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Research Article

Generalized eigenvectors of linear operators and biorthogonal systems

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ABSTRACT. The notions of a set of generalized eigenvalues and a set of generalized eigenvectors of a linear operator in Euclidean space are introduced. In addition, we provide a method to find a biorthogonal system of a subsystem of eigenvectors of some linear operators in a Hilbert space whose systems of canonical eigenvectors are over-complete. Related to our problem, we will show an example of a linear differential operator that is formally adjoint to Bessel-type differential operators. We also investigate the basic properties (completeness, minimality, basicity) of the systems of generalized eigenvectors of this differential operator.

Keywords: Linear operator, generalized eigenvector, Bessel function, complete system, minimal system, biorthogonal system.

2020 Mathematics Subject Classification: 33C10, 34B30, 34L10.

1. INTRODUCTION

Let $\mathcal{H}$ be an Euclidean space with inner product $\langle\cdot;\cdot\rangle : \mathcal{H}\times\mathcal{H}\to\mathbb{C}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $m\in\mathbb{N}_0$, $\overline{n;m}=[n;m]\cap\mathbb{N}_0$ and $\overline{n;m}=\emptyset$ if $n>m$. Suppose that a certain linear operator $A:\mathcal{H}\to\mathcal{H}$ has a countable set of simple eigenvalues $\{\lambda_k : k\in\mathbb{N}\}$ and a corresponding system of eigenvectors $\{\psi_k : k\in\mathbb{N}\}$ that is complete and minimal after removing, for example, the first $m\in\mathbb{N}$ members, or the adjoint operator of A has no eigenvalues. Such operators arise naturally in the study of some boundary value problems (see, for example, [3, 4, 10, 14, 16] and the reference therein), for instance, in the study of boundary value problems for Bessel's equation (see [8, 12, 13, 18, 19, 25, 26]). The problem is how to find a biorthogonal system $(U_n : n\in\mathbb{N}\backslash\overline{1;m})$. Such a biorthogonal system will be found if we can find the vectors U_n such that $\langle\psi_k;U_n\rangle=0$ for all $k\in\mathbb{N}\backslash\overline{1;m}$ and $n\in\mathbb{N}\backslash\overline{1;m}$.

Finding such biorthogonal systems often faces certain difficulties (see [3, 4, 8, 12, 13, 18, 19, 25, 26]). Sometimes, in the case of simple eigenvalues, such vectors U_n can be found by using a notion of a set of generalized eigenvectors which we propose in this paper (see Section 2). There are different methods to introduce the generalized eigenvectors with access to a wider space (for details, see [2, 3, 4, 5, 9]). The peculiarity of our interpretation of a set of generalized eigenvectors of a linear operator $B:\mathcal{H}\to\mathcal{H}$ with domain $\mathcal{D}(B)$ is that the generalized eigenvectors belong to $\mathcal{H}$ and the difference of eigenvectors belong to $\mathcal{D}(B)$. We show an example of a linear differential operator $B_\nu:\mathcal{H}^\nu\to\mathcal{H}^\nu$ in some Hilbert space $\mathcal{H}^\nu$ that has no eigenvectors, but has the generalized eigenvectors (see Section 3). In Sections 4 and 5, we will prove that this operator, B_ν, is formally adjoint to Bessel-type differential operators $\widetilde{A}_\nu:\mathcal{H}^\nu\to\mathcal{H}^\nu$ and

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$A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ whose systems of canonical eigenvectors are over-complete. We also investigate the basic properties (completeness, minimality, basicity) of the systems of generalized eigenvectors of an operator B_ν.

The introduced notions of the sets of generalized eigenvalues and eigenvectors probably are of interest in some sense for spectral theory.

2. Generalized eigenvectors

Let $\Omega \subseteq \mathbb{N}$ be some non-empty set.

Definition 2.1. *The set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is called a set of generalized eigenvalues of a linear operator $B : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(B)$ in a vector (linear) space $\mathcal{H}$ if there exists a set $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ of nonzero elements $U_j \in \mathcal{H}$ such that $U_n - U_k \in \mathcal{D}(B)$ and $B(U_n - U_k) = \mu_n U_n - \mu_k U_k$ for every $n \in \Omega$ and $k \in \Omega$. In this case, the set $\mathfrak{U}(B)$ is called a set of generalized eigenvectors of an operator B.*

We say that an operator $B : \mathcal{H} \to \mathcal{H}$ is a *formally adjoint* of an operator $A : \mathcal{H} \to \mathcal{H}$ in a Euclidean space $\mathcal{H}$ with inner product $\langle \cdot ; \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, if $\langle A\psi; u\rangle = \langle \psi; Bu\rangle$ for all $\psi \in \mathcal{D}(A)$ and $u \in \mathcal{D}(B)$.

Theorem 2.1. *Suppose that $A : \mathcal{H} \to \mathcal{H}$ be a linear operator with domain $\mathcal{D}(A)$ in a Euclidean space $\mathcal{H}$ with inner product $\langle \cdot ; \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ having a set of eigenvalues $\{\lambda_j : j \in \Omega\}$ and a set of eigenvectors $\{\psi_j : j \in \Omega\}$. Let each $\mu_j = \overline{\lambda_j}$ be a generalized eigenvalue of an operator $B : \mathcal{H} \to \mathcal{H}$ that is a formally adjoint of A, and let $\{U_j : j \in \Omega\}$ be a set of generalized eigenvectors of B. Then $\langle \psi_k; U_n\rangle = 0$ if $\lambda_k \neq \lambda_n$.*

Proof. Indeed,

$$\begin{aligned}
\lambda_k \langle \psi_k; U_n\rangle &= \lambda_k \langle \psi_k; U_n - U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle A\psi_k; U_n - U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle \\
&= \langle \psi_k; B(U_n - U_k)\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle \psi_k; \mu_n U_n - \mu_k U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle \\
&= \langle \psi_k; \mu_n U_n\rangle - \langle \psi_k; \mu_k U_k\rangle + \lambda_k \langle \psi_k; U_k\rangle = \langle \psi_k; \mu_n U_n\rangle \\
&= \lambda_n \langle \psi_k; U_n\rangle,
\end{aligned}$$

whence the theorem follows. Theorem 2.1 is proved. □

A linear operator can has several sets of generalized eigenvalues. The union of two such sets may not be a set of generalized eigenvalues. Every set of eigenvalues is a set of generalized eigenvalues. If for some $b \in \mathcal{H}$ and each $j \in \Omega$, and the numbers μ_j, the equation $B(u) = \mu_j u + b$ has a nonzero solution $u_j \in \mathcal{D}(B)$, then the set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is a set of generalized eigenvalues of an operator $B : \mathcal{H} \to \mathcal{H}$. If $\mathcal{D}(B) = \mathcal{H}$ and the set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ is a set of generalized eigenvalues of an operator $B : \mathcal{H} \to \mathcal{H}$, then there exists $b \in \mathcal{H}$ such that for every $k \in \Omega$ the equation $B(u) = \mu_k u + b$ has a nonzero solution $u_k \in \mathcal{H}$. In this case, $b = B(U_n) - \mu_n U_n$ and $n \in \Omega$ is arbitrary. If $\mathcal{D}(B) \neq \mathcal{H}$, then a linear operator $B : \mathcal{H} \to \mathcal{H}$ can has generalized eigenvectors of other kinds.

