ON SOME GAUSSIAN PELL AND PELL-LUCAS NUMBERS

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Abstract

In this study, we consider firstly the generalized Gaussian Fibonacci and Lucas sequences. Then we define the Gaussian Pell and Gaussian Pell-Lucas sequences. We give the generating functions and Binet formulas of Gaussian Pell and Gaussian Pell-Lucas sequences. Moreover, we obtain some important identities involving the Gaussian Pell and Pell-Lucas numbers.

Keywords. Recurrence Relation, Fibonacci numbers, Gaussian Pell and Pell-Lucas numbers.

Özet

Bu çalışmada, önce genelleştirilmiş Gaussian Fibonacci ve Lucas dizilerini dikkate aldık. Sonra, Gaussian Pell ve Gaussian Pell-Lucas dizilerini tanımladık. Gaussian Pell ve Gaussian Pell-Lucas dizilerinin Binet formüllerini ve üreteç fonksiyonlarını verdik. Üstelik, Gaussian Pell ve Gaussian Pell-Lucas sayılarını içeren bazı önemli özdeşlikler elde ettik.

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1. INTRODUCTION

From (Horadam 1961; Horadam 1963) it is well known Generalized Fibonacci sequence $\{U_n\}$,

$$U_{n+1} = pU_n + qU_{n-1} \ , \ U_0 = 0 \ and \ U_1 = 1,$$

and generalized Lucas sequence $\{V_n\}$ are defined by

$$V_{n+1} = pV_n + qV_{n-1} \ , \ V_0 = 2 \ and \ V_1 = p,$$

where p and q are nonzero real numbers and $n \geq 1$. For $p = q = 1$, we have classical Fibonacci and Lucas sequences. For $p = 2,\ q = 1$, we have Pell and Pell-Lucas sequences. For detailed information about Fibonacci and Lucas numbers one can see (Koshy 2001). Moreover, generalized Fibonacci and Lucas numbers with negative subscript can be defined as

$$U_{-n} = \frac{-U_n}{(-q)^n} \quad \text{and} \quad V_{-n} = \frac{V_n}{(-q)^n}$$

respectively. From the reccurence relation related with these sequences we can write $p^2 + 4q > 0$, $\alpha = \left(p + \sqrt{p^2 + 4q}\right)/2$ and $\beta = \left(p - \sqrt{p^2 + 4q}\right)/2$. So, the Binet formulas of generalized Fibonacci and Lucas sequences are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n.$$

Gaussian numbers are complex numbers $z = a + ib, a, b \in \mathbb{Z}$ were investigated by Gauss in 1832 and the set of these numbers is denoted by $\mathbb{Z}[i]$.

In Horadam (1963), introduced the concept the complex Fibonacci number called as the Gaussian Fibonacci number. And then, Jordan (1965) considered two of the complex Fibonacci sequences and extented some relationship which are known about the common Fibonacci sequences. Also the author gave many identities related with them. For example, for these sequences some of identities are given by

$$GF_n = F_n + iF_{n-1} \ , \quad GF_{-n} = iGF_n \ ,$$

$$GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n(2 - i),$$

$$GF_{n+1}^2 - GF_{n-1}^2 = F_{2n-1}(1+2i),$$

$$GF_n^2 + GF_{n+1}^2 = F_{2n}(1+2i), \quad \sum_{k=0}^{n} GF_k = GF_{n+2} - 1,$$

for some $n \geq 2$.

The above identities are known as the relationship between the usual Fibonacci and Gaussian Fibonacci sequences. Horadam investigated also the complex Fibonacci polynomials. In Berzsenyi (1977), presented a natural manner of extension of the Fibonacci numbers into the complex plane and obtained some interesting identities for the classical Fibonacci numbers. Moreover, the author gave a closed form to Gaussian Fibonacci numbers by the Fibonacci Q matrix.

In Harman 1981, gave an extension of Fibonacci numbers into the complex plane and generalized the methods given by Horadam (1963); Berzsenyi (1977). In Ascı&gurel (2013), the authors studied the generalized Gaussian Fibonacci numbers. Then they gave the sums of generalized Gaussian Fibonacci numbers by the matrix method. The authors studied also the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers.

In this study, we define and study the Gaussian Pell and Gaussian Pell-Lucas sequences. We give generating functions and Binet formulas for these sequences. Moreover, we obtain some important identities involving the Gaussian Pell and Pell-Lucas numbers.

