ and be *semi-regular* if it contains a finitely generated dense sub-ideal. Denote by $\mathcal{Q}$ the set of all finitely generated semi-regular ideals of R. Following from [20] that a ring R is called a DQ-*ring* if $\mathcal{Q} = \{R\}$. If R is an integral domain, the quotient field K is a very important R-module to study integral domains. However, the total quotient ring $T(R)$ is not always convenient to study commutative rings R with zero-divisors. For example, the polynomial ring $R[x]$ is not always integrally closed in $T(R[x])$ when R is integrally closed in $T(R)$ (see [3]). It is well-known that a finitely generated ideal $I = \langle a_0, a_1, \cdots, a_n\rangle$ is semi-regular if and only if the polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a regular element in $R[x]$ (see [17, Exercise 6.5] for example). So, to study the integrally closedness of $R[x]$, Lucas [10] introduced the ring of finite fractions of R:

$$Q_0(R) := \{\alpha \in T(R[x]) \mid \text{ there exists } I \in \mathcal{Q} \text{ such that } I\alpha \subseteq R\},$$

and showed that a reduced ring R is integrally closed in $Q_0(R)$ if and only if $R[x]$ is integrally closed in $T(R[x])$. Note that for any commutative ring R, we have $R \subseteq T(R) \subseteq Q_0(R)$. Recently, the authors [23,24] gave several homological characterizations of total quotients rings (i.e. $R = T(R)$) and DQ-rings utilizing certain generalized flat modules. There is a natural question to characterize commutative rings with $R = Q_0(R)$ (called WQ-rings from the star operation point of view). Actually, we show that WQ-rings are exactly those rings whose modules are all semi-regular w-flat (see Theorem 4.2).

Prüfer domains are well-known domains and have been studied by many algebraists. In order to generalize Prüfer domains to commutative rings with zero-divisors, Butts and Smith [4], in 1967, introduced the notion of *Prüfer rings* in which every finitely generated regular ideal is invertible. Later in 1985, Anderson et al. [1] introduced the notion of *strong Prüfer rings* whose finitely generated semi-regular ideals are all Q_0-invertible. Strong Prüfer rings have many nice properties. For example, the small finitistic dimensions of strong Prüfer rings are at most one (see [19]). To give a w-analogue of Prüfer rings, Huckaba and Papick [9] and Matsuda [13] called a ring R to be a PvMR (short for *Prüfer v-multiplication ring*) provided that any finitely generated regular ideal is t-invertible. For generalizing strong Prüfer rings, Lucas [12] said a ring R to be a Q_0-PvMR (short for *Q_0-Prüfer v-multiplication ring*) if any finitely generated semi-regular ideal is t-invertible, and then he considered the properties of polynomial rings $R[x]$ and Nagata rings $R(x)$ and w-Nagata rings $R\{x\}$ when R is a Q_0-PvMR. For studying Q_0-PvMRs, Qiao and Wang [15] introduced quasi-Q_0-PvMRs and showed that a ring R is a Q_0-PvMR if and only if R is a quasi-Q_0-PvMR, and R has Property B in $Q_0(R)$, i.e., $(IQ_0(R))_w = Q_0(R)_w$ for any $I \in \mathcal{Q}$. Wang and Kim [16] gave some module-theoretic properties of Q_0-PvMRs. Actually, they showed that Q_0-PvMRs are Q_0-w-coherent and each finite type semi-regular ideal is w-projective. The authors [23,24] also gave some homological characterizations of strong Prüfer rings and PvMRs utilizing the generalized flat modules. In this paper, we obtain several module-theoretic and homological characterizations of Q_0-PvMRs using w-projective modules, w-flat modules and semi-regular w-flat modules (see Theorem 4.4).