Definition 2.2. *Let $m \in \mathbb{N}_0$ and $\Omega = \mathbb{N} \backslash \overline{1;m}$. The set $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ of generalized eigenvalues of a linear operator $B : \mathcal{H} \to \mathcal{H}$ is called a set of generalized eigenvalues of width m (with respect to an operator $\widehat{B}$) if there exists a vector space $\widehat{\mathcal{H}}$ and a linear operator $\widehat{B} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ with domain $\mathcal{D}(\widehat{B})$ that has a countable set of eigenvalues $\{\mu_k : k \in \mathbb{N}\}$ and a set of eigenvectors $\{\widehat{u}_k : k \in \mathbb{N}\}$ such that $\widehat{\mathcal{H}} \cap \mathcal{H} \neq \emptyset$, $U_n - U_k \in \mathcal{D}(\widehat{B})$, $B(U_n - U_k) = \widehat{B}(U_n - U_k)$ for any $n \in \Omega$ and $k \in \Omega$, and*

$$U_s := \widehat{u}_s + \sum_{i \in \overline{1;m}} \omega_{i,s} \widehat{u}_i \in \mathcal{H}, \quad \omega_{i,s} := (\mu_s - \mu_i)^{-1},\ s \in \Omega.$$

In this case, the set $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ *is called a set of generalized eigenvectors of width* m.

Theorem 2.2. *Assume that* $m \in \mathbb{N}_0$, $\Omega = \mathbb{N}\backslash\overline{1;m}$, $B : \mathcal{H} \to \mathcal{H}$ *be a linear operator in a vector space* $\mathcal{H}$, *and* $\{U_j : j \in \Omega\}$ *be some set of nonzero elements of the space* $\mathcal{H}$. *Let there exist a vector space* $\widehat{\mathcal{H}}$ *and a linear operator* $\widehat{B} : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ *with a countable set of eigenvalues* $\{\mu_k : k \in \mathbb{N}\}$ *and a set of eigenvectors* $\{\widehat{u}_k : k \in \mathbb{N}\}$ *satisfying* $\widehat{\mathcal{H}} \cap \mathcal{H} \neq \emptyset$,

$$U_s := \widehat{u}_s + \sum_{i\in\overline{1;m}} \frac{1}{\mu_s - \mu_i}\widehat{u}_i \in \mathcal{H}, \quad s \in \Omega,$$

and $U_n - U_k \in \mathcal{D}(\widehat{B})$, $B(U_n - U_k) = \widehat{B}(U_n - U_k)$ *for every* $n \in \Omega$ *and* $k \in \Omega$. *Then* $\mathfrak{M}(B) = \{\mu_j : j \in \Omega\}$ *is a set of generalized eigenvalues of width* m *of an operator* B, *and* $\mathfrak{U}(B) = \{U_j : j \in \Omega\}$ *is a set of generalized eigenvectors of width* m.

Proof. Indeed, we have

$$\begin{aligned} B(U_n - U_k) &= \widehat{B}(U_n - U_k) \\ &= \mu_n\widehat{u}_n + \sum_{i\in\overline{1;m}} \frac{\mu_i}{\mu_n - \mu_i}\widehat{u}_i - \mu_k\widehat{u}_k - \sum_{i\in\overline{1;m}} \frac{\mu_i}{\mu_k - \mu_i}\widehat{u}_i \\ &= \mu_n\left(\widehat{u}_n + \sum_{i\in\overline{1;m}} \frac{1}{\mu_n - \mu_i}\widehat{u}_i\right) - \mu_k\left(\widehat{u}_k + \sum_{i\in\overline{1;m}} \frac{1}{\mu_k - \mu_i}\widehat{u}_i\right) \\ &+ \sum_{i\in\overline{1;m}} \frac{\mu_k - \mu_i}{\mu_k - \mu_i}\widehat{u}_i + \sum_{i\in\overline{1;m}} \frac{\mu_i - \mu_n}{\mu_n - \mu_i}\widehat{u}_i \\ &= \mu_n U_n - \mu_k U_k. \end{aligned}$$

Theorem 2.2 is proved. □

Remark 2.1. *Due to Theorem 2.2, if* U_k *and* U_n *are the generalized eigenvectors of width* m *of an operator* $B : \mathcal{H} \to \mathcal{H}$, *then*

$$\sum_{i\in\overline{1;m}} ((\omega_{i,n} - \omega_{i,k})\mu_i\widehat{u}_i - (\omega_{i,n}\mu_n - \omega_{i,k}\mu_k)\widehat{u}_i) = 0$$

for every $k \in \Omega$ *and* $n \in \Omega$, *because*

$$\begin{aligned} B(U_n - U_k) &= \widehat{B}(U_n - U_k) \\ &= \widehat{B}\left(\widehat{u}_n + \sum_{i\in\overline{1;m}} \omega_{i,n}\widehat{u}_i - \widehat{u}_k - \sum_{i\in\overline{1;m}} \omega_{i,k}\widehat{u}_i\right) \\ &= \mu_n\widehat{u}_n - \mu_k\widehat{u}_k + \sum_{i\in\overline{1;m}} (\omega_{i,n} - \omega_{i,k})\mu_i\widehat{u}_i, \end{aligned}$$

$$\mu_n U_n - \mu_k U_k = \mu_n\left(\widehat{u}_n + \sum_{i\in\overline{1;m}} \omega_{i,n}\widehat{u}_i\right) - \mu_k\left(\widehat{u}_k + \sum_{i\in\overline{1;m}} \omega_{i,k}\widehat{u}_i\right).$$

Theorems 2.1 and 2.2 indicate the method of finding a biorthogonal system that can be used in certain cases. In this paper, for illustrative purposes, we shall prove that there exists an operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ in some Hilbert space $\mathcal{H}^\nu$ that has no eigenvalues, but has generalized eigenvalues and corresponding eigenvectors of width $m \in \{0;1;2\}$ (see Theorem 3.3). We also study the properties of this operator B_ν (see Theorems 4.4 and 5.5).

To prove Theorems 3.3, 4.4 and 5.5, we need some preliminaries.

