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THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GODUNOVA-LEVIN CLASS OF FUNCTIONS IN HILBERT SPACE

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Abstract

In this paper, firstly we defined a new operator function class in Hilbert Space for Hermite-Hadamard type inequalities via Godunova-Levin functions, i.e., we introduce $S_Q O$ class. Secondly, we established some new theorems for them. Finally, we obtained The Hermite-Hadamard type inequalities for the product two operators Godunova-Levin functions in Hilbert Space.

Key words: *The Hermite-Hadamard inequality; Class of functions Godunova-Levin; Operator class of functions Godunova-Levin class of functions; Selfadjoint Operator; Inner product space; Hilbert Space.*

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HİLBERT UZAYINDA OPERATÖR GODUNOVA-LEVİN SINIFI FONKSİYONLAR IÇİN HERMİTE-HADAMARD TİPLİ EŞİTSİZLİKLER

Özet

Bu çalışmada biz ilk olarak Hilbert uzayında Godunava Levin fonksiyonlar yardımıyla Hermite-Hadamard tipli eşitsizlikler için yeni bir operatör sınıfı tanıttık, yani $S_Q O$ sınıfı. İkinci olarak bu sınıf için bazı yeni teoremler ispat edildi. Son olarak Hilbert uzayında iki çarpım durumunda operatör Godunova Levin fonksiyonlar için Hermite-Hadamard tipli eşitsizlikler elde edildi.

Anahtar Kelimeler : *Hermite-Hadamard eşitsizliği; Godunova-Levin fonksiyonları sınıfı operatör Godunova-Levin fonksiyonları sınıfı; Özeşlenik operatör; İç çarpım uzayı; Hilbert uzayı.*

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1.INTRODUCTION

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \qquad (1.1)$$

both inequalities hold in the reversed direction if f is concave. The inequality (1.1) was discovered by Hermite in 1881 in the journal $Mathesis$ (Mitrinovic′ et al 1985). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result (Pec'aric' et al 1992), (Beckenbach 1948) a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893. In 1974, (Mitronovic′ et al 1985). found Hermite's note in $Mathesis$. Since (1.1) was known as Hadamard's inequality, it is now commonly referred as the Hermite-Hadamard inequality (Pec'aric' et al 1992).

In 1985 Godunova et al (1985) (see also (Mitrinovic et al 1990) and (Mitrinovic' et al 1993)) introduced the following remarkable class of functions:

Definition 1.1 (Godunova et al 1985) *A map $f: I \to \mathbb{R}$ is said to belongs to the class $Q(I)$ if it is nonnegative and, for all $x, y \in I \text{ and } \lambda \in (0,1)$, satisfies the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \frac{1}{\lambda}f(x) + \frac{1}{1-\lambda}f(y) \qquad (1.2)$$

They also noted that all nonnegative monotone and nonnegative convex functions belong to this class and they also proved the following motivating result: If $f \in Q(I)$ and $x, y, z \in I$, then

$$f(x)(x-y)(x-z) + f(y)(y-x)(y-z) + f(z)(z-x)(z-y) \geq 0. \qquad (1.3)$$

In fact, (1.3) is even equivalent to (1.2) so it can alternatively be used in the definition of the class $Q(I)$ (see (Mitrinovic et al 1990)). Now, we associated with Linear Operator Theory and Inequality Theory. Therefore we firstly define operator Godunova-Levin class in Hilbert space, i.e., $S_Q O$ operator class. Secondly, we established some new theorems and finally, we obtained The Hermite-Hadamard type inequalities for the product two operators Godunova-Levin functions in Hilbert Space.

2. PRELIMINARY

Linear Operator Theory in Hilbert space plays a central role in contemporary mathematics with numerous applications for Partial Differential Equations, in Approximation Theory, Numerical Analysis, Probability Theory and Statistics and other fields. In this paper we present results concerning Hermite-Hadamard type inequalities for continuous functions of bounded selfadjoint operators on complex Hilbert spaces and give some new definitions and theorems. So we are intended for use by both researchers in various fields of Linear Operator Theory and Mathematical Inequalities, domains which have grown exponentially in the last decade, as well as by scientists applying inequalities in their specific areas.

Now, we give fundamental definitions.