As our work involves the w-operations, we give some reviews. A finitely generated ideal J of R is called a *Glaz-Vasconcelos ideal* (GV-*ideal* for short) if the natural homomorphism $R \to \mathrm{Hom}_R(J, R)$ is an isomorphism, and the set of all GV-ideals is denoted by $\mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}.$$

An R-module M is called GV-*torsion* (resp., GV-*torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). A GV-torsion-free module M is called a w-*module* if $\mathrm{Ext}^1_R(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$, and the w-*envelope* of M is given by

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\},$$

where $E(M)$ is the injective envelope of M. A fractional ideal I is said to be w-invertible if $(II^{-1})_w = R$. A DW *ring* R is a ring over which every module is a w-module, equivalently the only GV-ideal of R is R. A *maximal* w-*ideal* is an ideal of R which is maximal among all w-submodules of R. The set of all maximal w-ideals is denoted by w-$\mathrm{Max}(R)$, and any maximal w-ideal is prime.

An R-homomorphism $f : M \to N$ is said to be a w-*monomorphism* (resp., w-*epimorphism*, w-*isomorphism*) if for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$, $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a monomorphism (resp., an epimorphism, an isomorphism). A sequence $A \to B \to C$ is said to be w-*exact* if for any $\mathfrak{m} \in w$-$\mathrm{Max}(R)$, $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact. A class $\mathcal{C}$ of R-modules is said to be closed under w-isomorphisms provided that for any w-isomorphism $f : M \to N$, if one of the modules M and N is in $\mathcal{C}$, so is the other. An R-module M is said to be of *finite type* if there exist a finitely generated free module F and a w-epimorphism $g : F \to M$. Following from [16], an R-module M is said to be w-*flat* if for any w-monomorphism $f : A \to B$, the induced sequence $f \otimes_R 1 : A \otimes_R M \to B \otimes_R M$ is also a w-monomorphism. The classes of finite type modules and w-flat modules are all closed under w-isomorphisms, see [17, Corollary 6.7.4].

2. Semi-regular w-flat modules

Recall from [24], an R-module M is said to be a semi-regular flat module if, for any finitely generated semi-regular ideal I (i.e. $I \in \mathcal{Q}$), we have $\mathrm{Tor}^R_1(R/I, M) = 0$. Obviously, every flat module is semi-regular flat. We denote by $\mathcal{F}_{sr}$ the class of all semi-regular flat modules. Then the class $\mathcal{F}_{sr}$ of all semi-regular flat modules is closed under direct limits, pure submodules and pure quotients [24, Lemma 2.4]. Hence $\mathcal{F}_{sr}$ is a covering class (see [24, Theorem 2.6]). Now, we give a w-analogue of semi-regular flat modules.

Definition 2.1. An R-module M is said to be a *semi-regular w-flat module* if $\mathrm{Tor}^R_1(R/I, M)$ is GV-torsion for any $I \in \mathcal{Q}$. The class of all semi-regular w-flat modules is denoted by w-$\mathcal{F}_{sr}$.

Obviously, semi-regular flat modules and w-flat modules are all semi-regular w-flat. Following from [24] that an R-module M is said to be a semi-regular coflat module if for any $I \in \mathcal{Q}$, we have $\mathrm{Ext}_R^1(R/I, M) = 0$.

Lemma 2.2. *Let M be an R-module. Then the following statements are equivalent:*

(1) *M is a semi-regular w-flat module;*

(2) *for any $I \in \mathcal{Q}$, the natural homomorphism $I \otimes M \to R \otimes M$ is a w-monomorphism;*

(3) *for any $I \in \mathcal{Q}$, the natural homomorphism $\sigma_I : I \otimes M \to IM$ is a w-isomorphism;*

(4) *for any injective w-module E, $\mathrm{Hom}_R(M, E)$ is a semi-regular coflat module.*

Proof. (1) $\Leftrightarrow$ (2): Let I be a finitely generated semi-regular ideal. Then we have a long exact sequence:

$$0 \to \mathrm{Tor}_1^R(R/I, M) \to I \otimes_R M \to R \otimes_R M \to R/I \otimes_R M \to 0.$$

Consequently, $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion if and only if $I \otimes_R M \to R \otimes_R M$ is a w-monomorphism.