3. Operator B_ν

Let $C(\Delta)$ be a vector space of continuous functions $f : \Delta \to \mathbb{C}$ on the interval $\Delta \subset \mathbb{C}$, and $C^{(k)}(\Delta)$ be a set of functions $f \in C(\Delta)$ with $f^{(k)} \in C(\Delta)$. Let $\alpha \in \mathbb{R}$ and $L^2((0;1);x^\alpha dx)$ be the space of measurable functions $f : (0;1) \to \mathbb{C}$ such that $t^{\alpha/2} f(t) \in L^2(0;1)$; the inner product and the norm in $L^2((0;1);x^\alpha dx)$ are given by $\langle f_1; f_2\rangle = \int_0^1 t^\alpha f_1(t)\overline{f_2(t)}\,dt$ and $\|f\| = \sqrt{\int_0^1 t^\alpha |f(t)|^2\,dt}$, respectively. Let also

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$$

be a Bessel function of the first kind of index $\nu \in \mathbb{R}$, where Γ is the gamma function. The function J_ν is a solution (see, for instance, [1, 17, 27]) of the equation $y''+y'/x+(1-\nu^2/x^2)y = 0$, the function $y(x) = J_\nu(xs)$ is a solution of the equation $-y'' - y'/x + y\nu^2/x^2 = s^2 y$, and the functions $y(x) = \sqrt{xs}J_{\pm\nu}(xs)$ satisfy the equation

$$-y'' + \frac{\nu^2 - 1/4}{x^2} y = s^2 y.$$

For $\nu > -1$, the function J_ν has (see [1, p. 59], [17, p. 350], [27, p. 483]) an infinite set $\{\widetilde{s}_k : k \in \mathbb{Z}\}$ of real zeros, among them $\widetilde{s}_k$, $k \in \mathbb{N}$, are the positive zeros and $\widetilde{s}_{-k} := -\widetilde{s}_k$, $k \in \mathbb{N}$, are the negative zeros. All zeros are simple except, perhaps, the zero $\widetilde{s}_0 = 0$. For $\nu > 1$, the function $J_{-\nu}$ has (see [1, p. 59], [27, p. 483]) an infinity of real zeros and also $2[\nu]$ pairwise conjugate complex zeros, among them two pure imaginary zeros when $[\nu]$ is an odd integer. Let s_k, $k \in \mathbb{N}$, be the zeros of the function $J_{-\nu}$ for which $\operatorname{Im} s_k > 0$ if $s_k \in \mathbb{C}$ or $s_k > 0$ if $s_k \in \mathbb{R}$.

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu := L^2((0;1);x^{2\nu-1}dx)$ and B_ν is the operator generated by the formal differential operator

$$\ell_\nu^*(u) := -u'' - 2(2\nu-1)\frac{1}{x}u' - 3((\nu-1)^2 - 1/4)\frac{1}{x^2}u$$

with domain $\mathcal{D}(B_\nu)$ consisting of all functions $u \in C^{(2)}(0;1]$ satisfying the boundary conditions

$$u(x) = O(x^{-\nu+5/2}), \quad x \to 0+, \tag{3.1}$$

$$u(1) = 0, \tag{3.2}$$

and the asymptotic equality (3.1) can be twice differentiated termwise. Then $\ell_\nu^*(u) = O(x^{-\nu+1/2})$ as $x \to 0+$, and $B_\nu(u) \in \mathcal{H}^\nu$ if $u \in \mathcal{D}(B_\nu)$. Let also $\widehat{\mathcal{H}} = C(0;1]$ and $\widehat{B}_\nu$ is the operator generated by the formal differential operator $\ell_\nu^*(u)$ with domain $\mathcal{D}(\widehat{B}_\nu)$ consisting of all functions $u \in C^{(2)}(0;1]$ satisfying the boundary condition (3.2). Then $\widehat{B}_\nu(u) \in \widehat{\mathcal{H}}$ if $u \in \mathcal{D}(\widehat{B}_\nu)$.

In this section, we shall prove the following theorem.

Theorem 3.3. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then the operator B_ν has no eigenvalues. In this case, $\widetilde{\mathfrak{M}}(B_\nu) = \{\widetilde{\mu}_k : k \in \mathbb{N}\}$, $\widetilde{\mu}_k = \widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν, is the set of generalized eigenvalues of width $m = 0$ of an operator B_ν that corresponds to the operator $\widehat{B}_\nu$, and*

$$\widetilde{U}_{k,\nu}(x) := \frac{\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)}{\widetilde{s}_k^{\nu+1/2}x^{2\nu-1}}, \quad k \in \mathbb{N}$$

are the generalized eigenfunctions of width $m=0$ of the operator B_ν. Besides, the set $\mathfrak{M}(B_{3/2})=\{\mu_k : k\in\mathbb{N}\setminus\{1\}\}$, $\mu_k=s_k^2$, where s_k are the zeros of $J_{-\nu}$, is a set of generalized eigenvalues of width $m=1$ of the operator $B_{3/2}$ which correspond to the operator $\widehat{B}_{3/2}$, and

$$U_{k,3/2}(x):=\frac{s_k\sqrt{xs_k}J_{-3/2}(xs_k)-s_1\sqrt{xs_1}J_{-3/2}(xs_1)}{x^2(s_1^2-s_k^2)},\quad k\in\mathbb{N}\setminus\{1\}$$

are the generalized eigenfunctions of width $m=1$ of $B_{3/2}$. In addition, the set $\mathfrak{M}(B_{5/2})=\{\mu_k : k\in\mathbb{N}\setminus\{1;2\}\}$, $\mu_k=s_k^2$, is a set of generalized eigenvalues of width $m=2$ of an operator $B_{5/2}$ that corresponds to the operator $\widehat{B}_{5/2}$, and

$$\begin{aligned}U_{k,5/2}(x):=&\frac{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k)-s_1^2\sqrt{xs_1}J_{-5/2}(xs_1)}{x^4(s_k^2-s_1^2)}\\&-\frac{s_k^2\sqrt{xs_k}J_{-5/2}(xs_k)-s_2^2\sqrt{xs_2}J_{-5/2}(xs_2)}{x^4(s_k^2-s_2^2)},\quad k\in\mathbb{N}\setminus\{1;2\}\end{aligned}$$

are the generalized eigenfunctions of width $m=2$ of the operator $B_{5/2}$.

To prove Theorem 3.3, we need some auxiliary lemmas.

Lemma 3.1. *Let $l\in\mathbb{N}$ and $\nu=l+1/2$. Then the operator B_ν has no eigenvalues.*

Proof. In fact, in the case $s=0$, the functions $u_1(x)=x^{-\nu+3/2}$ and $u_2(x)=x^{-3\nu+3/2}$ are the linearly independent solutions of the equation $u''+2(2\nu-1)x^{-1}u'+3((\nu-1)^2-1/4)x^{-2}u=-s^2u$. In the case $s\neq 0$, the linearly independent solutions of this equation are the functions $v_1(x)=x^{-2\nu+1}\sqrt{xs}J_\nu(xs)$ and $v_2(x)=x^{-2\nu+1}\sqrt{xs}J_{-\nu}(xs)$. Using relation (see [15, p. 226], [17, p. 346], [27, p. 43])

$$J_\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}+O(x^{\nu+2}),\quad x\to 0,$$

we obtain

$$\frac{\sqrt{xs}J_\nu(xs)}{x^{2\nu-1}}=\frac{s^{\nu+1/2}}{2^\nu\Gamma(\nu+1)}x^{-\nu+3/2}+O(x^{-\nu+7/2}),\quad x\to 0+,\tag{3.3}$$

$$\frac{\sqrt{xs}J_{-\nu}(xs)}{x^{2\nu-1}}=\sum_{k\in\overline{0;\nu}}\frac{(-1)^k s^{-\nu+2k+1/2}x^{-3\nu+2k+3/2}}{2^{-\nu+2k}k!\Gamma(-\nu+k+1)}+O(x^{-3\nu+2[\nu]+7/2}),\quad x\to 0+.\tag{3.4}$$

