CONTENT ANALYSIS OF QUALITATIVE STUDIES ON IRRATIONAL NUMBERS IN TURKEY: A META-SYNTHESIS STUDY[1] [2]

TÜRKİYE'DE YAPILAN İRRASYONEL SAYILARA YÖNELİK NİTEL ÇALIŞMALARIN İÇERİK ANALİZİ: META SENTEZ ÇALIŞMASI

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(Araştırma Makalesi)

Abstract: The concept of number, which forms the basis of mathematics, is also an essential part of mathematics education. On the other hand, irrational numbers are one of the number sets that students have the most difficulty making sense of. In this study, descriptive content analysis and thematic content analysis (meta-synthesis) methods were used to determine the current status of qualitative studies on irrational numbers in Turkey, interpret them with a holistic perspective, and present systematic summary information. As a result of the research, although there are many studies in the national field to reveal conceptual knowledge about irrational numbers and to identify misconceptions, the scarcity of reflections on concept teaching has attracted attention. In addition, it has been seen that it will be more effective to emphasize the equivalence of different representations of numbers in instruction and make instructional designs by focusing on the fundamental properties of irrational numbers. The fact that the studies are mainly based on radical expressions, a form of representation, and few reflections on concept teaching, reveals the need for studies that deal only with irrational numbers in learning and teaching processes.

Keywords: *Irrational numbers, mathematics education, meta-synthesis, content analysis*

Özet: Matematiğin temelini oluşturan sayı kavramı, matematik eğitiminin de önemli bir parçasıdır. İrrasyonel sayılar ise öğrencilerin anlamlandırmada en çok zorlandığı sayı kümelerinden biridir. Bu araştırmada, Türkiye'de irrasyonel sayı kavramına yönelik yapılan nitel çalışmaların mevcut durumunu belirlemek, bütüncül bir bakış açısıyla yorumlamak ve sistematik özet bilgiler sunmak için betimsel içerik analizi ve tematik içerik analizi (meta sentez) yöntemi kullanılmıştır. Araştırma sonucunda ulusal alanda irrasyonel sayılara ilişkin kavramsal bilgiyi ortaya çıkarmaya ve kavram yanılgılarını belirlemeye yönelik çok sayıda çalışma yer almasına rağmen kavram öğretimine dair çalışmaların azlığı dikkat çekmiştir. Ayrıca sayıların farklı gösterim biçimlerinin denkliğinin öğretimde daha fazla vurgulanmasının ve irrasyonel sayıların temel özelliklerine odaklanılarak öğretim tasarımlarının yapılmasının daha etkili olacağı görülmüştür. Yapılan çalışmaların ağırlıklı olarak bir gösterim biçimi olan köklü ifadeler özelinde olması ve kavram öğretimine yönelik az sayıda çalışmanın olması, salt irrasyonel sayıları öğrenme ve öğretme süreçleri bağlamında ele alan çalışmalara olan ihtiyacı ortaya koymaktadır.

Anahtar Sözcükler: *İrrasyonel sayı, matematik eğitimi, meta sentez, içerik analizi*

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Introduction

Numbers, which form the basis of mathematics (Hossack, 2020), have an important role in scientific and technological contexts and daily life (Corry, 2015). The concept of number means *"mathematical objects used for actions such as counting, labeling, sorting, coding, measuring, etc."* (Argün, Arıkan, Bulut, & Halıcıoğlu, 2014, p. 443), while collections of objects, each element of which is a number, are known as the set of numbers. Due to their importance, the concept of number and number sets has found a vast place in mathematics teaching programs (MoNE, 2018a, 2018b; NCTM, 2000). Teaching of number concept starts from pre-school and is included in the secondary school curriculum with the "numbers and operations" learning area. When we look at the mathematics teaching programs (MoNE, 2018a, 2018b; NCTM, 2000), the teaching of number sets has spread to the curriculum gradually due to the cumulative nature of mathematics. Accordingly, the teaching of counting numbers, natural numbers, integers, and rational numbers sets until the eighth grade in the Ministry of National Education [MoNE] (2018a) mathematics curriculum is handled with different representations. The teaching of irrational numbers occurs in the eighth grade for the first time.

Number sense, which is at the center of *Principles and Standards for School Mathematics* (NCTM, 2000), develops when the individual discovers numbers and establishes relationships between numbers (Howden, 1989; Sowder, 1992). Number sense, which is a way of thinking about numbers and the multiple relations between numbers (Sowder, 1992), develops with the meaning of the number concept and number sets, which is called the "cornerstone" of mathematics (NCTM, 2000). Since number sets have a strong relationship, the difficulty in learning any number set will affect all number comprehension. One of the numbers sets that students have the most problem with is irrational numbers (Arbour, 2012; Fischbein, Jehiam, & Cohen, 1995).

Looking at the history of mathematics, irrational numbers were first discovered 2500 years ago (Havil, 2012) with the idea of incommensurability due to trying to calculate the diagonal length of the unit square with the Pythagorean theorem ($a^2 + b^2 = c^2$) (Arcavi, Bruckheimer, & Ben-Zvi, 1987). Indeed, the most crucial property of irrational numbers is indivisibility. In other words, irrational numbers emerged from the idea that non-rational numbers could also exist, and "numbers that cannot be expressed as the ratio of two integers (Havil, 2012)" or "numbers with a non-repeating decimal notation (Anatriello & Vincenzi, 2019; Havil, 2012; O'Connor, 2003)".