(2) $\Rightarrow$ (3): Let I be a finitely generated semi-regular ideal. Then we have the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I \otimes_R M & \longrightarrow & R \otimes_R M \\ & & \downarrow{\scriptstyle \sigma_I} & & \downarrow{\scriptstyle \cong} \\ 0 & \longrightarrow & IM & \longrightarrow & M. \end{array}$$

Then σ_I is a w-monomorphism. Since the multiplicative map σ_I is an epimorphism, σ_I is a w-isomorphism.

(3) $\Rightarrow$ (1): Let I be a finitely generated semi-regular ideal. Then we have a long exact sequence:

$$0 \longrightarrow \mathrm{Tor}_1^R(R/I, M) \longrightarrow IM \xrightarrow{f} M.$$

Since f is a natural embedding map, we have $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion.

(1) $\Rightarrow$ (4): Let I be a finitely generated semi-regular ideal and E an injective w-module. Then $\mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, M), E)$. Since M is a semi-regular w-flat module, $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion. Since E is a w-module, we have $\mathrm{Hom}_R(\mathrm{Tor}_1^R(R/I, M), E) = 0$. Thus $\mathrm{Ext}_R^1(R/I, \mathrm{Hom}_R(M, E)) = 0$, So $\mathrm{Hom}_R(M, E)$ is a semi-regular coflat module.

(4) $\Rightarrow$ (1): Let I be a finitely generated semi-regular ideal and E an injective w-module. Since $\mathrm{Hom}_R(M,E)$ is a semi-regular coflat module and

$$\mathrm{Ext}^1_R(R/I,\mathrm{Hom}_R(M,E))\cong \mathrm{Hom}_R(\mathrm{Tor}^R_1(R/I,M),E),$$

we have $\mathrm{Hom}_R(\mathrm{Tor}^R_1(R/I,M),E)=0$. By [25, Corollary 3.11], $\mathrm{Tor}^R_1(R/I,M)$ is GV-torsion. So M is a semi-regular w-flat module. □

Corollary 2.3. *Let R be a ring. The class of semi-regular w-flat R-modules is closed under w-isomorphisms.*

Proof. Let $f: M\to N$ be a w-isomorphism and I a finitely generated semi-regular ideal. There exist two exact sequences $0\to T_1\to M\to L\to 0$ and $0\to L\to N\to T_2\to 0$ with T_1 and T_2 GV-torsion. Consider the induced two long exact sequences $\mathrm{Tor}^R_1(R/I,T_1)\to \mathrm{Tor}^R_1(R/I,M)\to \mathrm{Tor}^R_1(R/I,L)\to R/I\otimes T_1$ and $\mathrm{Tor}^R_2(R/I,T_2)\to \mathrm{Tor}^R_1(R/I,L)\to \mathrm{Tor}^R_1(R/I,N)\to \mathrm{Tor}^R_1(R/I,T_2)$. By [17, Theorem 6.7.2], M is semi-regular w-flat if and only if N is semi-regular w-flat. □

Proposition 2.4. *Let R be a ring. Then R is a* DW*-ring if and only if any semi-regular w-flat module is semi-regular flat.*

Proof. Obviously, if R is a DW-ring, then every semi-regular w-flat module is semi-regular flat. On the other hand, let J be a GV-ideal of R, then R/J is GV-torsion and hence a semi-regular w-flat module by Corollary 2.3. So R/J is a semi-regular flat module. Note the GV-ideal J is finitely generated and semi-regular, so $\mathrm{Tor}^R_1(R/J,R/J)\cong J/J^2=0$ by [17, Exercise 3.20]. It follows that J is a finitely generated idempotent ideal of R, and thus J is projective by [6, Proposition 1.10]. Hence $J=J_w=R$. Consequently, R is a DW-ring. □