In view of this, every nonzero linear combination of these functions cannot satisfy (3.1), and hence this operator has no eigenfunctions. Lemma 3.1 is proved. □

Let $l\in\mathbb{N}$, $\nu=l+1/2$, $\widehat{\mathcal{H}}=C(0;1]$ and $\widehat{B}_\nu$ is the operator generated by the formal differential operator $\ell_\nu^*(u)$ with domain $\mathcal{D}(\widehat{B}_\nu)$ consisting of all functions $u\in C^{(2)}(0;1]$ satisfying the boundary condition (3.2). Then $\widehat{B}_\nu(u)\in\widehat{\mathcal{H}}$ if $u\in\mathcal{D}(\widehat{B}_\nu)$.

Lemma 3.2. *Let $l\in\mathbb{N}$ and $\nu=l+1/2$. Then $\widetilde{\mathfrak{M}}(B_\nu)=\{\widetilde{\mu}_k : k\in\mathbb{N}\}$, $\widetilde{\mu}_k=\widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν, is the set of generalized eigenvalues of width $m=0$ of an operator B_ν which correspond to the operator $\widehat{B}_\nu$, and $\widetilde{U}_{k,\nu}(x)$, $k\in\mathbb{N}$, are the generalized eigenfunctions of width $m=0$ of B_ν.*

Proof. Indeed, the numbers $\widetilde{\mu}_k=\widetilde{s}_k^2$ are the eigenvalues of the operator $\widehat{B}_\nu$, and $\widehat{u}_{k,\nu}(x)=\widetilde{U}_{k,\nu}(x)=\widetilde{s}_k^{-\nu-1/2}x^{-2\nu+1}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)$ are the eigenfunctions of this operator. Further, $\widetilde{U}_{k,\nu}\in\mathcal{H}^\nu$, by using (3.3)

$$\widetilde{U}_{k,\nu}(x)=\frac{1}{2^\nu\Gamma(\nu+1)}x^{-\nu+3/2}+O(x^{-\nu+7/2}),\quad x\to 0+.$$

Besides,

$$\widetilde{U}_{k,\nu}(x) - \widetilde{U}_{n,\nu}(x) = O(x^{-\nu+7/2}) = O(x^{-\nu+5/2}), \quad x \to 0+.$$

Therefore, $\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu} \in \mathcal{D}(B_\nu)$ and $\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu} \in \mathcal{D}(\widehat{B}_\nu)$. In addition,

$$B_\nu(\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu}) = \widehat{B}_\nu(\widetilde{U}_{k,\nu} - \widetilde{U}_{n,\nu}) = \ell^*_\nu(\widehat{u}_{k,\nu} - \widehat{u}_{n,\nu}) = \widetilde{s}^2_k\widehat{u}_{k,\nu} - \widetilde{s}^2_n\widehat{u}_{n,\nu} = \widetilde{s}^2_k\widetilde{U}_{k,\nu} - \widetilde{s}^2_n\widetilde{U}_{n,\nu}.$$

Hence, $\widetilde{\mathfrak{M}}(B_\nu) = \{\widetilde{\mu}_k : k \in \mathbb{N}\}$ is a set of generalized eigenvalues of width $m = 0$ of the operator B_ν, and $\widetilde{\mathfrak{U}}(B_\nu) = \{\widetilde{U}_{k,\nu} : k \in \mathbb{N}\}$ is the set of generalized eigenfunctions of width $m = 0$. Lemma 3.2 is proved. □

Lemma 3.3. *Let s_k, $k \in \mathbb{N}$ be the zeros of the function $J_{-\nu}$. Then $\mathfrak{M}(B_{3/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1\}\}$, $\mu_k = s^2_k$, is a set of generalized eigenvalues of width $m = 1$ of the operator $B_{3/2}$ which correspond to the operator $\widehat{B}_{3/2}$, and $U_{k,3/2}(x)$, $k \in \mathbb{N} \setminus \{1\}$, are the generalized eigenfunctions of width $m = 1$ of $B_{3/2}$.*

Proof. Indeed, the numbers $\mu_k = s^2_k$ are the eigenvalues of the operator $\widehat{B}_{3/2}$, and the functions $\widehat{u}_{k,3/2}(x) = x^{-2}(s^2_1 - s^2_k)^{-1}s_k\sqrt{xs_k}J_{-3/2}(xs_k)$, $k \neq 1$, and $\widehat{u}_{1,3/2}(x) = x^{-2}s_1\sqrt{xs_1}J_{-3/2}(xs_1)$ are their corresponding eigenfunctions. Moreover, $U_{k,3/2}(x) = \widehat{u}_{k,3/2}(x) + \omega_{1,k}\widehat{u}_{1,3/2}(x)$ if $\omega_{1,k} = (s^2_k - s^2_1)^{-1}$. Using (3.4), we obtain

$$U_{k,3/2}(x) = \frac{1}{\sqrt{2\pi x}} + O(x), \quad x \to 0+.$$

Therefore, $U_{k,3/2} \in \mathcal{H}^{3/2}$. Besides, $U_{k,3/2}(x) - U_{n,3/2}(x) = O(x)$ as $x \to 0+$. Hence, $U_{k,3/2} - U_{n,3/2} \in \mathcal{D}(B_{3/2})$, $U_{k,3/2} - U_{n,3/2} \in \mathcal{D}(\widehat{B}_{3/2})$, and

$$\begin{aligned}
B_{3/2}(U_{k,3/2} - U_{n,3/2}) &= \widehat{B}_{3/2}(U_{k,3/2} - U_{n,3/2}) \\
&= \ell^*_{3/2}(U_{k,3/2} - U_{n,3/2}) \\
&= \ell^*_{3/2}(\widehat{u}_{k,3/2} + \omega_{1,k}\widehat{u}_{1,3/2} - \widehat{u}_{n,3/2} - \omega_{1,n}\widehat{u}_{1,3/2}) \\
&= s^2_k\widehat{u}_{k,3/2} + \omega_{1,k}s^2_1\widehat{u}_{1,3/2} - s^2_n\widehat{u}_{n,3/2} - \omega_{1,n}s^2_1\widehat{u}_{1,3/2} \\
&= s^2_k(\widehat{u}_{k,3/2} + \omega_{1,k}\widehat{u}_{1,3/2}) - s^2_n(\widehat{u}_{n,3/2} + \omega_{1,n}\widehat{u}_{1,3/2}) \\
&\quad + (s^2_1(\omega_{1,k} - \omega_{1,n}) - (\omega_{1,k}s^2_k - \omega_{1,n}s^2_n))\widehat{u}_{1,3/2} \\
&= s^2_kU_{k,3/2} - s^2_nU_{n,3/2}.
\end{aligned}$$