Now, let we define the generalized Gaussian Fibonacci sequence $U_n(p,q;a,b\ )$ as follows.

$$GU_{n+1} = pGU_n + \mathrm{q}\, GU_{n-1} \quad GU_0 = a,\ GU_1 = b \qquad (2.1)$$

where a and b are initial values. If we take $p = q = 1$, $a = i,\ b = 1$ in the equation (2.1), then we get the Gaussian Fibonacci sequence $\mathrm{GU_n}(1{,}1;\mathrm{i},1\ )$ that is

$$\{GF_n\} = \{i, 1, 1+i, 2+i, 3+2i, \dots\}.$$

If we take $p = q = 1$, $a = 2 - i, b = 1 + 2i$ in the equation (2.1), then we get the Gaussian Lucas sequence

$$\{GL_n\} = \{2-i, 1+2i,\ 3+i,\ 4+3i,\ 7+4i, \dots\}.$$

If we take $p = 2,\ q = 1\ ,\ a = i, b = 1$ in the equation (2.1), then we get the Gaussian Pell sequence

$$\{GP_n\} = \{i, 1, 2 + i, 5 + 2i, 12 + 5i, \dots\}.$$

If we take $p = 2,\ q = 1\ ,\ a = 2 - 2i,\ \ b = 2 + 2i$ in (2.1), then we get the Gaussian Pell-Lucas sequence

$$\{GQ_n\} = \{2 - 2i, 2 + 2i,\ 6 + 2i,\ 14 + 6i,\ 34 + 14i, \dots\}.$$

Also we have $GP_n = P_n + iP_{n-1}$ and $GQ_n = Q_n + iQ_{n-1}$,

where P_n and Q_n are the nth Pell and Pell-Lucas numbers, respectively.

2. GAUSSIAN PELL AND GAUSSIAN PELL-LUCAS SEQUENCES

In this section, we consider Gaussian Pell and Gaussian Pell-Lucas sequences. We give the Binet formulas for these sequences. Then we obtain the generating functions and we give some identities involving these sequences.

Binet formulas are well known formulas in the theory Fibonacci numbers . These formulas can also be carried out to the Gaussian Pell numbers.

In the following theorem we give the Binet formulas for Gaussian Pell numbers.

THEOREM 1. Binet formulas for Gaussian Pell and Gaussian Pell-Lucas sequences are given by

$$GP_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta}, \quad n \geq 0$$

and

$$GQ_n = (\alpha^n + \beta^n) - i(\alpha\beta^n + \beta\alpha^n), \quad n \geq 0$$

respectively.

PROOF. From the theory of difference equations we know the general term of Gaussian Pell numbers can be expressed in the following form

$$GP_n(x) = c\alpha^n(x) + d\beta^n(x),$$

where c and d are the coefficients. Using the values $n = 0, 1$

$$c = \frac{1+(\sqrt{2}-1)i}{2\sqrt{2}}, d = \frac{-1+(\sqrt{2}+1)i}{2\sqrt{2}}.$$

can be written. Considering the values c, d and making some calculations, we obtain

$$GP_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\,\frac{\alpha\beta^n - \beta\alpha^n}{\alpha - \beta},$$

In addition to this, we get

$$GQ_n = (\alpha^n + \beta^n) - i(\alpha\beta^n + \beta\alpha^n).$$

For Gaussian Pell-Lucas sequence $\{GQ_n\}$ generating function $g(t)$ is a formal power series. The generating function $g(t)$ of the sequence $\{GP_n\}$ is defined by

$$g(t) = \sum_{n=0}^{\infty} GP_n t^n.$$

Then we can give the generating functions for the Gaussian Pell and Gaussian Pell-Lucas sequences in the following theorem.

THEOREM 2. The generating functions to the Gaussian Pell and Gaussian Pell-Lucas sequences are

$$g(t) = \frac{t + i(1-2t)}{1-2t-t^2},$$

and

$$g(t) = \frac{(2-2t) + i(6t-2)}{1-2t-t^2},$$

respectively.