We say a class $\mathcal{F}$ of R-modules is precovering provided that for any R-module M, there is a homomorphism $f:F\to M$ with $F\in\mathcal{F}$ such that $\mathrm{Hom}_R(F',F)\to \mathrm{Hom}_R(F',M)$ is an epimorphism for any $F'\in\mathcal{F}$. If, moreover, any homomorphism h such that $f=f\circ h$ is an isomorphism, $\mathcal{F}$ is said to be covering. It is well-known that the class of flat modules is a covering class (see [2, Theorem 3]). It was also proved in [22, Theorem 3.5] that the class of w-flat modules is a covering class. For the class of semi-regular flat modules, we have the following similar result.

Proposition 2.5. *Let R be a ring. Then the class w-$\mathcal{F}_{sr}$ of all semi-regular flat modules is closed under direct limits, pure submodules and pure quotients. Consequently, w-$\mathcal{F}_{sr}$ is a covering class.*

Proof. For the direct limits, suppose $\{M_i\}_{i\in\Gamma}$ is a direct system consisting of semi-regular w-flat modules. Then, for any finitely generated semi-regular ideal I, we have $\mathrm{Tor}_1^R(R/I, \varinjlim M_i) = \varinjlim \mathrm{Tor}_1^R(R/I, M_i)$ is GV-torsion. So $\varinjlim M_i$ is a semi-regular w-flat module.

For pure submodules and pure quotients, let I be a finitely generated semi-regular ideal. Suppose $0 \to M \to N \to L \to 0$ is a pure exact sequence. We have the following commutative diagram with rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_R I & \longrightarrow & N \otimes_R I & \longrightarrow & L \otimes_R I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M \otimes_R R & \longrightarrow & N \otimes_R R & \longrightarrow & L \otimes_R R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_R R/I & \longrightarrow & N \otimes_R R/I & \longrightarrow & L \otimes_R R/I & \longrightarrow & 0
\end{array}$$

By the generalized Five Lemma (see [17, Lemma 6.3.6]), the natural homomorphism $f : M \otimes_R I \to M \otimes_R R$ and $g : L \otimes_R I \to L \otimes_R R$ are all w-monomorphisms. Consequently, M and L are all semi-regular w-flat. Consequently, w-$\mathcal{F}_{sr}$ is a covering class by [8, Theorem 3.4]. □

3. On the homological dimension of semi-regular w-flat modules

The author [23] introduced the notions of homological dimensions of regular w-flat modules for the homological characterizations of total quotient rings and PvMRs. In order to characterize WQ rings and Q_0-PvMRs, we introduce the homological dimensions using semi-regular w-flat modules in this section.

Definition 3.1. Let R be a ring and M an R-module. We write sr-w-$\mathrm{fd}_R(M) \leq n$ (sr-w-fd abbreviates *semi-regular w-flat dimension*) if there is a w-exact sequence of R-modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad (\diamondsuit)$$

with each F_i w-flat ($i = 0, \cdots, n-1$) and F_n semi-regular w-flat. The w-exact sequence ($\diamondsuit$) is said to be a semi-regular w-flat w-resolution of length n of M. The semi-regular w-flat dimension sr-w-$\mathrm{fd}_R(M)$ is defined to be the length of the shortest semi-regular w-flat w-resolution of M. If such finite w-resolution ($\diamondsuit$) does not exist, then we say sr-w-$\mathrm{fd}_R(M) = \infty$.

It is obvious that an R-module M is semi-regular w-flat if and only if sr-w-$\mathrm{fd}_R(M) = 0$ and sr-w-$\mathrm{fd}_R(N) \leq w$-$\mathrm{fd}_R(N)$ for any R-module N.