Thus, $\mathfrak{M}(B_{3/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1\}\}$ is the set of generalized eigenvalues of the operator $B_{3/2}$, and $\mathfrak{U}(B_{3/2}) = \{U_{k,3/2} : k \in \mathbb{N} \setminus \{1\}\}$ is a set of generalized eigenfunctions of width $m = 1$. Lemma 3.3 is proved. □

Lemma 3.4. *Let s_k, $k \in \mathbb{N}$, be the zeros of the function $J_{-\nu}$. Then $\mathfrak{M}(B_{5/2}) = \{\mu_k : k \in \mathbb{N} \setminus \{1; 2\}\}$, $\mu_k = s^2_k$, is a set of generalized eigenvalues of width $m = 2$ of an operator $B_{5/2}$ which corresponds to the operator $\widehat{B}_{5/2}$, and $U_{k,5/2}(x)$, $k \in \mathbb{N} \setminus \{1; 2\}$, are the generalized eigenfunctions of width $m = 2$ of $B_{5/2}$.*

Proof. In fact, the numbers $\mu_k = s^2_k$ are the eigenvalues of the operator $\widehat{B}_{5/2}$, and the functions

$$\widehat{u}_{k,5/2}(x) = \frac{s^2_k(s^2_1 - s^2_2)\sqrt{xs_k}J_{-5/2}(xs_k)}{x^4(s^2_k - s^2_1)(s^2_k - s^2_2)}, \quad k \in \mathbb{N} \setminus \{1; 2\},$$

$\widehat{u}_{1,5/2}(x) = -x^{-4}s^2_1\sqrt{xs_1}J_{-5/2}(xs_1)$ and $\widehat{u}_{2,5/2}(x) = x^{-4}s^2_2\sqrt{xs_2}J_{-5/2}(xs_2)$ are their corresponding eigenfunctions. Moreover, $U_{k,5/2}(x) = \widehat{u}_{k,5/2}(x) + \omega_{1,k}\widehat{u}_{1,5/2}(x) + \omega_{2,k}\widehat{u}_{2,5/2}(x)$ if

$\omega_{i,k} = (s_k^2 - s_i^2)^{-1}$, $i \in \{1;2\}$. Using (3.4), we get

$$U_{k,5/2}(x) = \frac{s_1^2 - s_2^2}{4\sqrt{2\pi}x^2} + O(1), \quad x \to 0+.$$

Therefore, $U_k \in \mathcal{H}^{5/2}$. Furthermore, $U_{k,5/2}(x) - U_{n,5/2}(x) = O(1)$ as $x \to 0+$. Hence, $U_{k,5/2} - U_{n,5/2} \in \mathcal{D}(B_{5/2})$, $U_{k,5/2} - U_{n,5/2} \in \mathcal{D}(\widehat{B}_{5/2})$ and

$$\begin{aligned} B_{5/2}(U_{k,5/2} - U_{n,5/2}) &= \widehat{B}_{5/2}(U_{k,5/2} - U_{n,5/2}) \\ &= \ell^*_{5/2}(U_{k,5/2} - U_{n,5/2}) \\ &= \ell^*_{5/2}(\widehat{u}_{k,5/2} + \omega_{1,k}\widehat{u}_{1,5/2} + \omega_{2,k}\widehat{u}_{2,5/2} - \widehat{u}_{n,5/2} - \omega_{1,n}\widehat{u}_{1,5/2} - \omega_{2,n}\widehat{u}_{2,5/2}) \\ &= s_k^2\widehat{u}_{k,5/2} + \omega_{1,k}s_1^2\widehat{u}_{1,5/2} + \omega_{2,k}s_2^2\widehat{u}_{2,5/2} - s_n^2\widehat{u}_{n,5/2} - \omega_{1,n}s_1^2\widehat{u}_{1,5/2} - \omega_{2,n}s_2^2\widehat{u}_{2,5/2} \\ &= s_k^2(\widehat{u}_{k,5/2} + \omega_{1,k}\widehat{u}_{1,5/2} + \omega_{2,k}\widehat{u}_{2,5/2}) - s_n^2(\widehat{u}_{n,5/2} + \omega_{1,n}\widehat{u}_{1,5/2} + \omega_{2,n}\widehat{u}_{2,5/2}) \\ &+ (s_1^2(\omega_{1,k} - \omega_{1,n}) - (\omega_{1,k}s_k^2 - \omega_{1,n}s_n^2))\widehat{u}_{1,5/2} + (s_2^2(\omega_{2,k} - \omega_{2,n}) \\ &- (\omega_{2,k}s_k^2 - \omega_{2,n}s_n^2))\widehat{u}_{2,5/2} \\ &= s_k^2 U_{k,5/2} - s_n^2 U_{n,5/2}. \end{aligned}$$

Thus, $\mathfrak{M}(B_{5/2})$ is the set of generalized eigenvalues of an operator $B_{5/2}$, and $U_{k,5/2}$ are the generalized eigenfunctions of width $m = 2$. Lemma 3.4 is proved. □

Remark 3.2. *$U_{k,\nu} - \widetilde{U}_{n,\nu} \notin \mathcal{D}(B_\nu)$ if $\nu = 3/2$ or $\nu = 5/2$. Lemmas 3.2–3.4 are leaving aside the existence of other sets of generalized eigenvalues. We have not been able to extend Lemma 3.4 to an arbitrary $\nu = l + 1/2$ with $l \in \mathbb{N}$.*

Theorem 3.3 is an immediate consequence of Lemmas 3.1–3.4.

4. Operator $\widetilde{A}_\nu$ and Approximation Properties of the System $(\widetilde{U}_k : k \in \mathbb{N})$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{H}^*$ its dual space, i.e., the space of linear continuous functionals on $\mathcal{H}$. The system of elements $(e_k : k \in \mathbb{N})$ is called *complete* ([11, p. 4258]) in $\mathcal{H}$ if $\overline{\text{span}}\,(e_k : k \in \mathbb{N}) = \mathcal{H}$. The system of elements $(e_k : k \in \mathbb{N})$ is said to be *minimal* ([11, p. 4258]) in $\mathcal{H}$ if $e_{k_0} \notin \overline{\text{span}}\,(e_k : k \in \mathbb{N} \setminus \{k_0\})$ for each $k_0 \in \mathbb{N}$. The system $(e_k : k \in \mathbb{N})$ is called ([11, p. 4258]) a *basis* for the space $\mathcal{H}$ if, for every $f \in \mathcal{H}$, there exists a unique series with respect to the system $(e_k : k \in \mathbb{N})$ which converges to f (in $\mathcal{H}$): $f = \sum_{k=1}^{\infty} d_k e_k$, $d_k \in \mathbb{C}$. Minimality of the system $(e_k : k \in \mathbb{N})$ in $\mathcal{H}$ is equivalent (see [11, p. 4258]) to the existence of the system of conjugate functionals $(f_k : k \in \mathbb{N}) \in \mathcal{H}^*$, i.e., $f_k(e_n) = \delta_{kn}$, where δ_{kn} is the Kronecker delta. The system $(f_k : k \in \mathbb{N})$ is also called a *biorthogonal system* with respect to the system $(e_k : k \in \mathbb{N})$. A system $(e_k : k \in \mathbb{N})$ is said to be *uniformly minimal* ([11, p. 4258]) in $\mathcal{H}$ if there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ the distance of e_n to the closure of the linear span of the system $(e_k : k \in \mathbb{N} \setminus \{n\})$ is greater than $\delta\|u_n\|$. A complete system $(e_k : k \in \mathbb{N})$ that has a biorthogonal system $(f_k : k \in \mathbb{N})$ is uniformly minimal if and only if (see [11, p. 4258])

