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p -QUASİ-KONVEKS FONKSİYONLAR İÇİN GENELLEŞTİRİLMİŞ HERMİTE-HADAMARD TİPLİ EŞİTSİZLİKLER

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Özet

Bu çalışmada, yazar türevlenebilir fonksiyonlar için yeni genel bir özdeşlik verir ve bu özdeşliği kullanarak p -quasi konveks fonksiyonlar için bazı yeni genelleştirilmiş Hermite-Hadamard tipli eşitsizlikler elde eder.

Mathematics Subject Classification: 26D15, 26A51

Anahtar Kelimeler: Hermite-Hadamard eşitsizliği, p -quasi konveks fonksiyon

Generalized Hermite-Hadamard Type Inequalities for p -Quasi-Convex Functions

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Abstract

In this paper, the author gives a new general identity for diferentiable functions and establishes some new generalized Hermite-Hadamard type inequalities for p -quasi convex functions by using this identity.

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Keywords: Hermite-Hadamard's inequality, p -quasi-convex function

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1 Introduction

Let $f: I \subset R \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

In Dragomir & Agarwal (1998**),** gave the following Lemma. By using this Lemma, Dragomir obtained the following Hermite-Hadamard type inequalities for convex functions:

Lemma 1 Let $f: I^{\circ} \subset R \to R$ be a differentiable mapping on I° and $a, b \in I^{\circ}$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)dt. \tag{2}$$

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f:[a,b] \to R$ is said quasi-convex on $[a,b]$ if

$$f(\alpha x + (1-\alpha)y) \leq \sup\{f(x), f(y)\},$$

for any $x, y \in a,b]$ and $\alpha \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see Ion 2007).

For some results which generalize, improve and extend the inequalities(1) related to quasi-convex functions we refer the reader to see (Alomari et al 2010; Alomari et al 2011; Ion 2007; İşcan 2013; 2013; 2013, İşcan et al 2014,Zehang 2013) and plenty of references therein.

In (İşcan 2014), the author, gave definition Harmonically convex and concave functions as follow.

Definition 1 Let $I \subset R \setminus \{0\}$ be a real interval. A function $f: I \to R$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in 0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

Zhang et al (2013) defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 2 A function $f : I \subseteq (0,\infty) \to [0,\infty)$ is said to be harmonically quasi-convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in 0,1]$.

We would like to point out that any harmonically convex function on $I \subseteq (0,\infty)$ is a harmonically quasi-convex function, but not conversely. For example, the function

$$f(x) = \begin{cases} 1, & x \in (0,1]; \\ (x-2)^2, & x \in 1,4]. \end{cases}$$

is harmonically quasi-convex on $(0,4]$, but it is not harmonically convex on $(0,4]$.
In [16], Zhang and Wan gave definition of p -convex function as follow:

Definition 3 Let I be a p-convex set. A function $f : I \to R$ is said to be a p-convex function or belongs to the class PC(I), if

$$f\left(\left[\alpha x^p + (1-\alpha) y^p\right]^{1/p}\right) \leq \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in I$ and $\alpha \in 0,1]$.

Remark 1 An interval I is said to be a p-convex set if $\left[\alpha x^p + (1-\alpha) y^p\right]^{1/p} \in I$ for all $x, y \in I$ and $\alpha \in 0,1]$, where $p = 2k+1$ or $p = n/m$, $n = 2r+1$, $m = 2t+1$ and $k, r, t \in \mathrm{N}$.

Remark 2 If $I \subset (0,\infty)$ be a real interval and $p \in R \setminus \{0\}$, then

$$\left[\alpha x^p + (1-\alpha) y^p\right]^{1/p} \in I \text{ for all } x, y \in I \text{ and } \alpha \in 0,1].$$

According to Remark 2, we can give a different version of the definition of p -convex function as follow:

Definition 4 [10,11,12] Let $I \subset (0,\infty)$ be a real interval and $p \in R\setminus\{0\}$. A function $f : I \to \mathrm{P}$ is said to be a p-convex function, if

$$f\left(\left[\alpha x^p + (1-\alpha)y^p\right]^{1/p}\right) \leq \alpha f(x) + (1-\alpha)f(y) \qquad (4)$$

for all $x, y \in I$ and $\alpha \in 0,1]$. If the inequality in (4) is reversed, then f is said to be p-concave.