Proposition 3.2. *Let R be a ring. The following statements are equivalent for an R-module M:*

(1) sr-w-$fd_R(M) \leq n$;
(2) $\mathrm{Tor}^R_{n+1}(M, R/I)$ *is* GV*-torsion for all finitely generated semi-regular ideals* I;
(3) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat R-modules, then* F_n *is semi-regular w-flat;*
(4) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an w-exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are w-flat R-modules, then* F_n *is semi-regular w-flat;*
(5) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are w-flat R-modules, then* F_n *is semi-regular w-flat;*
(6) *if* $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ *is an w-exact sequence, where* $F_0, F_1, \ldots, F_{n-1}$ *are flat R-modules, then* F_n *is semi-regular w-flat.*

Proof. (1) ⇒ (2): We prove (2) by induction on n. The case $n = 0$ is trivial. If $n > 0$, then there is a w-exact sequence $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i w-flat $(i = 0, \cdots, n-1)$ and F_n is semi-regular w-flat. Let $K_0 = \ker(F_0 \to M)$. We have two w-exact sequences $0 \to K_0 \to F_0 \to M \to 0$ and $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to K_0 \to 0$. Note that sr-w-$\mathrm{fd}_R(K_0) \leq n-1$. Let I be a finitely generated semi-regular ideal. By induction, we have $\mathrm{Tor}^R_n(K_0, R/I)$ is GV-torsion. It follows from [18, Lemma 2.2] that $\mathrm{Tor}^R_{n+1}(M, R/I)$ is GV-torsion.

(2) ⇒ (4): Let $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ be a w-exact sequence with each F_i w-flat $(i = 0, \cdots, n-1)$. Set $L_n = F_n$ and $L_i = \mathrm{Im}(F_i \to F_{i-1})$, where $i = 1, \ldots, n-1$. Then both $0 \to L_{i+1} \to F_i \to L_i \to 0$ and $0 \to L_1 \to F_0 \to M \to 0$ are w-exact sequences. By using [18, Lemma 2.2] repeatedly, we can obtain that $\mathrm{Tor}^R_1(F_n, R/I)$ is GV-torsion for all finitely generated semi-regular ideals I. Thus F_n is semi-regular w-flat.

(4) ⇒ (5) ⇒ (3) and (4) ⇒ (6) ⇒ (3) ⇒ (1): Trivial. □

Definition 3.3. The *sr-w-weak global dimension* of a ring R is defined by

$$sr\text{-}w\text{-}\mathrm{w.gl.dim}(R) = \sup\{sr\text{-}w\text{-}\mathrm{fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Obviously, by definition, sr-w-w.gl.dim$(R) \leq w$-w.gl.dim(R) for any ring R. We can easily deduce the following results from Proposition 3.2.

Corollary 3.4. *The following statements are equivalent for a ring R.*

(1) sr-w-*w.gl.dim*$(R) \leq n$.
(2) sr-w-$fd_R(M) \leq n$ *for all R-modules M.*

(3) $\mathrm{Tor}_{n+1}^R(M, R/I)$ *is* GV*-torsion for all R-modules M and all finitely generated semi-regular ideals I of R.*

4. Rings with sr-w-weak global dimensions at most one

Recently, the authors in [27] introduced the notion of q-operation on a commutative ring R. We give some reviews here. An R-module M is said to be $\mathcal{Q}$-torison-free if $Im = 0$ with $I \in \mathcal{Q}$ and $m \in M$ can deduce $m = 0$. Let M be a $\mathcal{Q}$-torison-free R-module. The Lucas envelope

$$M_q = \{x \in E(M) \mid \text{ there exists } I \in \mathcal{Q} \text{ such that } Ix \subseteq M\}$$