PROOF. Let $g(t)$ be the generating function of sequence $\{GP_n\}$. Then we can write

$$g(t) = \sum_{n=0}^{\infty} GP_n t^n = GP_0 + GP_1 t + GP_2 t^2 + \ldots + GP_n t^n + \ldots$$

If we use the recursive relation of this sequence, then we get

$$g(t)\left[1-2t-t^2\right]=GP_0+\left(GP_1-2GP_0\right)t\ .$$

Thus, we obtain

$$g(t)=\frac{t+(1-2t)i}{1-2t-t^2}\ ,$$

which is desired.

Similarly, the generating function of Gaussian Pell-Lucas sequence $\{GQ_n\}$

$$g(t)=\frac{(2-2t)+i(6t-2)}{1-2t-t^2}.$$

can be obtained. Moreover, we have the negatively subscripted terms of the sequences $\{GP_n\}$ and $\{GQ_n\}$ by using the recursive relation,

$$GP_{-n}=2GP_{-(n+1)}+GP_{-(n+2)}$$

and

$$GQ_{-n}=2GQ_{-(n+1)}+GQ_{-(n+2)}$$

respectively. Notice that

$$GP_{-n}=P_{-n}+iP_{-n-1} \text{ and } GQ_{-n}=Q_{-n}+iQ_{-n-1}.$$

In the following theorem, we give the relations between the Gaussian Pell and Gaussian Pell-Lucas sequences involving the negative indices.

THEOREM 3. For $n\geq 1$, we have the following identities.

$$i)\ GP_{-n}=(-1)^{n-1}(P_n-iP_{n+1})$$

$$ii)\ GQ_{-n}=(-1)^{n}(Q_n-iQ_{n+1})$$

PROOF. The proof can be seen by the mathematical induction on n .

In the following Corollary, we give some useful identities concerning the Gaussian Pell and Gaussian Pell-Lucas numbers, and also give some sum formulas for these numbers without proof.

COROLLARY 1. For $n \geq 1$, we have the following equations

$$i)\ 4GP_n = GQ_n + GQ_{n-1},$$

$$ii)\ GP_n^2 + GP_{n+1}^2 = 2P_{2n}(1+i),$$

$$iii)\ GQ_n^2 + GQ_{n+1}^2 = 16P_n\, Q_n(1+i).$$

It is well known that the Cassini identity is one of the oldest identities involving the Fibonacci numbers. In the following theorem, we give the Cassini formula related with the Gaussian Pell and Pell-Lucas numbers.

THEOREM 4. (Cassini Formula) Let $n \geq 1$. Then we have

$$i)\ GP_{n+1}GP_{n-1} - {GP_n}^2 = (-1)^n 2(1-i)$$

$$ii)\ GQ_{n+1}GQ_{n-1} - {GQ_n}^2 = (-1)^{n+1}16(1-i),$$

respectively.

PROOF. By using the mathematical induction method we get

$$GP_2GP_0 - GP_1^2 = 2(1-i); GP_0 = i, GP_2 = (2+i)$$

$$GP_{k+1}GP_{k-1} - {GP_k}^2 = (-1)^k\, 2(1-i)$$

$$GP_{k+2}GP_k - {GP_{k+1}}^2 = (2GP_{k+1} + GP_k)\left(\frac{1}{2}GP_{k+1} - \frac{1}{2}GP_{k-1}\right) - {GP_{k+1}}^2$$

$$= \frac{1}{2}GP_kGP_{k+1} - \frac{1}{2}GP_kGP_{k-1} - \left[(-1)^k\, 2(1-i) + {GP_k}^2\right]$$

$$= \frac{1}{2}GP_kGP_{k+1} - GP_k(\frac{1}{2}GP_{k-1} + GP_k) + (-1)^{k+1}\, 2(1-i)$$

$$= \left(-1\right)^{k+1} 2\left(1-i\right).$$

Similarly, we can prove the other formula by the mathematical induction method. Thus, the proof is completed.