$$\limsup_{k\to\infty} \|e_k\|^2\|f_k\|^2 < +\infty.$$

Every basis is uniformly minimal system (see [11, p. 4258]).

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu = L^2((0;1); x^{2\nu-1}dx)$, and $\widetilde{A}_\nu$ is the operator generated by the formal differential operator $\ell_\nu(\psi) := -\psi'' + (\nu^2 - 1/4)x^{-2}\psi$ with domain $\mathcal{D}(\widetilde{A}_\nu)$ consisting of those functions $\psi \in C^{(2)}[0;1]$ which satisfy the boundary conditions $\psi(0) = \psi(1) = 0$. Then $\ell_\nu(\psi) = O(x^{-1})$ as $x \to 0+$, and $\widetilde{A}_\nu(\psi) \in \mathcal{H}^\nu$ if $\psi \in \mathcal{D}(\widetilde{A}_\nu)$.

In this section, we prove that the operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ is formally adjoint in H^ν of an operator $\widetilde{A}_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$. We also investigate completeness, minimality and basicity of the system $(\widetilde{U}_k : k \in \mathbb{N})$ of generalized eigenfunctions of width $m = 0$ of a operator B_ν.

Lemma 4.5. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the operator B_ν is formally adjoint in H^ν of an operator $\widetilde{A}_\nu$.*

Proof. Let $a_0 = (\nu^2 - 1/4)x^{2\nu-3}$, $a_2 = -x^{2\nu-1}$ and $\widetilde{\ell}_\nu(\psi) = a_2\psi'' + a_0\psi$. Then $\widetilde{\ell}^*_\nu(u) = (\overline{a}_2 u)'' + \overline{a}_0 u$ is formally adjoint operator of the operator $\widetilde{\ell}_\nu(\psi)$ (see [9, p. 97]). Moreover, $\widetilde{\ell}_\nu(\psi) = -x^{2\nu-1}\psi'' + (\nu^2 - 1/4)x^{2\nu-3}\psi = x^{2\nu-1}\ell_\nu(\psi)$ and $\widetilde{\ell}^*_\nu(u) = \overline{a}_2 u'' + 2\overline{a}'_2 u' + (\overline{a}_0 + \overline{a}''_2)u = -x^{2\nu-1}u'' - 2(2\nu-1)x^{2\nu-2}u' - 3((\nu-1)^2 - 1/4)x^{2\nu-3}u = x^{2\nu-1}\ell^*_\nu(u)$. Furthermore, according to the Lagrange identity (see [9, p. 97]), for every $\psi \in \mathcal{D}(\widetilde{A}_\nu)$ and $u \in \mathcal{D}(B_\nu)$,

$$\begin{aligned} x^{2\nu-1}(\ell_\nu(\psi)\overline{u} - \psi\overline{\ell^*_\nu(u)}) &= \widetilde{\ell}_\nu(\psi)\overline{u} - \psi\overline{\widetilde{\ell}^*_\nu(u)} \\ &= \frac{d}{dx}((a_2\psi' - \psi a'_2)\overline{u} - \psi a_2 \overline{u}') \\ &= \frac{d}{dx}((-x\psi' + (2\nu-1)\psi)x^{2\nu-2}\overline{u} + x^{2\nu-1}\psi\overline{u}'). \end{aligned} \tag{4.5}$$

Hence,

$$\int_0^1 x^{2\nu-1}\ell_\nu(\psi)\overline{u}\,dx = \int_0^1 x^{2\nu-1}\psi\overline{\ell^*_\nu(u)}\,dx.$$

Lemma 4.5 is proved. □

Lemma 4.6 ([21, 6, 7])**.** *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$, $\widetilde{U}_{k,\nu}(x) = \widetilde{s}_k^{-\nu-1/2} x^{-2\nu+1}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)$ is complete in the space $\widetilde{\mathcal{H}}^\nu := L^2((0;1); x^{4\nu-4}dx)$.*

Lemma 4.7. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$ in the space $\mathcal{H}^\nu$ has a biorthogonal system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ that is formed by the functions*

$$\widetilde{\gamma}_{k,\nu}(x) := \frac{2\widetilde{s}_k^{\nu-1/2}}{J^2_{\nu+1}(\widetilde{s}_k)}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k).$$

The system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is a system of eigenfunctions of an operator $\widetilde{A}_\nu$ which correspond to their eigenvalues $\widetilde{\mu}_k = \widetilde{s}_k^2$, where $\widetilde{s}_k$ are the zeros of J_ν.

Proof. Since (see [17, p. 347], [27, p. 482])

$$\int_0^1 xJ_\nu(x\widetilde{s}_k)J_\nu(x\widetilde{s}_n)\,dx = \begin{cases} \frac{1}{2}J^2_{\nu+1}(\widetilde{s}_n), & k = n, \\ 0, & k \neq n, \end{cases}$$

it follows that

$$\begin{aligned} \int_0^1 x^{2\nu-1}\widetilde{U}_{k,\nu}(x)\overline{\widetilde{\gamma}_{n,\nu}(x)}\,dx &= \frac{2\sqrt{\widetilde{s}_k\widetilde{s}_n}\,\widetilde{s}_n^{\nu-1/2}}{\widetilde{s}_k^{\nu+1/2}J^2_{\nu+1}(\widetilde{s}_n)}\int_0^1 xJ_\nu(x\widetilde{s}_k)J_\nu(x\widetilde{s}_n)\,dx \\ &= \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases} \end{aligned}$$

Furthermore, since $\ell_\nu(\widetilde{\gamma}_{k,\nu}) = \widetilde{s}_k^2 \widetilde{\gamma}_{k,\nu}$ and $J_\nu(x) = O(x^\nu)$ as $x \to 0$, we conclude that the numbers $\widetilde{\mu}_k = \widetilde{s}_k^2$ are the eigenvalues of an operator $\widetilde{A}_\nu$, and $\widetilde{\gamma}_{k,\nu}(x)$, $k \in \mathbb{N}$ are the corresponding eigenfunctions of this operator. Lemma 4.7 is proved. □