where $E(M)$ is the injective envelope of M. An $\mathcal{Q}$-torison-free R-module M is said to be a Lucas module provided that $M_q = M$. By [21, Proposition 2.2], a ring is a DQ-ring if and only if every R-module is Lucas module. Since any GV-ideal is finitely generated semi-regular, we have Lucas modules are all w-modules. However, R itself is not always a Lucas module. It was proved in [20, Proposition 3.8] that a ring R is a Lucas module if and only if the q- and w-operations on R coincide, if and only if every finitely generated semi-regular ideal is a GV-ideal, if and only if $Q_0(R) = R$. For convenience, we say a ring R is a WQ-ring if every finitely generated semi-regular ideal is a GV-ideal. Obviously, a ring R is a DQ-ring if and only if it is both a DW-ring and a WQ-ring. It was proved in [23, Theorem 4.1] that a ring R is a total quotient ring (i.e. any regular element of R is a unit) if and only if every R-module is regular w-flat, if and only if reg-w-w.gl.dim$(R) = 0$. Next, we will give a homological characterization of WQ rings utilizing sr-w-weak global dimensions.

Lemma 4.1. *Let $I = \langle a_1, a_2, \cdots, a_n\rangle$ be a finitely generated ideal of R. Suppose m is a positive integer and $K = \langle a_1^m, a_2^m, \cdots, a_n^m\rangle$. Then $I^{mn} \subseteq K$.*

Proof. Note that I^{mn} is generated by $\{\prod\limits_{i=1}^n a_i^{k_i} \mid \sum\limits_{i=1}^n k_i = mn\}$. By the pigeonhole principle, there exists some k_i such that $k_i \geq m$. So each $\prod\limits_{i=1}^n a_i^{k_i} \in K$, and thus $I^{mn} \subseteq K$. □

Theorem 4.2. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a* WQ*-ring;*

(2) *every R-module is semi-regular w-flat;*

(3) *sr-w-w.gl.dim$(R) = 0$;*

(4) *for every finitely generated semi-regular ideal I, R/I is a w-flat module;*

(5) *$I \subseteq (I^2)_w$ for any finitely generated semi-regular ideal I of R;*

(6) *every w-module is a Lucas module;*

(7) $Q_0(R) = R$.

Proof. (1) $\Rightarrow$ (2): Let I be a finitely generated semi-regular ideal of R and M an R-module. Then I is a GV-ideal of R. So $\mathrm{Tor}_1^R(R/I, M)$ is GV-torsion. Hence M is semi-regular w-flat.

(1) $\Rightarrow$ (6) and (2) $\Leftrightarrow$ (3): Trivial.

(2) $\Rightarrow$ (4): Let I be a finitely generated semi-regular ideal of R and K a finitely generated ideal of R. Then R/K is a semi-regular w-flat module. So $\mathrm{Tor}_1^R(R/K, R/I)$ is GV-torsion. Hence R/I is a w-flat module by [17, Theorem 6.7.3].

(4) $\Rightarrow$ (5): Let I be a finitely generated semi-regular ideal of R. Then $\mathrm{Tor}_1^R(R/I, R/I)$ is GV-torsion since R/I is a w-flat module by (4). That is, I/I^2 is GV-torsion, and thus $I \subseteq (I)_w = (I^2)_w$.

(5) $\Rightarrow$ (1): Let $I = \langle a_1, \dots, a_n\rangle$ is finitely generated semi-regular ideal. There exists a GV-ideal J such that $JI \subseteq I^2$. We claim that I is also a GV-ideal. Indeed, suppose J is generated by $\{j_1, \cdots, j_m\}$. For each $k = 1, \cdots, m$, we have $j_k a_i = \sum\limits_{j=1}^{n} r_{ij} a_j$ for some suitable $r_{ij} \in I$. The column vector $\mathbf{a} \in R^n$ whose i-th coordinate is a_i, and the matrix $\mathbf{A} = \|j_k \delta_{ij} - r_{ij}\|$, where δ_{ij} is the Kronecker symbol, satisfy $\mathbf{A}\mathbf{a} = 0$. Hence $\det(\mathbf{A})\mathbf{a} = 0$. Since I is semi-regular, we have $\det(\mathbf{A}) = 0$. So $j_k^n + j_k^{n-1} r_1 + \cdots + r_n = 0$ for some $r_i \in I$. Thus $j_k^n \in I$ for each $k = 1, \cdots, m$. By Lemma 4.1, we have $J^{nn} \subseteq \langle j_k^n \mid k = 1, \cdots, m\rangle \subseteq I$. Since J^{nn} is a GV-ideal, the finitely generated semi-regular ideal I is also a GV-ideal (see [17, Proposition 6.1.9]).