According to Definition 4, It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Example 1 Let $f : (0,\infty) \to R,\ f(x) = x^p, p \neq 0,$ and $g : (0,\infty) \to R, g(x) = c,\ c \in R,$ then f and g are both p-convex and p-concave functions.

In [4, Theorem 5], if we take $I \subset (0,\infty)$, $p \in R\setminus\{0\}$ and $h(t) = t$, then we have the following Theorem.

Theorem 1 Let $f : I \subset (0,\infty) \to R$ be a p-convex function, $p \in R\setminus\{0\}$, and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then we have

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) \leq \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \leq \frac{f(a)+f(b)}{2}. \qquad (5)$$

In[11], İşcan defined the p-quasi-convex function and supplied several properties of this kind of functions as follow:

Definition 5 Let $I \subset (0,\infty)$ be a real interval and $p \in R\setminus\{0\}$. A function $f : I \to R$ is said to be p-quasi-convex, if

$$f\left(\left[tx^p + (1-t)y^p\right]^{1/p}\right) \leq \max\{f(x), f(y)\} \qquad (6)$$

for all $x, y \in I$ and $t \in 0,1]$. If the inequality in (6) is reversed, then f is said to be p-quasi-concave.

It can be easily seen that for $r=1$ and $r=-1$, p-quasi convexity reduces to ordinary quasi convexity and harmonically quasi convexity of functions defined on $I \subset (0,\infty)$, respectively. Morever every p-convex function is a p-quasi-convex function.

Example 2 Let $f:(0,\infty)\to R,\ f(x)=x^p,\ p\in R\setminus\{0\},$ and $g:(0,\infty)\to R,\ g(x)=c,\ c\in R,$ then f and g are p-quasi-convex functions.

Proposition 1 *Let $I\subset(0,\infty)$ be a real interval, $p\in R\setminus\{0\}$ and $f:I\to R$ is a function, then ;*

1. If $p\le 1$ and f is quasi-convex and nondecreasing function then f is p-quasi-convex.
2. If $p\ge 1$ and f is p-quasi-convex and nondecreasing function then f is quasi-convex.
3. If $p\le 1$ and f is p-quasi-concave and nondecreasing function then f is quasi-concave.
4. If $p\ge 1$ and f is quasi-concave and nondecreasing function then f is p-quasi-concave.
5. If $p\ge 1$ and f is quasi-convex and nonincreasing function then f is p-quasi-convex.
6. If $p\le 1$ and f is p-quasi-convex and nonincreasing function then f is quasi-convex.
7. If $p\ge 1$ and f is p-quasi-concave and nonincreasing function then f is quasi-concave.
8. If $p\le 1$ and f is quasi-concave and nonincreasing function then f is p-quasi-concave.

Proposition 2 If $f:[a,b]\subseteq(0,\infty)\to R$ and if we consider the function $g:[a^p,b^p]\to R,$ defined by $g(t)=f\left(t^{1/p}\right),\ p\neq 0$, then f is p-quasi-convex on $[a,b]$ if and only if g is quasi-convex on $[a^p,b^p]$.

For some results related to p-convex functions and its generalizations, we refer the reader to see (Fang 2014; İşcan 2016; 2016; 2016,Noor 2015; Zhang et al 2015).
The main purpose of this paper is to establish some new general results connected with the right-hand side of the inequalities (5) for p-quasi-convex functions.

2 Main Results

In order to prove our main results we need the following lemma:

Lemma 2 Let $f : I \subset (0,\infty) \to R$ be a differentiable mapping on I°, $a,b \in I$ with $a < b$. If $f' \in L[a,b]$, $p \in R \setminus \{0\}$ and $\lambda, \mu \in [0,\infty)$, $\lambda + \mu > 0$, then the following equality holds:

$$\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx = \frac{b^p-a^p}{p(\lambda+\mu)}\int_0^1 \frac{[(\lambda+\mu)t-\lambda]}{\left[tb^p+(1-t)a^p\right]^{1-1/p}} f'(M_{p,t}(a,b))dt$$

where $M_{p,t}(a,b) = \left[tb^p+(1-t)a^p\right]^{1/p}$.