There are many studies on irrational numbers in the national literature. Presenting the qualitative studies on irrational numbers, which is the aim of this research, in the form of a systematic summary with a holistic view will help determine the need for research on irrational numbers. In this context, the study problem is "How is the current situation of qualitative studies on irrational numbers in Turkey?" determined. The sub-problems of the research are as follows:

1. How does qualitative studies' distribution on irrational numbers in terms of research type, year, design, study group, and size?
2. How do the qualitative studies on irrational numbers distribute in terms of the subject, the aim of the study, the teaching method, and the use of theory?
3. How do the results of qualitative studies on irrational numbers distribute in terms of characterizing the concept and the causes of difficulties?

Method

In this study, descriptive and thematic content analysis methods (meta-synthesis) were used to determine the current status of qualitative studies on irrational numbers in the national literature. The descriptive content analysis includes systematic studies in which the general tendencies of qualitative and quantitative studies on the same subject are evaluated descriptively (Çalık & Sözbilir, 2014). Meta-synthesis, which is used as *"evaluation of evaluation* (Weed, 2006, p. 6)*"*, is a method that involves combining the results of qualitative studies on a specific subject under common themes (Douglas et al., 2008). The essential point of systematic review in meta-synthesis studies is to define the inclusion and exclusion criteria from the beginning to limit the scope of the review (Weed, 2006). The requirements for obtaining the studies examined in this study are as follows:

i. To take place in Council of Higher Education Thesis Center and DergiPark databases.
ii. To be in the field of mathematics education.
iii. It is made about irrational numbers and related topics (such as radical expressions).
iv. To be published by October 2021.

v. To have a qualitative research method.
vi. Be open to online access.
vii. Include the thesis in the study if there is a thesis and an article belonging to the same author and review the article if the thesis is not accessible.

In line with the above criteria, the studies were examined under various categories through irrational numbers and related concepts, and general approaches and tendencies were determined. Deficiencies that will provide insight into future research have been tried to be revealed, and systematic analyzes have been presented.

Data Collection

The data of this research consists of 28 studies, 12 theses, and 16 articles published between 2000-2021, focusing on irrational numbers in mathematics education. The distribution of the types of qualitative research in the national literature is given in Figure 1.

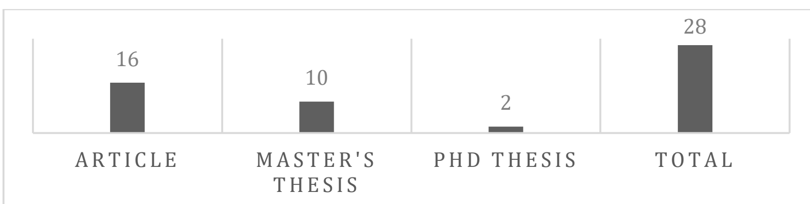

Figure 1. Distribution of data used in the study according to research types

To obtain the study's data, the Council of Higher Education Thesis Center and DergiPark databases were scanned four times in total, and the scans were ended on 24.10.2021. The databases were searched with the keywords *"irrational number, square root, square root expression, radical expression, square root number, radical number, pi number, e number."* By the literature on irrational numbers, each research was examined by preparing a "Data collection tool form (Table 1)" by both descriptive and thematic content analysis (meta-synthesis). In the coding process, the themes were arranged in line with the information obtained from the studies and the main template is given in Table 1.

Table 1

Data Collection Tool Form

INFO ON THE STUDY	
Research Type	Article [ ] Master's thesis [ ] Ph.D. thesis [ ]
Year of publication	
Title of the study	
Author (s)	
RESEARCH QUESTION	
SUB-RESEARCH QUESTIONS	

STUDY GROUP		**METHOD**
Secondary School ( )	Study group size	Case Study ( )
High School ( )		Action Research ( )
Undergraduate ( )		Material Development ( )
Graduate ( )		Mixed Methods Research ( )
Teachers ( )		

TEACHING METHOD (If applicable)	**SUBJECT OF RESEARCH**
Concept Cartoons ( )	Real numbers and Complex numbers ( )
Realistic Mathematics Education ( )	Real numbers ( )
	Irrational numbers ( )
	Radical/Square root expression ( )
	e and/or π numbers ( )

AIM OF THE STUDY	**THEORY USED (If applicable)**
To reveal conceptual knowledge ( )	APOS Theory ( )
To design instruction ( )	RBC + C Theory ( )
To identify misconception ( )	Concept Image and Concept Definition Theoretical Framework ( )
To review textbooks ( )	Object-Process Duality Framework ( )
To examine teacher knowledge ()	

RESULTS FOR CHARACTERIZING THE CONCEPT OF IRRATIONAL NUMBERS

Acceptable Characterizations	**Incorrect Characterizations**
Non-rational number ( )	Non-real numbers ( )
Numbers that cannot go out of the root ( )	Radical numbers ( )
Numbers that cannot be written in the form of a/b ( )	Numbers whose numerator and denominator are not integers ( )
Numbers with an approximate value ( )	Examples of numbers like i, $\frac{22}{7}$, $\frac{5}{0}$, $(0,\bar{9})$, $(3{,}14)$ ( )
Examples of numbers like $\sqrt{2}$, $\sqrt{3}$, π, e ( )	Numbers that involve rational numbers ( )
Numbers with non-repeating decimal notation ( )	Complex numbers ( )
Real numbers ( )	Repeating numbers ( )
	Decimal numbers ( )
	Numbers that do not have a place on the number line ( )

CAUSES OF DIFFICULTIES WITH THE CONCEPT OF AN IRRATIONAL NUMBER
Causes arising from incorrect or incomplete characterizations ( )
Causes arising from relationships between number sets and representations ( )
Causes arising from the curriculum ( )

Analysis of Data

First, descriptive content analysis was used to analyze the data of this study. Then, thematic content analysis (meta-synthesis) was used to examine, interpret, and evaluate the qualitative findings in depth. For descriptive content analysis, data such as the type of research, publication year, study group, research method were discussed, and descriptive findings were presented. Based on the data for the meta-synthesis, three main themes were determined by the authors of this study: "the subject addressed," "the aim of the study," and "the results of the studies." Explanations on these themes are given in Table 2.