Let $(H,\langle .,.\rangle)$ be a Hilbert space over the complex numbers field $\mathbb{C}$. A bounded linear operator A defined on H is selfadjoint, i.e. $A=A^*$ if and only if $\langle Ax,x\rangle \in \mathbb{R}$ for all $x \in H$ and if A is selfadjoint, then

$$\| A \| = sup_{\|x\|=1} |\langle Ax,x\rangle| = sup_{\|x\|,\|y\|=1} |\langle Ax,x\rangle|.$$

We assume in what follows that all operators are bounded on defined on the whole Hilbert space H. We denote by $B(H)$ the Banach algebra of all bounded linear operators defined on H. First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B (\text{or } B \geq A) \text{ if } \langle Ax,x\rangle \leq \langle Bx,x\rangle (\text{or } \langle Bx,x\rangle \geq \langle Ax,x\rangle)$$

we call it the operator order.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (Furuta et al 2005): For any $f,g \in C(Sp(A))$ and any $\alpha,\beta \in \mathbb{C}$ we have [*i.]*

1. $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
2. $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f^*)=\Phi(f)^*$;
3. $\| \Phi(f) \| = \| f \| := sup_{t\in Sp(A)} |f(t)|$;
4. $\Phi(f_0)=1$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t \in Sp(A)$.

If f is a continuous complex-valued functions on $C(Sp(A))$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A. If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order $B(H)$. A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

3. THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GODUNOVA-LEVIN CLASS OF FUNCTIONS IN HILBERT SPACE

3.1. Operator Godunova-Levin class of functions in Hilbert space

The following definition and function class are firstly defined by Seren Salas.

Definition 3.1 *Let I be an interval in $\mathbb{R}$ and K be a convex subset of $B(H)^+$. A continuous function $f: I \to \mathbb{R}$ is said to be operator Godunova-Levin class of function on I, operators in K if*

$$f(\lambda A + (1-\lambda)B) \leq \frac{1}{\lambda} f(A) + \frac{1}{1-\lambda} f(B)$$

(3.1)

in the operator order in $B(H)$, for all $\lambda \in (0,1)$ and for every positive operators A and B in K whose spectra are contained in I .

In the other words, if f is an operator Godunova-Levin class of functions on I, we denote by $f \in S_Q O$.

Lemma 3.1 *If $f \in S_Q O$ in $K \subseteq B^+(H)$, then $f(A)$ is positive for every $A \in K$.*

Proof. For $A \in K$, we have

$$f(A) = f(\frac{A}{2} + \frac{A}{2}) \leq 2f(A) + 2f(A) = 4f(A).$$

This implies that $f(A) \geq 0$. (Moslehian et al 2011) proved the following theorem for positive operators as follows :

Theorem 3.1 (Moslehian et al 2011) *Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator functions f on $[0, \infty)$.*

Dragomir in (Dragomir 2013) has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 3.2 (Dragomir 2013) *Let $f: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I, we have the inequality*

$$(f(\frac{A+B}{2}) \leq) \frac{1}{2}[f(\frac{3A+B}{4}) + f(\frac{A+3B}{4})]$$
$$\leq \int_0^1 f((1-t)A + tB))dt$$
$$\leq \frac{1}{2}[f(\frac{A+B}{2}) + \frac{f(A)+f(B)}{2}](\leq (\frac{f(A)+f(B)}{2})]$$

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0,1].$$

We consider the function $f: [x, y]: \to \mathbb{R}$ and the associated function

$$g(x, y): [0,1] \to \mathbb{R}$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f(\frac{x+y}{2}) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x)+f(y)}{2}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex $g(x, y): [0,1] \to \mathbb{R}$.

Lemma 3.2 *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I. The function $f \in S_Q O$ for operators in*

$$[A, B] := \{(1-t)A + tB; \ t \in [0,1]\}$$

if and only if the function $\varphi_{x,A,B}: [0,1] \to \mathbb{R}$ defined by

$$\varphi_{x,A,B}:=< f((1-t)A+tB)x,x >$$

is Gudonava-Levin on $[0,1]$ for every $x \in H$ with $\| x \| = 1$.