(6) $\Rightarrow$ (1): Since R is a w-module, then it is a Lucas module. So R is a WQ-ring by [20, Proposition 3.8].

(1) $\Leftrightarrow$ (7): See [20, Proposition 3.8]. $\square$

It was proved in [24, Theorem 3.1] that a ring R is a DQ-ring (i.e. the only finitely generated semi-regular ideal of R is R itself) if and only if every R-module is semi-regular flat. Hence rings with sr-w-weak global dimensions equal to 0 and sr-weak global dimensions equal to 0 do not coincide.

Example 4.3. [11, Example 12] Let $D = L[X^2, X^3, Y]$, $\mathcal{P} = Spec(D) - \{\langle X^2, X^3, Y\rangle\}$, $B = \bigoplus\limits_{\mathfrak{p} \in \mathcal{P}} K(R/\mathfrak{p})$ and $R = D(+)B$ where L is a field and $K(R/\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$. Since $Q_0(R) = R$, R is a WQ-ring by [20, Proposition 3.8]. Since every finitely generated R-ideal of the form $J(+)B$ with $\sqrt{J} = \langle X^2, X^3, Y\rangle$ is a GV-ideal, R is not a DW-ring. Hence R is not a DQ-ring by [21, Proposition 2.2].

Recall from Lucas [12] that an ideal I of R is said to be t-invertible if there is an R-submodule J of $Q_0(R)$ such that $(IJ)_t = R$, and R is called a Q_0-PvMR if every finitely generated semi-regular ideal of R is t-invertible. From [16, Proposition 4.17], a semi-regular ideal is t-invertible if and only if it is w-invertible. So a ring R is a Q_0-PvMR if and only if every finitely generated semi-regular ideal is w-invertible. Recall from [16] that an R-module M is said to be a *w-projective module* if $\mathrm{Ext}^1_R((M/\mathrm{Tor}_{\mathrm{GV}}(M))_w, N)$ is a GV-torsion module for any torsion-free w-module N. Recall from [16] that an R-module M is said to be semi-regular if there are a positive integer n and a chain of submodules of M:

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that every factor module M_i/M_{i-1} is w-isomorphic to a semi-regular ideal of R.

Theorem 4.4. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a Q_0-*PvMR*;*

(2) *any submodule of a semi-regular w-flat R-module is semi-regular w-flat;*

(3) *any submodule of a w-flat R-module is semi-regular w-flat;*

(4) *any ideal of R is semi-regular w-flat;*

(5) *any finitely generated* (*resp., finite type*) *ideal of R is semi-regular w-flat;*

(6) *any finitely generated* (*resp., finite type*) *semi-regular ideal of R is w-flat;*

(7) *any finitely generated* (*resp., finite type*) *semi-regular ideal of R is w-projective;*

(8) *sr-w-$w.gl.dim(R) \leq 1$;*

(9) *any finitely generated* (*resp., finite type*) *semi-regular R-module is w-projective.*

Proof. Since the classes of semi-regular w-flat modules, w-flat modules, w-projective modules and w-invertible ideals are closed under w-isomorphism and every finite type ideal is isomorphic to a finitely generated sub-ideal, we just need to consider the "finitely generated" cases in (5), (6), (7) and (9).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), (7) $\Rightarrow$ (6) and (9) $\Rightarrow$ (5): Trivial.