Table 2

Explanations on the Themes and Sub-Themes Created in the Study

Themes	Sub-themes	Explanations
The subject addressed	Real numbers and complex numbers	Includes studies focusing on real numbers and complex number sets.
	Real numbers	Includes studies focusing on rational and irrational number sets.
	Irrational numbers	Includes studies focusing only on the set of irrational numbers.
	Radical expression	Includes studies focusing on radical or square root expressions, which are one of the ways of representing numbers.
	e and/or π numbers	Includes studies focusing on examples of irrational numbers.
The aim of the study	To identify misconception	Includes studies to identify the study group's misconceptions/difficulties and views about the subject.
	To reveal conceptual knowledge	Includes studies to determine the conceptual understanding of the study group about the subject and to examine their conceptual knowledge.
	To design instruction	Includes studies on teaching the subject addressed.
	To review textbooks	Includes studies on the analysis of textbooks on the subject addressed.
	To examine teacher knowledge	Includes studies on teachers' or prospective teachers' knowledge of teaching mathematics.
The results of the studies	Results for characterizing the concept of irrational numbers	Includes studies intend for the results of the study group regarding the acceptable or incorrect characterization of the irrational number concept.
	Causes of difficulties with the concept of an irrational number	Includes studies that show why the study group has difficulty with irrational numbers.

Results

This section is divided into two parts: the findings obtained from descriptive analysis and meta-synthesis.

Findings Obtained as a Result of Descriptive Content Analysis

As a result of the descriptive content analysis, conclusions regarding the type of research (Figure 1), year, research design, study group, and size were obtained. In this research, the distribution of qualitative studies on the concept of irrational numbers in the national literature published between 2000-2021 across years is given in Figure 2.

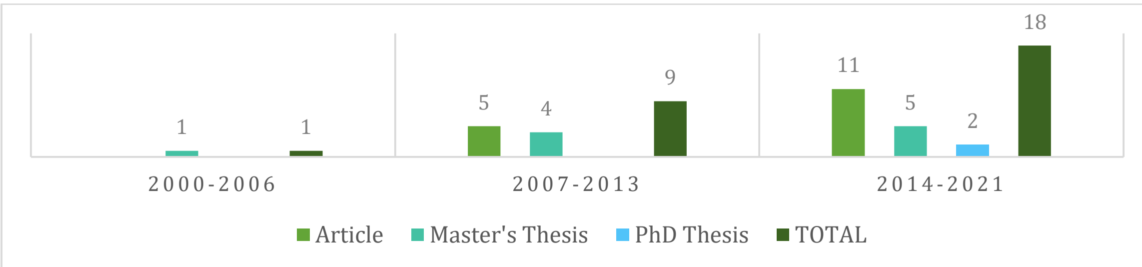

Figure 2. Distribution of studies across years

When Figure 2 is examined, it is seen that qualitative studies on irrational numbers have increased in the last eight years (n=18) and doctoral dissertation studies (n=2) started in this period. The distribution of the studies across the qualitative research designs is given in Figure 3.

*Figure 3***.** Distribution of studies across qualitative research design

As seen in Figure, case studies (n=22) were generally used in the studies. Apart from these, it is possible to come across mixed method research (n=3) in which qualitative and quantitative research methods are used together and studies that make the instructional design (n=2) in the literature. The distribution of the studies across the study groups is given in Figure 4.

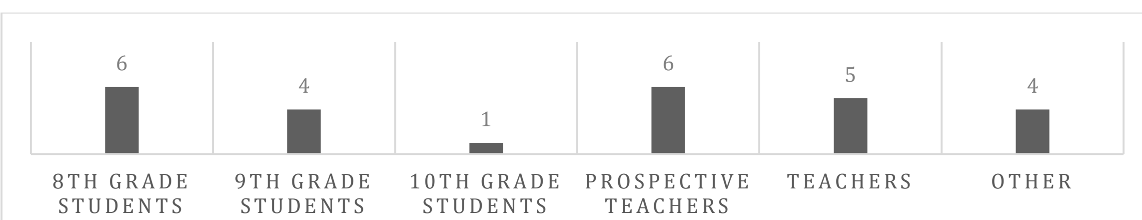

Figure 4. Distribution of studies across the study group

When Figure 4 is examined, it is seen that the studies were mainly conducted with prospective mathematics teachers (n=6) and eighth-grade students (n=6). More than one study group was used in the "other" group studies. Apart from this, in studies aiming to design instruction (n=2), no application was made with any study group. The distribution of the sizes of the study groups is given in Figure 5.

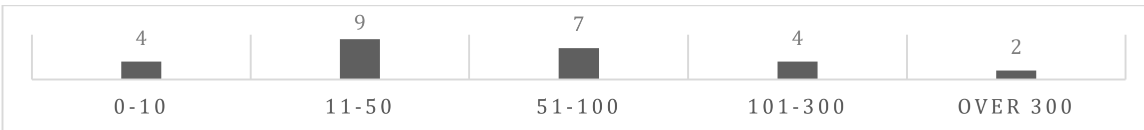

Figure 5. Distribution of study groups across their size

When Figure 5 is examined, it is seen that the size of the qualitative study groups is mostly chosen between 11-50 (n=9). No study group was used in studies (n=2) aiming to design instruction.