Proof. Since $f \in S_Q O$ operator in $[A,B]$, then for any $t_1, t_2 \in [0,1]$ and $\lambda \in (0,1)$, we have

$$\begin{aligned}
&\varphi_{x,A,B}(\lambda t_1 + (1-\lambda)t_2) \\
&=< f((1-(\lambda t_1 + (1-\lambda)t_2)A + (\lambda t_1 + (1-\lambda)t_2)B)x,x > \\
&=< f(\lambda[(1-t_1)A + t_1 B] + (1-\lambda)[(1-t_2)A + t_2 B])x,x > \\
&\leq \tfrac{1}{\lambda} < f((1-t_1)A + t_1 B)x,x > + \tfrac{1}{1-\lambda} f((1-t_2)A + t_2 B)x,x > \\
&\leq \tfrac{1}{\lambda}\varphi_{x,A,B}(t_1) + \tfrac{1}{1-\lambda}\varphi_{x,A,B}(t_2).
\end{aligned}$$

Theorem 3.3 *Let $f \in S_Q O$ on the interval $I \subseteq [0,\infty)$ for operators $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality*

$$\frac{1}{4} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(\lambda A + (1-\lambda)B)\,d\lambda \tag{3.2}$$

Proof. For $x \in H$ with $\| x \| = 1$ and $\lambda \in (0,1)$, we have

$$< ((1-\lambda)A + \lambda B)x,x >= (1-\lambda) < Ax,x > + \lambda < Bx,x > \in I, \tag{3.3}$$

Since $< Ax,x > \in Sp(A) \subseteq I$ and $< Bx,x > \in Sp(B) \subseteq I$. Continuity of f and 3.3 imply that the operator-valued integral $\int_0^1 f(\lambda A + (1-\lambda)B)d\lambda$ exists. Since $f \in S_Q O$, therefore for λ in $(0,1)$, and $A, B \in K$ we have

$$f(\lambda A + (1-\lambda)B) \leq \frac{1}{\lambda} f(A) + \frac{1}{1-\lambda} f(B) \tag{3.4}$$

To prove the first inequality of 3.2, we observe that

$$f\left(\frac{A+B}{2}\right) \leq 2f(\lambda A + (1-\lambda)B) + 2f((1-\lambda)A + \lambda B) \tag{3.5}$$

Integrating the inequality 3.5, over $\lambda \in (0,1)$ and taking into account that

$$\int_0^1 f(\lambda A + (1-\lambda)B)d\lambda = \int_0^1 f((1-\lambda)A + \lambda B)d\lambda$$

then we deduce the first part of 3.2.

Remark 3.1 *We note that, the Beta functions is defined as follows:*

$$\beta(x,y)=\int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}d\lambda x>0, y>0.$$

Theorem 3.4 *Let $f\in S_Q O$ on the interval $I\subseteq[0,\infty)$ for operators $K\subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality*

$$\int_0^1 \lambda(1-\lambda)f(\lambda A+(1-\lambda)B)d\lambda\leq\frac{f(A)+(B)}{2} \tag{3.6}$$

Proof. For $x\in H$ with $\| x \|=1$ and $\lambda\in(0,1)$, we have

$$<((1-\lambda)A+\lambda B)x,x>=(1-\lambda)<Ax,x>+\lambda<Bx,x>\in I, \tag{3.7}$$

Since $<Ax,x>\in Sp(A)\subseteq I$ and $<Bx,x>\in Sp(B)\subseteq I$. Continuity of f and 3.7 imply that the operator-valued integral $\int_0^1 f(\lambda A+(1-\lambda)B)d\lambda$ exists. Since $f\in S_Q O$, therefore for $\lambda\in(0,1)$, and $A,B\in K$ we have

$$f(\lambda A+(1-\lambda)B)\leq\frac{1}{\lambda}f(A)+\frac{1}{1-\lambda}f(B) \tag{3.8}$$

For the proof of (3.6), we first note that if $f\in S_Q O$, then for all $A,B\in K$ and $\lambda\in(0,1)$, it yields

$$\lambda(1-\lambda)f(\lambda A+(1-\lambda)B)\leq(1-\lambda)f(A)+\lambda f(B) \tag{3.9}$$

and

$$\lambda(1-\lambda)f((1-\lambda)A+\lambda B)\leq\lambda f(A)+(1-\lambda)f(B) \tag{3.10}$$