Proof: integration by parts we have

$$\begin{aligned} I &= \frac{b^p-a^p}{p(\lambda+\mu)}\int_0^1 \frac{[(\lambda+\mu)t-\lambda]}{\left[tb^p+(1-t)a^p\right]^{1-1/p}} f'(M_{p,t}(a,b))dt \\ &= \frac{1}{(\lambda+\mu)}\int_0^1 [(\lambda+\mu)t-\lambda]df(M_{p,t}(a,b)) \\ &= \frac{[(\lambda+\mu)t-\lambda]}{(\lambda+\mu)} f(M_{p,t}(a,b))\Big|_0^1 - \int_0^1 f(M_{p,t}(a,b))dt \\ &= \frac{\lambda f(a)+\mu f(b)}{(\lambda+\mu)} - \int_0^1 f(M_{p,t}(a,b))dt \end{aligned}$$

Setting $x^p = tb^p + (1-t)a^p$, and $px^{p-1}dx = (b^p - a^p)dt$ gives

$$I = \frac{\lambda f(a)+\mu f(b)}{(\lambda+\mu)} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx.$$

which completes the proof.

Remark 3 If we take $\lambda = \mu = p = 1$ in Lemma 2, then we obtain the inequality (2) in Lemma 1.

Theorem 2 Let $f : I \subset (0,\infty) \to R$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, $p \in R \setminus \{0\}$ and $f' \in L[a,b]$. f $|f'|$ is p-convex on $[a,b]$, then

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right|$$

$$\leq \frac{b^p - a^p}{p(\lambda+\mu)}\left(\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)C_{\lambda,\mu}(a,b;p)$$

where

$$C_{\lambda,\mu}(a,b;p) = \frac{2p(\lambda+\mu)}{(p+1)(b^p-a^p)^2}\left[A_{\frac{\lambda}{\lambda+\mu},p}\left(M_{\frac{\lambda}{\lambda+\mu},p} - A\right)\right.$$

$$\left. -(p+1)\left(M^{p+1}_{\frac{\lambda}{\lambda+\mu},p} - A^{p+1}_{\frac{1}{2},p}\right)\right],\ p \in \mathrm{P} \setminus \{-1,0\},$$

$$C_{\lambda,\mu}(a,b;-1) = \frac{2(\lambda+\mu)}{(b^p-a^p)^2}\left[A_{\frac{\lambda}{\lambda+\mu},-1}\left(A - M_{\frac{\lambda}{\lambda+\mu},-1}\right) - \ln\left(\frac{G}{M_{\frac{\lambda}{\lambda+\mu},-1}}\right)\right],$$

and

$M_{p,t}(a,b) = \left[tb^p + (1-t)a^p\right]^{1/p}$, $A_{t,p} = tb^p + (1-t)a^p$, $M_{t,p} = A^{1/p}_{t,p}$, $A = (a+b)/2$ and $G = \sqrt{ab}$.

Proof: From Lemma 2 and using the Hölder integral inequality and p -quasi-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\left|\frac{\lambda f(a) + \mu f(b)}{\lambda+\mu} - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right|$$

$$\leq \frac{b^p - a^p}{p(\lambda+\mu)}\int_0^1 \frac{\left|(\lambda+\mu)t - \lambda\right|}{\left[tb^p + (1-t)a^p\right]^{1-1/p}}\left|f'(M_{p,t}(a,b))\right|dt$$

$$\leq \frac{b^p - a^p}{p(\lambda+\mu)}\left(\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)\int_0^1 \frac{\left|(\lambda+\mu)t - \lambda\right|}{\left[tb^p + (1-t)a^p\right]^{1-1/p}}dt$$

It is easily check that

$$\int_0^1 \frac{\left|(\lambda+\mu)t - \lambda\right|}{\left[tb^p + (1-t)a^p\right]^{1-1/p}}dt = C_{\lambda,\mu}(a,b;p).$$