Findings Obtained as a Result of Meta-Synthesis

The findings obtained as a result of the meta-synthesis were combined under three themes as "subject addressed," "the aim of the study," and "results of the study" (See Table 2).

Findings related to the subject addressed. The distribution regarding the subject addressed in the studies is given in Figure 6.

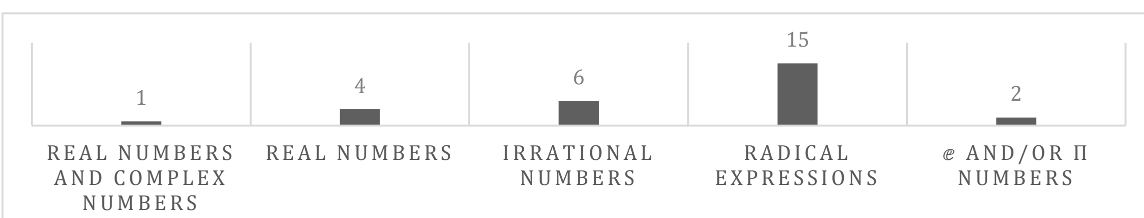

Figure 6. Distribution of studies across the subject addressed

When Figure 6 is examined, it is seen that the majority of studies (n=15) in which radical expressions (and square root expressions), which are a form of representation of numbers, are the focal point about the concept of an irrational number. Studies focusing only on irrational numbers (n=6), studies focusing on irrational number examples such as e and π (n=2), studies dealing with both rational and irrational numbers under the general title of real numbers set (n=4), and studies examining real numbers and complex numbers together (n=1) are also included in the literature.

Findings related to the aim of the study. The distribution of the studies across their aims is given in Figure 7.

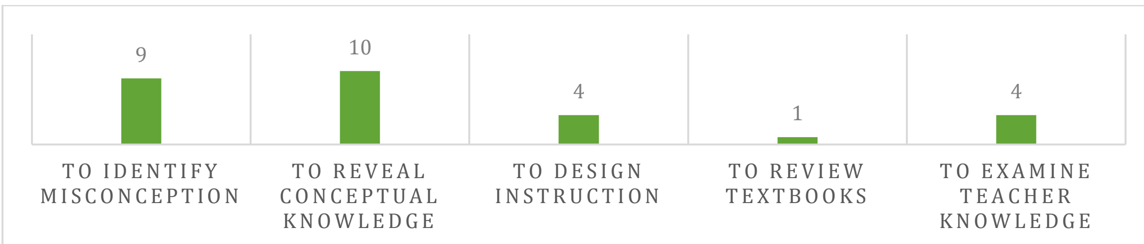

Figure 7. Distribution of studies according to their aims

When Figure 7 is examined, it is seen that studies related to the concept of irrational number in the national literature are primarily carried out to reveal conceptual knowledge (n=10) and identify misconceptions (n=9). Studies conducted to design instruction about irrational numbers or related subjects (n=4), examining teachers' knowledge of teaching mathematics teachers (n=4), and reviewing mathematics textbooks (n=1) are also included in the literature.

On the other hand, teaching methods such as realistic mathematics education (Ocakbaşı, 2019) and concept cartoons (Taşkın-Gültekin, 2013) were used in two studies. They were considering the use cases of theory in studies (n=4), RBC+C theory (Dinç, 2018), concept definition-concept image theoretical framework (Ercire, 2014; Tavşan & Pusmaz, 2020), object-process duality framework (Ercire, 2014), and APOS theory (Ercire, 2014; Ocakbaşı, 2019).

Findings related to results of the study. Finally, when the results of the qualitative studies on irrational numbers were considered, two sub-themes were determined as "results on characterizing irrational numbers" and "results on the causes of difficulties with the concept of an irrational number" (See Table 2). The results for characterizing irrational numbers are given in Table 3.

Table 3

Frequency Table of Findings and Sample Studies for Characterizing Irrational Numbers

Acceptable characterizations	Frequency	Sample studies	Incorrect characterizations	Frequency	Sample studies
Examples of numbers like $\sqrt{2}$, $\sqrt{3}$, π, e	9	(Çevikbaş & Argün, 2017)	Radical numbers	8	(Çevikbaş & Argün, 2017)
Non-rational number	8	(Taşkın-Gültekin, 2013)	Repeating numbers	6	(Adıgüzel, 2013)
Numbers that cannot go out of the root	8	(Toluk-Uçar, 2016)	Examples of numbers like i, $\frac{22}{7}$, $\frac{5}{0}$, $(0,\overline{9})$, $(3,14)$	6	(Temel & Eroğlu, 2014)
Numbers with non-repeating decimal notation	6	(Güler, 2017)	Decimal numbers	2	(Ercire, 2014)
Numbers that cannot be written in the form of a/b	4	(Çevikbaş & Argün, 2017)	Complex numbers	2	(Toluk-Uçar, 2016)
Numbers with an approximate value	3	(Adıgüzel, 2013)	Numbers whose numerator and denominator are not integers	1	(Yiğitcan-Nayir, Erhan, Koştur, Türkoğlu, & Mirasyedioğlu, 2018)
Real numbers	1	(Ercire, 2014)	Numbers that do not have a place on the number line	1	(Ercire, 2014)
			Non-real numbers	1	(Yiğitcan-Nayir et al., 2018)
			Numbers that involve rational numbers	1	(Yiğitcan-Nayir et al., 2018)