By adding (3.9) and (3.10) inequalities,

$$\lambda(1-\lambda)(f(\lambda A+(1-\lambda)B)+f((1-\lambda)A+\lambda B))\leq f(A)+f(B) \tag{3.11}$$

and integrating both sides of 3.11 over $[0,1]$, we get the following inequality

$$\int_0^1 \lambda(1-\lambda)(f(\lambda A+(1-\lambda)B)+f((1-\lambda)A+\lambda B))d\lambda\leq f(A)+f(B) \tag{3.12}$$

Integrating the inequality 3.12 over $\lambda\in(0,1)$ and taking into account that

$$\int_0^1 \lambda(1-\lambda)f(\lambda A+(1-\lambda)B)d\lambda=\int_0^1 \lambda(1-\lambda)f((1-\lambda)A+\lambda B)d\lambda$$

then we deduce the first part of 3.6.

3.2 The Hermite-Hadamard type inequality for the product two operators Godunova-Levin functions

Let f, g belong to $S_Q O$ on the interval in I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $M(A,B)$ and $N(A,B)$ on H by

$$M(A,B)(x) = \langle f(A)x,x\rangle\langle g(A)x,x\rangle + \langle f(B)x,x\rangle\langle g(B)x,x\rangle \;\; (x \in H),$$

$$N(A,B)(x) = \langle f(A)x,x\rangle\langle g(B)x,x\rangle + \langle f(B)x,x\rangle\langle g(A)x,x\rangle \;\; (x \in H).$$

Theorem 3.5 *Let $f, g \in S_Q O$ be function on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality*

$$\int_0^1 \lambda^2(1-\lambda)^2\langle f(\lambda A + (1-\lambda)B)x,x\rangle\langle g(\lambda A + (1-\lambda)B)x,x\rangle d\lambda \qquad (3.13)$$
$$\leq \frac{1}{3}M(A,B) + \frac{1}{6}N(A,B)$$

hold for any $x \in H$ with $\| x \| = 1$.

***Proof*.** For $x \in H$ with $\| x \| = 1$ and $t \in [0,1]$, we have

$$< (A+B)x,x > = < Ax,x > + < Bx,x > \in I \qquad (3.14)$$

since $< Ax,x > \in Sp(A) \subseteq I$ and $< Bx,x > \in Sp(B) \subseteq I$. Continuity of f, g and imply (3.14) that the operator-valued integral

$$\int_0^1 f(\lambda A + (1-\lambda)B)d\lambda, \; \int_0^1 g(\lambda A + (1-\lambda)B)d\lambda \text{ and } \int_0^1 (fg)(\lambda A + (1-\lambda)B)d\lambda$$

exist. Since f, g belong to $S_Q O$, therefore for λ in $(0,1)$ and $x \in H$ we have

$$\langle f(\lambda A + (1-\lambda)B)x,x\rangle \leq \langle \frac{1}{\lambda}f(A) + \frac{1}{1-\lambda}f(B)x,x\rangle \qquad (3.15)$$

$$\langle g(\lambda A + (1-\lambda)B)x,x\rangle \leq \langle \frac{1}{\lambda}g(A) + \frac{1}{1-\lambda}g(B)x,x\rangle. \qquad (3.16)$$

From 3.14 and 3.15, we obtain

$$\langle f(\lambda A+(1-\lambda)B)x,x\rangle\langle g(\lambda A+(1-\lambda)B)x,x\rangle \leq \frac{1}{\lambda^2}\langle f(A)x,x\rangle\langle g(A)x,x\rangle$$
$$+\frac{1}{\lambda(1-\lambda)}\langle f(A)x,x\rangle\langle g(B)x,x\rangle$$
$$+\frac{1}{\lambda(1-\lambda)}\langle f(B)x,x\rangle\langle g(A)x,x\rangle$$
$$+\frac{1}{(1-\lambda)^2}\langle f(B)x,x\rangle\langle g(B)x,x\rangle$$

Integrating both sides of ?? over $(0,1)$, we get the required inequality 3.13.

Theorem 3.6 *Let f,g belong to S_QO on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality*

$$\beta(3,3)\langle f(\tfrac{A+B}{2})x,x\rangle\langle g(\tfrac{A+B}{2})x,x\rangle$$
$$\leq 8\int_0^1 \lambda^2(1-\lambda)^2\langle f(\lambda A+(1-\lambda)B)x,x\rangle\langle g(\lambda A+(1-\lambda)B)x,x\rangle d\lambda$$
$$+\frac{4}{3}M(A,B)+\frac{8}{3}N(A,B) \tag{3.17}$$

hold for any $x \in H$ with $\| x \| = 1$.