Lemma 4.8. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is complete in the space $\mathcal{H}^\nu$.*

Proof. Assume the contrary. Then, according to the Hahn-Banach theorem ([11, p. 4258]), there exists a nonzero function $h \in \mathcal{H}^\nu$ such that

$$\frac{2\widetilde{s}_k^{\nu-1/2}}{J_{\nu+1}^2(\widetilde{s}_k)} \int_0^1 x^{2\nu-1} \sqrt{x\widetilde{s}_k} J_\nu(x\widetilde{s}_k) h(x)\, dx = 0, \quad k \in \mathbb{N}.$$

Let $q(x) = x^{2\nu-1}h(x)$. Then $q \in L^2(0;1)$ and, therefore, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is incomplete in the space $L^2(0;1)$. We have a contradiction, because it is well known that the system $(\sqrt{x}J_\nu(x\widetilde{s}_k) : k \in \mathbb{N})$ is complete in $L^2(0;1)$ (see [15, p. 223], [17, p. 357]). Thus, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is complete in $\mathcal{H}^\nu$. Lemma 4.8 is proved. □

We remark that Lemma 4.7 also follows from Lemmas 4.5, 4.8, 3.2 and Theorem 2.1.

Lemma 4.9. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is not a basis in the space $\mathcal{H}^\nu$.*

Proof. Using relations (see [15, p. 226], [17, pp. 346, 352], [27, pp. 43, 618])

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right), \quad x \to \infty,$$

$$J_\nu(x) = O(x^\nu), x \to 0, \widetilde{s}_k = \pi k + \frac{\pi\nu}{2} - \frac{\pi}{4} + O(k^{-1})$$

and

$$|\sqrt{\widetilde{s}_k} J_{\nu+1}(\widetilde{s}_k)| = \sqrt{2/\pi}(1 + O(k^{-1})) \text{ as } k \to \infty,$$

we get

$$\begin{aligned}
\|\widetilde{U}_{k,\nu}\|^2_{\mathcal{H}^\nu} \|\widetilde{\gamma}_{k,\nu}\|^2_{\mathcal{H}^\nu} &= \frac{4}{J_{\nu+1}^4(\widetilde{s}_k)} \int_0^1 x|J_\nu(x\widetilde{s}_k)|^2\, dx \int_0^1 x^{2\nu}|J_\nu(x\widetilde{s}_k)|^2\, dx \\
&= \frac{O(\widetilde{s}_k^{4\nu})}{J_{\nu+1}^4(\widetilde{s}_k)} \\
&= O(\widetilde{s}_k^{4\nu+2}) \longrightarrow +\infty, \quad k \to \infty.
\end{aligned}$$

Hence, the system $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ is not uniformly minimal in the space $\mathcal{H}^\nu$ and therefore is not a basis in this space. □

From Lemmas 4.5–4.9, we obtain the following assertion.

Theorem 4.4. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\widetilde{U}_{k,\nu} : k \in \mathbb{N})$ of the generalized eigenfunctions of width $m = 0$ of an operator B_ν is complete in the space $\widetilde{\mathcal{H}}^\nu$ and minimal in $\mathcal{H}^\nu$. Moreover, the operator B_ν is formally adjoint in $\mathcal{H}^\nu$ of an operator $\widetilde{A}_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ which has a complete and minimal system of eigenfunctions $(\widetilde{\gamma}_{k,\nu} : k \in \mathbb{N})$ such that is not a basis in $\mathcal{H}^\nu$.*

Remark 4.3. *Basis properties (completeness, minimality, basicity) of more general systems $(\Theta_{k,\nu,p} : k \in \mathbb{N})$ with $\Theta_{k,\nu,p}(x) = x^{-p-1}\sqrt{x\widetilde{s}_k}J_\nu(x\widetilde{s}_k)$ in the space $L^2((0;1); x^{2p}dx)$, where $\nu \geq 1/2$, $p \in \mathbb{R}$ and $(\widetilde{s}_k)_{k\in\mathbb{N}}$ is a sequence of distinct nonzero complex numbers, have been studied in* [6, 7, 20, 21, 22, 23, 24].

5. Operator A_ν

Let $\nu = l + 1/2$ with $l \in \mathbb{N}$, $\mathcal{H}^\nu = L^2((0;1); x^{2\nu-1}dx)$, and A_ν is the operator generated by the formal differential operator $\ell_\nu(\psi)$ and the boundary conditions

$$\psi(1) = 0, \tag{5.6}$$

$$\psi(x) = \sum_{j\in\overline{0;\nu}} c_j x^{-\nu+2j+1/2} + o(x^{\nu+1/2}), \quad x \to 0+ \tag{5.7}$$

for some constants $c_j \in \mathbb{C}$, $j \in \overline{0;\nu}$. Suppose that the domain $\mathcal{D}(A_\nu)$ consists of those functions $\psi \in C^{(2)}(0;1]$ that satisfy these boundary conditions and the asymptotic equality (5.7) can be twice differentiated termwise. Then $\ell_\nu(\psi) = 4c_1(-1+\nu)x^{-\nu+1/2} + o(x^{-\nu+1/2}) + o(x^{\nu-3/2}) = O(x^{-\nu+1/2})$ as $x \to 0+$, and $A_\nu(\psi) \in \mathcal{H}^\nu$ if $\psi \in \mathcal{D}(A_\nu)$.

In this section, we show that the operator $B_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ is formally adjoint in $\mathcal{H}^\nu$ of an operator $A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ whose systems of canonical eigenfunctions are over-complete. We also remark about basis properties of the systems of generalized eigenfunctions of width $m \in \{1;2\}$ of an operator B_ν.

Lemma 5.10 ([26]). *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. The operator A_ν has a finite set $\{\mu_k : k \in \mathbb{N}\}$ of eigenvalues, where $\mu_k = s_k^2$ and s_k are the zeros of the function $J_{-\nu}$. Moreover, the functions $\psi_{k,\nu}(x) := s_k^{\nu-1/2}\sqrt{xs_k}J_{-\nu}(xs_k)$, $k \in \mathbb{N}$, are the eigenfunctions of this operator.*

Lemma 5.11 ([13]). *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. Then, the system $(\psi_{k,\nu} : k \in \mathbb{N} \setminus \{1;2;\ldots;l\})$ is complete in $\mathcal{H}^\nu$.*

Lemma 5.12. *Let $l \in \mathbb{N}$ and $\nu = l + 1/2$. The operator B_ν is formally adjoint in $\mathcal{H}^\nu$ of an operator A_ν.*

Proof. Using relations (3.1), (3.2), (5.6), (5.7) and

$$\psi'(x) = \sum_{j\in\overline{0;\nu}} c_j(-\nu + 2j + 1/2)x^{-\nu+2j-1/2} + o(x^{\nu-1/2}), \quad x \to 0+,$$

from (4.5), it follows that

$$\int_0^1 x^{2\nu-1}\ell_\nu(\psi)\overline{u}\,dx = \int_0^1 x^{2\nu-1}\psi\overline{\ell_\nu^*(u)}\,dx.$$