When Table 3 is examined, it can be seen that irrational numbers are mostly tried to be explained with numerical examples such as $\sqrt{2}$, $\sqrt{3}$, π, e (n=9), non-rational numbers (n=8), and numbers that cannot go out of the root (n=8) with correct or acceptable characterizations. In addition, there are also numbers with non-repeating decimal notation (n=6), numbers that cannot be written in the form of a/b (n=4), numbers with approximate values (n=3), and real numbers (n=1). On the other hand, it is seen that irrational numbers tried to be misdescribed, mostly with radical numbers (n=8), repeating numbers (n=6), and numerical examples such as i, $\frac{22}{7}$, $\frac{5}{0}$, $(0,\overline{9})$, $(3,14)$

by the study groups. Incorrect characterizations include decimal numbers (n=2), complex numbers (n=2), numbers whose numerator and denominator are not integers (n=1), numbers that have no place on the number line (n=1), non-real numbers (n=1), and numbers that involve rational numbers (n=1) ideas are also encountered.

The results regarding the causes of the difficulties with irrational numbers can be combined under three sub-themes: "causes arising from incorrect or incomplete characterizations," "causes arising from the relationship between number sets and representations," and "causes arising from the curriculum." The explanations of these sub-themes are given in Table 4.

Table 4

Sub-Themes and Explanations for the Causes of Difficulties with Irrational Numbers

Causes arising from incorrect or incomplete characterizations	Causes arising from the relationship between number sets and representations	Causes arising from the curriculum
The inability of the definition of "non-rational number" to highlight the essential properties of irrational numbers (Güler, 2017)	Contemplating of representations as a set of numbers (Dinç, 2018)	The lack of clarity of the relationship between the representations in the curriculum (Bakır, 2011)
Perceiving expressions such as 5/0 as irrational as a result of incorrectly generalizing the definition of "non-rational number" (Ercire, 2014)	Believing that the representation determines the number of characters (Çevikbaş & Argün, 2017)	Perception as the irrational of the numbers 22/7 and 3,14 as a result of the use of rational approaches to the number of π in teaching (Tavşan & Pusmaz, 2020)
Deficiencies in defining and making sense of rational numbers, which is a prerequisite for learning (Güler, 2017)	Inability to make sense of the equivalence of different representations of the same number (Çevikbaş & Argün, 2017)	Teachers being limited to the textbook in teaching or not being aware of the activities in the textbook (Shabanifar, 2014)
Thinking that there is an intersection of rational and irrational number sets (Temel & Eroğlu, 2014)	Consubstantiating the square root expression with irrational numbers (Ercire, 2014)	Teaching irrational numbers after square root expressions in the curriculum (Adıgüzel, 2013)
Not expressing the domain of a and b in the definition of "numbers that cannot be written in the form of a/b" or elucidating faulty (Çiftçi, Akgün, & Soylu, 2015)	Consubstantiating fraction representation with rational numbers (Temel & Eroğlu, 2014)	Concentrating more on square root expressions in the curriculum (Leylek, 2020) and less time allocated to irrational numbers (Çevikbaş & Argün, 2017)
Using the same type of numerical examples to describe irrational numbers (Yiğitcan-Nayir et al., 2018)	Consubstantiating decimal notation with irrational numbers (Toluk-Uçar, 2016)	Teachers are not aware of the mathematical properties of the concept or have erroneous knowledge (Özkaya, Konyalıoğlu, & Gedik, 2013)
Inability to make sense of that numbers with repeating decimal notation are rational numbers (Temel & Eroğlu, 2014)	Inability to understand that the quotient of two integers will always have a finite or repeating decimal notation (Toluk-Uçar, 2016)	Ignoring students' prior knowledge in the transition between concepts in textbooks (Leylek, 2020)
Failure to approximate irrational numbers (Şandır, Ubuz, &Argün, 2007)		Failure to place irrational numbers on the number line as a result of focusing more on operations with numbers in teaching (Yiğitcan-Nayir et al., 2018)

When Table 4 is examined, it is seen that the difficulties with irrational numbers arise from various causes. Among these, it can be said that the most fundamental cause of the difficulties is emphasized in the literature that irrational numbers are expressed only as "non-rational numbers" and the lack of understanding in the knowledge of equivalence of different representations of the same number.

Discussion Conclusion and Suggestions

This research has been tried to draw the framework of the current situation of qualitative studies on irrational numbers in Turkey. As a result of the descriptive analysis of this research, which is thought to shed light on what needs to be done to achieve conceptual understanding, it was seen that the studies on irrational numbers increased in the last eight years. Still, very few studies (n=2), especially at the doctoral level. This shows that more comprehensive studies such as postgraduate theses on irrational numbers should be done in the national field.

Qualitative research is preferred because it allows for the detailed examination of complex issues and understanding the context in which the subject is addressed (Creswell & Poth, 2018). Ethnography, phenomenology, grounded theory, case study, and action research patterns can be used in conducting qualitative research (Creswell & Poth, 2018; Yıldırım & Şimşek, 2018). Since it is seen that the case study design (n=22) is mainly adopted in the qualitative studies examined in this research, it can be said that the studies to be conducted with other qualitative research designs will contribute to the knowledge in the literature to obtain different perspectives and detailed understandings on the concept of an irrational number.

Since irrational numbers are involved in the eighth grade for the first time in the mathematics curriculum (MoNE, 2018a), it is expected that at most eighth-grade students (n=6) will be selected as participants in the studies examined. In addition, although studies were conducted with prospective mathematics teachers (n=5), primary school teacher candidates (n=1), mathematics teachers (n=5), ninth-grade students (n=4), and tenth-grade students (n=1), it is seen that no study with eleventh and twelfth-grade students. In this case, it can be said that the data obtained from different study groups will contribute to the expansion of knowledge on irrational number comprehension.