(5) $\Leftrightarrow$ (6): Let I be a finitely generated semi-regular ideal of R and J a finitely generated ideal of R. Then we have $\mathrm{Tor}^R_1(R/J, I) \cong \mathrm{Tor}^R_2(R/I, R/J) \cong \mathrm{Tor}^R_1(R/I, J)$. Consequently, J is semi-regular w-flat if and only if I is w-flat.

(6) $\Rightarrow$ (1): Let I be a finitely generated semi-regular ideal of R and $\mathfrak{m}$ a maximal w-ideal of R. Then $I_{\mathfrak{m}}$ is finitely generated flat $R_{\mathfrak{m}}$-ideal. By [7, Lemma 4.2.1] and [14, Theorem 2.5], we have $I_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-ideal. So the rank of $I_{\mathfrak{m}}$ is at most 1. Hence I is w-invertible by [16, Theorem 4.13].

(1) $\Rightarrow$ (6): Let I be a finitely generated semi-regular ideal of R and $\mathfrak{m}$ a maximal w-ideal of R. Then $I_{\mathfrak{m}}$ is a principal $R_{\mathfrak{m}}$-ideal by [16, Theorem 4.13]. Suppose $I_{\mathfrak{m}} = \langle \frac{x}{s} \rangle$. Then $(0 :_{R_{\mathfrak{m}}} \frac{x}{s}) = (0 :_{R_{\mathfrak{m}}} I_{\mathfrak{m}}) = (0 :_R I)_{\mathfrak{m}} = 0$ by [17, Exercise 1.72]. Thus $\frac{x}{s}$ is regular element. So $I_{\mathfrak{m}} \cong R_{\mathfrak{m}}$. Consequently, I is a w-flat R-ideal.

(6) $\Rightarrow$ (2): Let M be a semi-regular w-flat module and N a submodule of M. Suppose I is a finitely generated semi-regular ideal, then I is a w-flat ideal. Thus $w\text{-fd}_R(R/I) \leq 1$. Consider the exact sequence

$$\mathrm{Tor}_2^R(R/I, M/N) \to \mathrm{Tor}_1^R(R/I, N) \to \mathrm{Tor}_1^R(R/I, M).$$

Since $\mathrm{Tor}_2^R(R/I, M/N)$ and $\mathrm{Tor}_1^R(R/I, M)$ are GV-torsion, we have $\mathrm{Tor}_1^R(R/I, N)$ is GV-torsion. So N is a semi-regular w-flat module.

(1) $\Rightarrow$ (7): Let I be a finitely generated semi-regular ideal of R. Then I is w-invertible, and hence w-projective by [16, Theorem 4.13].

(3) $\Leftrightarrow$ (8): By Proposition 3.2 and Corollary 3.4.

(1) $\Rightarrow$ (9): See [16, Theorem 4.23]. $\square$

The following examples show that regular w-flat ideals are not necessary semi-regular w-flat and semi-regular w-flat ideals are not necessary semi-regular flat.

Example 4.5. [12, Example 8.10] Let $D = \mathbb{Z} + (Y, Z)\mathbb{Q}[[Y, Z]]$ and let $\mathcal{P}$ be the set of height one primes of D. Let $R = B + B$ be the $A + B$ ring corresponding to D and $\mathcal{P}$. It was showed that R is a PvMR but not a Q_0-PvMR. Hence there exists a regular w-flat ideal which is not semi-regular w-flat by Theorem 4.4 and [23, Theorem 4.8].

Example 4.6. [12, Example 8.11] Let $E = D[Z]$ where D is a Dedekind domain with a maximal ideal $N = \langle a, b \rangle$ for which no power of N is principal. Let $\mathcal{P}$ be the set of primes of E which contain neither Z nor NE. Set $R = B + B$. It was showed that R is a Q_0-PvMR but not a strong Prüfer ring. Hence there exists a semi-regular w-flat ideal which is not semi-regular flat by Theorem 4.4 and [24, Theorem 3.4].

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