Lemma 5.12 is proved. □

Remark 5.4. *From Lemmas 3.3, 5.10, 5.12 and Theorem 2.1, it follows that $\langle \psi_{k,3/2}; U_{n,3/2}\rangle = 0$, if $k \neq n$, $k \in \mathbb{N}\backslash\{1\}$ and $n \in \mathbb{N}\backslash\{1\}$. By direct calculations, we get $\langle \psi_{n,3/2}; U_{n,3/2}\rangle = 1$ (see also [18, 19, 25]). Lemma 5.11 implies that the system $(\psi_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ is complete in the space $\mathcal{H}^{3/2}$. Moreover, in [25, 26] the authors proved that this system is minimal and is not a basis in $\mathcal{H}^{3/2}$. Furthermore, the biorthogonal system is formed by the functions $g_{k,3/2}(x) = \pi s_k^{-4}(1+s_k^2)(s_1^2 - s_k^2)U_{k,3/2}(x)$. In [19], it was shown that the system $(g_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ is also complete in $\mathcal{H}^{3/2}$. In addition, in [19] it has been established that the system $(U_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ has in the space $\mathcal{H}^{3/2}$ a biorthogonal system $(\gamma_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ that is formed by the functions $\gamma_{k,3/2}(x) = \pi s_k^{-4}(1+s_k^2)(s_1^2 - s_k^2)\psi_{k,3/2}(x)$.*

Since $s_k = \pi k - \frac{1}{\pi k} + o(k^{-3})$ *as* $k \to \infty$ *(see [1, 27]), and*

$$\begin{aligned}&\|U_{k,3/2}\|^2_{\mathcal{H}^{3/2}}\|\gamma_{k,3/2}\|^2_{\mathcal{H}^{3/2}}\\ =&\frac{\pi^2(1+s_k^2)^2}{s_k^9}\int_0^{s_k}|t\sqrt{t}J_{-3/2}(t)|^2\,dt\int_0^1\frac{|s_k\sqrt{ts_k}J_{-3/2}(ts_k)-s_1\sqrt{ts_1}J_{-3/2}(ts_1)|^2}{t^2}\,dt\\ =&\frac{\pi(1+s_k^2)^2}{9s_k^3}(1+o(1))\longrightarrow+\infty,\quad k\to\infty,\end{aligned}$$

the system $(U_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ *is not uniformly minimal in* $\mathcal{H}^{3/2}$ *and, hence, is not a basis in this space. Lemma 5.11 implies that the system* $(\psi_{k,5/2} : k \in \mathbb{N}\setminus\{1; 2\})$ *is complete in the space* $\mathcal{H}^{5/2}$*. From Lemmas 3.4, 5.10, 5.12 and Theorem 2.1, it follows that* $\langle\psi_{k,5/2}; U_{n,5/2}\rangle = 0$ *if* $k \neq n$*,* $k \in \mathbb{N} \setminus \{1; 2\}$ *and* $n \in \mathbb{N}\setminus\{1; 2\}$*. In [12], it was proven by some other method that the system* $(\psi_{k,5/2} : k \in \mathbb{N})$ *has in* $\mathcal{H}^{5/2}$ *a biorthogonal system* $(U_{k,5/2} : k \in \mathbb{N} \setminus \{1; 2\})$*. However, the problem of finding a biorthogonal system* $(U_{k,\nu} : k \in \mathbb{N} \setminus \{1; 2; \ldots; l\})$ *to the system* $(\psi_{k,\nu} : k \in \mathbb{N} \setminus \{1; 2; \ldots; l\})$ *for an arbitrary* $\nu = l + 1/2$ *with* $l \in \mathbb{N} \setminus \{1; 2\}$ *remains open.*

From Lemmas 5.10–5.12 and Remark 5.4, we obtain the following statement.

Theorem 5.5. *Let* $l \in \mathbb{N}$ *and* $\nu = l + 1/2$*. Then the system* $(U_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ *of the generalized eigenfunctions of width* $m = 1$ *of an operator* $B_{3/2}$ *is complete, minimal and is not a basis in the space* $\mathcal{H}^{3/2}$*. The biorthogonal system* $(\gamma_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ *also is complete in* $\mathcal{H}^{3/2}$*. Furthermore, the system* $(U_{k,5/2} : k \in \mathbb{N} \setminus \{1; 2\})$ *of the generalized eigenfunctions of width* $m = 2$ *of an operator* $B_{5/2}$ *is minimal in the space* $\mathcal{H}^{5/2}$*, and its biorthogonal system* $(\psi_{k,5/2} : k \in \mathbb{N}\setminus\{1; 2\})$ *is complete in* $\mathcal{H}^{5/2}$*. Moreover, the operator* B_ν *is also formally adjoint in* $\mathcal{H}^\nu$ *of an operator* $A_\nu : \mathcal{H}^\nu \to \mathcal{H}^\nu$ *whose system of eigenfunctions* $(\psi_{k,\nu} : k \in \mathbb{N})$ *is complete after removing a finite number of eigenfunctions, i.e., the system* $(\psi_{k,\nu} : k \in \mathbb{N} \setminus \{1; 2; \ldots; l\})$ *is complete in* $\mathcal{H}^\nu$*.*

Remark 5.5. *Let* $f \in \mathcal{H}^{3/2}$ *and* $d_k = \int_0^1 t^2 f(t)\overline{g_{k,3/2}(t)}\,dt$*. Since the system* $(g_{k,3/2} : k \in \mathbb{N} \setminus \{1\})$ *is complete in the space* $\mathcal{H}^{3/2}$*, the numbers* d_k *determine the function* $f \in \mathcal{H}^{3/2}$ *uniquely. But, the series*

$$\sum_{k=2}^{\infty} d_k\psi_{k,3/2}(x), \quad \psi_{k,3/2}(x) = s_k\sqrt{xs_k}J_{-3/2}(xs_k)$$

does not converge for each function $f \in \mathcal{H}^{3/2}$ *in* $\mathcal{H}^{3/2}$ *to the function* f*. We do not know whether it converges in some sense, for example, whether a given series converges in* $\mathcal{H}^{3/2}$ *to* f *in the sense of Cesàro. Similar questions arise for the other series that can be constructed by using the above considered biorthogonal systems.*

6. Concluding remarks

In this paper, the notions of a set of generalized eigenvalues and a set of generalized eigenvectors of a linear operator in an Euclidean space are introduced. A method is described to find a biorthogonal system of a subsystem of eigenvectors of linear operators in a Hilbert space whose systems of canonical eigenvectors are over-complete. This is illustrated by an example of a linear differential operator that is formally adjoint to Bessel-type differential operators. Also, basic properties of the systems of generalized eigenvectors of those differential operators are studied. Those results can be used for the investigations in spectral theory and nonharmonic analysis.

Remark that there are other points of view on how to study similar problems (see, for example, [2, 3, 4, 5, 9, 10, 16] and the bibliography in them).

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