Qualitative studies limit the size of the study group as much as possible to have in-depth information about the problem and thus aim to obtain a large number of data from a small number of participants (Creswell & Poth, 2018; Yıldırım & Şimşek, 2018). Although there is no standard for the size of the study group, it is recommended to keep the number of participants small to increase the level of detail, especially in case studies (Creswell & Poth, 2018; Patton, 2015). In the national literature, it is observed that the group sizes of qualitative studies on irrational numbers are mostly chosen between 11-50 (n=9) and 51-100 (n=7). In this context, it can be said that there is a need for studies aiming to obtain richer data by choosing a smaller number of study group sizes.

In this research, although the result of meta-synthesis is a representation of numbers in the context of irrational numbers, it is seen that radical expressions occur in a large number of studies (n=15) in the literature. The scarcity of studies focusing only on irrational numbers (n=6) draws attention. Therefore, it can be said that studies should be conducted in which irrational numbers, which form an essential part of real numbers, are the focus in the national literature.

Considering the aims of the studies on irrational numbers, although most studies reveal conceptual knowledge (n=10) and identify misconceptions (n=9), it is seen that solution suggestions to provide the conceptual understanding and what kind of instructional design should be made have not been clarified in consequence of these studies. The scarcity of studies on how to teach the concept of irrational numbers (n=4) both supports this claim and shows that more studies are needed in this area. Similarly, although learning and teaching theories are essential to support conceptual understanding (Arnon et al., 2014), it is seen that studies conducted under a theory framework (n=4) are very few in the literature. In this regard, it can be said that the answers to the questions *“how are irrational numbers structured in the mind of the individual or how the individual learns irrational numbers”* and *“how to teach the concept of irrational numbers”* cannot be obtained from the national literature.

When the results of the qualitative studies on irrational numbers are examined, two sub-themes were determined as "results on characterizing irrational numbers" and "results on the causes of difficulties with irrational numbers.”. Among the results for characterizing irrational numbers, numerical examples such as $\sqrt{2}$, $\sqrt{3}$, π, e (n=9), non-rational numbers (n=8,) and numbers that cannot go out of the root (n=8) are seen as acceptable or correct characterizations. In contrast, rooted numbers (n=8), repeating numbers (n=6) and numerical examples such as $(3{,}14)$, $\frac{22}{7}$, i, $(0,\bar{9})$, $\frac{5}{0}$ (n=6) are included in incorrect characterizations. Incorrect characterizations are also one of the causes for the difficulties experienced in irrational numbers. In particular, the definition of "non-rational number" is insufficient to emphasize the basic features of irrational numbers (Çevikbaş & Argün, 2017), and statements such as 5/0 as a result of the wrong generalization are perceived as irrational numbers (Ercire, 2014). Parallel to this, the deficiencies in rational number comprehension, which is a prerequisite for learning (Güler, 2017), cause both to think that the intersection of rational and irrational numbers exist (Temel & Eroğlu, 2014) and to carry the difficulties experienced in rational numbers to irrational numbers (Ercire, 2014). On the other hand, using the same type of number examples in teaching (Yiğitcan-Nayir et al., 2018) negatively affects meaningful learning of the concept of irrational numbers. In this context, some approaches suggest that decimal notation (Arcavi et al., 1987) or positioning on the number line (Sirotic & Zazkis, 2007) are more effective in teaching irrational numbers.

The second cause of the difficulties experienced in irrational numbers stems from the relationship between number sets and their representations. Thinking of representational forms (such as fractions, decimals, square roots) as a set of numbers (Dinç, 2018) and believing that representation styles determine the character of numbers (Çevikbaş & Argün, 2017) both cause difficulties in making sense of the equivalence of different representations of the same number (Çevikbaş & Argün, 2017) and consubstantiating representations with number sets (such as consubstantiating square root expressions with irrational numbers or fraction representation with rational numbers) (Ercire, 2014; Temel & Eroğlu, 2014; Toluk-Uçar, 2016). To overcome these difficulties, it is necessary to design teaching environments that allow students to switch between different representations and present the transition between concepts relationally, as mentioned in many mathematics curriculums (MoNE, 2018a; NCTM, 2000). Teaching should be planned with a design in which a rich network of relations will be established instead of mathematical structures presented separately by ignoring students' prior knowledge (Leylek, 2020). In this context, support can be obtained from learning and teaching theories such as APOS theory (Arnon et al., 2014), which was developed specifically for mathematics education and interprets how mathematical concepts are structured in the individual's mind with a relational understanding.

The last of the causes of the difficulties experienced in irrational numbers stems from the curriculum. In particular, the reason why irrational numbers are associated with the representation of square root expressions is that irrational numbers are behind the subject of square root expressions in the eighth-grade mathematics curriculum (Adıgüzel, 2013), more focus on square root expressions (Leylek, 2020) and less time allocated to irrational numbers (Çevikbaş & Argün, 2017) can be seen. In other respects, numbers such as 22/7 or 3.14 are perceived as irrational due to rational approaches to the number of π in teaching (Tavşan & Pusmaz, 2020). At the same time, the fact that the teachers, who are the implementers of the curriculum, are limited to the textbook in teaching or are not aware of the activities in the textbook (Shabanifar, 2014), and their incomplete or erroneous knowledge about the mathematical properties of the concepts also cause student difficulties (Özkaya et al., 2013).