Proof. Since $f,g \in S_QO$, therefore for any $t \in I$ and any $x \in H$ with $\| x \| = 1$, we observe that

$$\langle f(\tfrac{A+B}{2})x,x\rangle\langle g(\tfrac{A+B}{2})x,x\rangle$$
$$\leq \langle f(\frac{\lambda A+(1-\lambda)B}{2}+\frac{(1-\lambda)A+\lambda B}{2})x,x\rangle$$
$$\times \langle g(\frac{\lambda A+(1-\lambda)B}{2}+\frac{(1-\lambda)A+\lambda B}{2})x,x\rangle$$

$$\leq 4\{\langle f(\lambda A+(1-\lambda)B)x,x\rangle+\langle f((1-\lambda)A+\lambda B)x,x\rangle$$
$$\times \langle g(\lambda A+(1-\lambda)B)x,x\rangle+\langle g((1-\lambda)A+\lambda B)x,x\rangle\}$$

$$\leq 4\{[\langle f(\lambda A+(1-\lambda)B)x,x\rangle\langle g(\lambda A+(1-\lambda)B)x,x\rangle]$$
$$+[\langle f((1-\lambda)A+tB)x,x\rangle\langle g((1-\lambda)A+\lambda B)x,x\rangle]$$
$$+[\frac{1}{\lambda}\langle f(A)x,x\rangle+\frac{1}{1-\lambda}\langle f(B)x,x\rangle]$$

$$\times [\tfrac{1}{1-\lambda}\langle g(A)x,x\rangle + \tfrac{1}{\lambda}\langle g(B)x,x\rangle]$$
$$+[\tfrac{1}{1-\lambda}\langle f(A)x,x\rangle + \tfrac{1}{\lambda}\langle f(B)x,x\rangle]$$
$$\times [\tfrac{1}{\lambda}\langle g(A)x,x\rangle + \tfrac{1}{1-\lambda}\langle g(B)x,x\rangle]\}$$

$$= 4\{\langle f(\lambda A + (1-\lambda)B)x,x\rangle g(\lambda A + (1-\lambda)B)x,x\rangle]$$
$$+[\langle f((1-\lambda)A + \lambda B)x,x\rangle\langle g((1-\lambda)A + \lambda B)x,x\rangle]$$
$$+\tfrac{1}{\lambda(1-\lambda)}[\langle f(A)x,x\rangle\langle g(A)x,x\rangle] + \tfrac{1}{(\lambda)^2}[\langle f(A)x,x\rangle\langle g(B)x,x\rangle]$$
$$+\tfrac{1}{(1-\lambda)^2}[\langle f(B)x,x\rangle\langle g(A)x,x\rangle] + \tfrac{1}{\lambda(1-\lambda)}[\langle f(B)x,x\rangle\langle g(B)x,x\rangle]$$
$$+\tfrac{1}{\lambda(1-\lambda)}[\langle f(A)x,x\rangle\langle g(A)x,x\rangle] + \tfrac{1}{(1-\lambda)^2}[\langle f(A)x,x\rangle\langle g(B)x,x\rangle]$$
$$+\tfrac{1}{(1-\lambda)^2}[\langle f(B)x,x\rangle\langle g(A)x,x\rangle] + \tfrac{1}{\lambda(1-\lambda)}[\langle f(B)x,x\rangle\langle g(B)x,x\rangle]\}$$

By integration over $[0,1]$, we obtain

$$\int_0^1 \lambda^2(1-\lambda)^2\langle f(\tfrac{A+B}{2})x,x\rangle\langle g(\tfrac{A+B}{2})x,x\rangle d\lambda$$
$$\leq 4\int_0^1 \lambda^2(1-\lambda)^2[\langle f((1-\lambda)A + \lambda B)x,x\rangle + g((\lambda A + (1-\lambda)B)x,x\rangle$$
$$+\langle f(\lambda A + (1-\lambda)B)x,x\rangle + \langle g((1-\lambda)A + \lambda B)x,x\rangle)]d\lambda$$
$$+\tfrac{4}{3}M(A,B) + \tfrac{8}{3}N(A,B)$$

This implies the inequality 3.17.

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