THEOREM 5. For the Gaussian Pell numbers, we have the following formula.

$$\sum_{j=0}^{n} GP_j = \frac{1}{2}[GP_{n+1} + GP_n - (1-i)].$$

PROOF. From the recursive relation we can write

$$GP_n = \frac{1}{2}GP_{n+1} - \frac{1}{2}GP_{n-1}$$

and

$$\begin{aligned}
GP_0 &= \frac{1}{2}GP_1 - \frac{1}{2}GP_{-1} \\
GP_1 &= \frac{1}{2}GP_2 - \frac{1}{2}GP_0 \\
GP_2 &= \frac{1}{2}GP_3 - \frac{1}{2}GP_1 \\
&\vdots \\
GP_n &= \frac{1}{2}GP_{n+1} - \frac{1}{2}GP_{n-1}
\end{aligned}$$

Then, we obtain

$$\sum_{j=0}^{n} GP_j = \frac{1}{2}\left(GP_{n+1} + GP_n\right) - \frac{1}{2}\left(GP_{-1} + GP_0\right)$$

$$\begin{aligned}
&= \frac{1}{2}\left(GP_{n+1} + GP_n\right) - \frac{1}{2}\left(1-i\right) \\
&= \frac{1}{2}\left[GP_{n+1} + GP_n - \left(1-i\right)\right]
\end{aligned}$$

This completes the proof.

THEOREM 6. For all $n \in \mathbb{N}$, we have the following sum formula

$$\sum_{j=0}^{n} GQ_n = \frac{1}{2}(GQ_{n+1} + GQ_n) - 2i.$$

In the following corollary, we give some summation formulas for the Gaussian Pell and Pell-Lucas numbers.

COROLLARY 2. For $n \geq 1$, we have

i) $\sum_{j=0}^{n} GP_{2j} = \frac{1}{2}(GP_{2n+1} + 2i - 1)$,

ii) $\sum_{j=0}^{n} GQ_{2j} = \frac{1}{2}GQ_{2n+1} + (1 - 3i)$.

iii) $\sum_{j=1}^{n} GP_{2j-1} = \frac{1}{2}(GP_{2n} - i)$,

and

iv) $\sum_{j=1}^{n} GQ_{2j-1} = \frac{1}{2}GQ_{2n} - (1 - i)$.

The above equalities can be seen by Theorem 6.

THEOREM 7 (Catalan Formulas) For nonzero positive integers n, k we have

$$i)\ \ GP_{n+k}GP_{n-k} - {GP_n}^2 = (-1)^n(1-i)\left[1 + \frac{(-1)^{k+1}(\alpha^k + \beta^k)^2}{4}\right],$$

ii) $GQ_{n+k}GQ_{n-k} - {GQ_n}^2 = 2(-1)^{n+1}(1-i)[4 + (-1)^{k+1}(\alpha^k + \beta^k)^2]$.

PROOF. For the first equality, from the Binet formula

$$[\frac{\alpha^{n+k}-\beta^{n+k}}{\alpha-\beta}+\frac{\alpha\beta^{n+k}-\beta\alpha^{n+k}}{\alpha-\beta}i][\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}+\frac{\alpha\beta^{n-k}-\beta\alpha^{n-k}}{\alpha-\beta}i]-[\frac{\alpha^{n}-\beta^{n}}{\alpha-\beta}+\frac{\alpha\beta^{n}-\beta\alpha^{n}}{\alpha-\beta}i]^2$$

$$=4\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}+\frac{(\alpha\beta-1)(\alpha\beta)^{n-k}(\alpha^{2k}+\beta^{2k})}{(\alpha-\beta)^2}-[4\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}+\frac{(\alpha\beta-1)(\alpha\beta)^{n-k}(\alpha^{2k}+\beta^{2k})}{(\alpha-\beta)^2}]i$$

$$=(-1)^n(1-i)[1+\frac{(-1)^{k+1}(\alpha^k+\beta^k)^2}{4}]$$

can be written which is desired. Using the same method the other formula can be given easily.

THEOREM 8 (d'Ocagne's Identity) For all $m,n\in\mathbb{Z}$ we have

i) $GP_{m+1}GP_n-GP_mGP_{n+1}=2(-1)^{n+1}(1-i)P_{m-n}$

ii) $GQ_{m+1}GQ_n-GQ_mGQ_{n+1}=16(-1)^n(1-i)P_{m-n}$

where P_n is the nth Pell number.

PROOF. By using the Binet formula fort he Gaussian Pell-Lucas sequence, the proof can be easily seen.

3. CONCLUSION

In conclusion, we firstly consider the generalized Gaussian Fibonacci and Lucas sequences. Then we introduce the Gaussian Pell and Gaussian Pell-Lucas sequences. We give the generating functions and Binet formulas of Gaussian Pell and Gaussian Pell-Lucas sequences. Furthermore, we obtain some important identities involving the terms of these sequences.

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