In summary, there is a need for studies on how the concept of an irrational number is learned and how to teach it, together with all its components (definition, representations, teaching method, misconceptions, conceptual understanding).

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Genişletilmiş Özet

Giriş

Matematiğin temelini oluşturan sayılar hem bilimsel ve teknolojik bağlamlarda hem de günlük hayatta önemli bir yere sahip olmasından dolayı matematik öğretim programlarında geniş bir yer bulmuştur. Sayı kavramı öğretimi, okul öncesi dönemden başlamakta ve "sayılar ve işlemler" öğrenme alanıyla ortaokul müfredatında yer almaktadır. Matematik öğretim programında sekizinci sınıfa kadar sayma sayıları, doğal sayılar, tam sayılar ve rasyonel sayılar kümelerinin öğretimi farklı gösterim biçimleri ile ele alınmakta ve irrasyonel sayı kavramının öğretimi ise ilk defa sekizinci sınıfta yer almaktadır. Sayı kümeleri, birbiriyle güçlü bir ilişkiye sahip olduğundan herhangi bir sayı kümesinin öğreniminde yaşanan zorluk bütün sayı kavrayışını etkileyecektir. Öğrencilerin en çok zorlandığı sayı kümelerinden biri, irrasyonel sayılardır. İrrasyonel sayılar rasyonel olmayan sayıların da var olabileceği fikrinden ortaya çıkmış ve "iki tam sayının oranı olarak ifade edilemeyen sayılar" veya "sonsuza kadar tekrar etmeyen devirli olmayan ondalık gösterime sahip sayılar" olarak tanımlanabilir.

Ulusal alanyazında irrasyonel sayılar ile ilgili birçok çalışma bulunmaktadır. Bu araştırmanın amacı olan irrasyonel sayılar ile ilgili nitel çalışmaların bütüncül bakış ile sistematik özet biçiminde sunulması, irrasyonel sayılar ile ilgili yapılacak araştırmalara yönelik ihtiyaçların belirlenmesini sağlayacaktır.

Yöntem

Bu araştırmada, ulusal alanyazında yer alan irrasyonel sayılar ile ilgili nitel çalışmaların mevcut durumunu belirlemek amacıyla betimsel içerik analizi ve tematik içerik analizi yöntemi (meta sentez) kullanılmıştır. Araştırmanın verilerini matematik eğitiminde irrasyonel sayılara odaklanan 2000-2021 yılları arasında yayınlanmış 12 adet tez ve 16 adet makale olmak üzere toplam 28 adet çalışma oluşturmaktadır. Verileri elde etmek için Yöktez ve Dergipark veri tabanları toplam 4 kez taranmış ve taramalar 24.10.2021 tarihinde sonlanmıştır. Veri tabanlarında "*irrasyonel sayı, karekök, kareköklü ifade, köklü ifade, kareköklü sayı, köklü sayı, pi sayısı, e sayısı*" anahtar kelimeleri ile taramalar yapılmıştır. İrrasyonel sayılar ile ilgili alanyazın doğrultusunda hem betimsel hem de tematik içerik analizine (meta sentez) uygun olacak biçimde bir "Veri toplama aracı formu (Tablo 1)" hazırlanarak her bir araştırma incelenmiştir.

Bulgular, Sonuç ve Tartışma

Bu araştırmada, Türkiye'de yapılan irrasyonel sayılara yönelik nitel çalışmaların mevcut durumunun çerçevesi çizilmeye çalışılmıştır. Kavramsal anlamanın sağlanabilmesi için neler yapılması gerektiğine ışık tutacağı düşünülen bu araştırmada betimsel analiz sonucu, ilk olarak son sekiz yılda irrasyonel sayılar ile ilgili çalışmaların artış gösterdiği ancak özellikle doktora seviyesinde çok az çalışma (n=2) olduğu görülmüştür. Bu durum ulusal alanda irrasyonel sayılara yönelik doktora tezleri gibi kapsamlı çalışmaların daha çok yapılması gerektiğini göstermektedir.

Meta sentez sonucu, ilk olarak irrasyonel sayılar bağlamında sayıların bir gösterim biçimi olmasına rağmen köklü ifadeler ilgili çalışmaların (n=15) alanyazında çok sayıda yer aldığı görülmektedir. Salt irrasyonel sayılara odaklanan çalışmaların azlığı (n=6) ise dikkat çekmektedir. O nedenle ulusal alanyazında gerçel sayıların önemli bir parçasını oluşturan irrasyonel sayıların odak noktası olduğu çalışmaların yapılması gerektiği söylenebilir.

İrrasyonel sayılar ile ilgili çalışmaların hedefleri göz önüne alındığında kavramsal bilgiyi ortaya çıkarmaya (n=10) ve kavram yanılgılarını belirlemeye (n=9) yönelik çalışmaların çoğunluğu oluşturmasına rağmen bu çalışmalar neticesinde kavramsal anlamayı sağlamaya yönelik çözüm önerileri ve nasıl bir öğretim tasarımı yapılması gerektiğinin henüz aydınlatılmadığı görülmektedir. Nitekim irrasyonel sayı kavramı öğretiminin nasıl yapılması gerektiğine ilişkin çalışmaların azlığı (n=4) hem bu iddiayı desteklemekte hem de bu alanda daha fazla çalışmaya ihtiyaç olduğunu göstermektedir. Benzer şekilde kavramsal anlamanın sağlanabilmesi için öğrenme ve öğretme teorileri önemli bir destek olmasına rağmen alanyazında bir teori çatısı altında yürütülen çalışmaların (n=4) oldukça az olduğu görülmektedir. Bu hususta "*irrasyonel sayılar bireyin zihninde nasıl yapılandırılır* veya *birey irrasyonel sayıları nasıl öğrenir*" ve "*irrasyonel sayı kavramı nasıl öğretilir*" sorularının yanıtının ulusal alanyazından elde edilemediği söylenebilir.