In Theorem 2, if we put $p=1$, then we obtain the following corollary for quasi-convex functions:

Corollary 1 Under the conditions of Theorem 2, if we take $p=1$, then we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{\lambda+\mu}\left(\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)C_{\lambda,\mu}(a,b;1).$$

In Theorem 2, if we put $p=-1$, then we obtain the following corollary for harmonically quasi-convex functions:

Corollary 2 Under the conditions of Theorem 2, if we take $p=-1$, then we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{b-a}{\lambda+\mu}\left(\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)C_{\lambda,\mu}(a,b;-1).$$

Theorem 3 Let $f:I\subset(0,\infty)\to R$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in R\setminus\{0\}$ and $f'\in L[a,b]$. f $\left|f'\right|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leq \frac{b^p-a^p}{2p}K_{\lambda,\mu}^{1/r}(r)D^{1/q}(a,b;p;q)\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{1/q}$$

where

$$K_{\lambda,\mu}(r)=\frac{\lambda^{r+1}+\mu^{r+1}}{(r+1)(\lambda+\mu)},$$

$$D(a,b;p;q)=\begin{cases}\left(L_{p-1}^{p-1}\right)^{-1}L_{q-qp+p-1}^{q-qp+p-1}, & p\in R\setminus\{0,1,q/(q-1)\}\\ L^{-1}(a^p,b^p), & p=q/(q-1)\\ 1, & p=1\end{cases}$$

$L_p = L_p\left(a,b\right) := \left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $p \in R \setminus \{-1,0\}$, is the p- logarithmic mean and $L(a,b) := \dfrac{b-a}{\ln b - \ln a}$ is logarithmic mean.

Proof: From Lemma 2 and using the Hölder integral inequality and p -quasi-convexity of $\left|f'\right|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
&\le \frac{b^p-a^p}{p(\lambda+\mu)}\int_0^1 \frac{\left|(\lambda+\mu)t-\lambda\right|}{\left[tb^p+(1-t)a^p\right]^{1-1/p}}\left|f'(M_{p,t}(a,b))\right|dt \\
&\le \frac{b^p-a^p}{2p}\left(\int_0^1 \left|(\lambda+\mu)t-\lambda\right|^r dt\right)^{1/r}\left(\int_0^1 \frac{\left|f'(M_{p,t}(a,b))\right|^q}{\left[tb^p+(1-t)a^p\right]^{q-q/p}}dt\right)^{1/q} \\
&\le \frac{b^p-a^p}{2p}\left(\int_0^1 \left|(\lambda+\mu)t-\lambda\right|^r dt\right)^{1/r}\left(\int_0^1 \frac{\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}}{\left[tb^p+(1-t)a^p\right]^{q-q/p}}dt\right)^{1/q} \\
&\le \frac{b^p-a^p}{2p}K_{\lambda,\mu}^{1/r}.D^{1/q}\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{1/q}.
\end{aligned}$$

It is easily check that

$$\int_0^1 \left|(\lambda+\mu)t-\lambda\right|^r dt = K_{\lambda,\mu}(r),$$

$$\int_0^1 \frac{1}{\left[tb^p+(1-t)a^p\right]^{q-q/p}}dt = D(a,b;p;q).$$

In Theorem 3, if we put $p=1$, then we obtain the following corollary for quasi-convex functions:

Corollary 3 Under the conditions of Theorem 2, if we take $p=1$, then we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{b-a}{2}K_{\lambda,\mu}^{1/r}(r)\left(\max\left\{\left|f^{'}(a)\right|^q,\left|f^{'}(b)\right|^q\right\}\right)^{1/q}.$$

In Theorem 3, if we put $p=-1$, then we obtain the following corollary for harmonically quasi-convex functions:

Corollary 4 Under the conditions of Theorem 2, if we take $p=-1$, then we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{\lambda+\mu}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{b^p-a^p}{2p}K_{\lambda,\mu}^{1/r}(r)D^{1/q}(a,b;-1;q)\left(\max\left\{\left|f^{'}(a)\right|^q,\left|f^{'}(b)\right|^q\right\}\right)^{1/q}.$$

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