İrrasyonel sayılar ile ilgili nitel çalışmaların sonuçları incelendiğinde ise "irrasyonel sayıları nitelemeye yönelik sonuçlar" ve "irrasyonel sayılarda yaşanan zorlukların nedenlerine yönelik sonuçlar" olmak üzere iki alt tema belirlenmiştir. İrrasyonel sayıları nitelemeye yönelik sonuçlar içerisinde $\sqrt{2}$, $\sqrt{3}$, π, e gibi sayı örnekleri (n=9), rasyonel olmayan sayı (n=8) ve kök dışına çıkamayan sayılar (n=8) ifadelerinin kabul edilebilir veya doğru nitelemeler biçiminde yer aldığı görülürken köklü sayılar (n=8), devirli sayılar (n=6) ve $(3{,}14)$, $\frac{22}{7}$, i, $(0,\bar{9})$, $\frac{5}{0}$ gibi sayı örnekleri (n=6) hatalı nitelemeler içerisinde yer almaktadır. Hatalı nitelemeler aynı zamanda irrasyonel sayılarda yaşanan zorlukların nedenlerinden biridir. Özellikle "rasyonel olmayan sayı" tanımının irrasyonel sayıların temel özelliklerini vurgulamada yetersiz olması ve yanlış genelleme sonucu $\frac{5}{0}$ gibi ifadeler irrasyonel sayı olarak algılanmaktadır. Buna paralel olarak ön koşul öğrenme olan rasyonel sayı kavrayışındaki eksiklikler hem rasyonel ve irrasyonel sayıların kesişiminin olduğunu düşünmeye hem de rasyonel sayılarda yaşanan zorlukların irrasyonel sayılara taşınmasına neden olmaktadır. Diğer yandan öğretimde aynı tip sayı örneklerinin kullanılması, irrasyonel sayı kavramını anlamlı öğrenmeyi olumsuz etkilemektedir. Bu kapsamda, irrasyonel sayı öğretiminde ondalık gösterimin veya sayı doğrusunda yer belirlemenin daha etkili olduğuna yönelik yaklaşımlar bulunmaktadır.

İrrasyonel sayılarda yaşanan zorlukların nedenlerinden ikincisi, sayı kümeleri ve gösterim biçimleri arasındaki ilişkiden kaynaklanmaktır. Gösterim biçimlerini (kesir, ondalık, karekök gibi) sayı kümesi olarak düşünme ve gösterim biçimlerinin sayı karakterini belirlediğine inanma hem aynı sayının farklı gösterimlerinin denkliğini anlamlandırmada zorlanmaya hem de gösterimleri sayı kümeleri ile özdeşleştirmeye (kareköklü ifadeler ile irrasyonel sayıları veya kesir gösterimi ile rasyonel sayıları özdeşleştirme gibi) neden olmaktadır. Bu zorlukların üstesinden gelebilmek için matematik öğretim programlarında da çokça değinildiği üzere öğrencilerin farklı gösterimler arası geçiş yapabilmesine izin verecek öğretim ortamlarının tasarlanması ve kavramlar arası geçişin ilişkisel biçimde sunulması gereklidir. Öğrencilerin ön bilgilerinin göz ardı edilerek birbirinden kopuk şekilde sunulan matematiksel yapılar yerine zengin ilişkiler ağının kurulacağı bir tasarım ile öğretim planlanmalıdır. Bu kapsamda matematik eğitimine özgü geliştirilen ve matematiksel kavramların bireyin zihninde nasıl yapılandırıldığını ilişkisel bir anlayışla yorumlayan APOS teorisi gibi öğrenme ve öğretme teorilerinden destek alınabilir.

İrrasyonel sayılarda yaşanan zorlukların nedenlerinden sonuncusu ise öğretim programından kaynaklanmaktadır. Özellikle irrasyonel sayıların kareköklü ifade gösterimi ile özdeşleştirilmesinin nedeni, irrasyonel sayıların sekizinci sınıf matematik öğretim programında kareköklü ifadeler konusunun ardında yer alması, kareköklü ifadelere daha fazla odaklanılması ve irrasyonel sayılara ayrılan sürenin az olması olarak görülebilir. Öte yandan öğretimde π sayısının rasyonel yaklaşımlarının kullanılması sonucu $\frac{22}{7}$ veya 3,14 gibi sayılar irrasyonel olarak algılanmaktadır. Aynı zamanda öğretim programının uygulayıcısı olan öğretmenlerin öğretimde ders kitabı ile sınırlı kalması veya ders kitabındaki etkinliklerin farkında olmaması, kavramların matematiksel özellikleri hakkındaki bilgilerinin eksik veya hatalı olması da öğrenci zorluklarına neden olmaktadır.

Özetle, bütün bileşenleriyle (tanım, gösterim biçimleri, öğretim yöntemi, kavram yanılgıları, kavramsal anlama) birlikte irrasyonel sayı kavramının nasıl öğrenildiği ve nasıl öğretileceği üzerine çalışmalara ulusal alanyazında ihtiyaç duyulmaktadır.

ETİK BEYAN: *"Türkiye'de Yapılan İrrasyonel Sayılara Yönelik Nitel Çalışmaların İçerik Analizi: Meta Sentez Çalışması"* başlıklı çalışmamızın yazım sürecinde bilimsel, etik ve alıntı

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