Avrupa Bilim ve Teknoloji Dergisi
Özel Sayı 34, S. 181-192, Mart 2022

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
Special Issue 34, pp. 181-192, March 2022

Research Article

Adjoint Approach between a Spatial Curve and a Ruled Surface Based on the Bishop Frame

Vahide Bulut

İzmir Katip Çelebi University, Departmant of Engineering Sciences, İzmir, Turkey, (ORCID: 0000-0002-0786-8860), vahide.bulut@ikcu.edu.tr

(2nd International Conference on Applied Engineering and Natural Sciences ICAENS 2022, March 10-13, 2022)

(**DOI:** 10.31590/ejosat.1079225)

ATIF/REFERENCE: Bulut, V. (2022). Adjoint Approach between a Spatial Curve and a Ruled Surface Based on the Bishop Frame. *European Journal of Science and Technology*, (34), 181-192.

Abstract

The adjoint approach is usually used to study the crank-rocker linkages' coupler curves and the geometry of rigid objects in spatial motion. In this paper, the adjoint approach between a spatial curve and a ruled surface based on the Bishop frame is presented. Also, a ruled surface by using the components of the Type-1 and Type-2 Bishop frames is expressed. Moreover, for a curve that adjoint to a ruled surface the fixed point conditions concerning the Bishop frame are determined. Finally, we presented four examples to show the relationship between the ruled surface and its adjoint curve.

Keywords: Type-1 Bishop frame, Type-2 Bishop frame, Serret-Frenet frame, adjoint curve, ruled surface.

Bir Uzay Eğrisi ve Regle Yüzey Arasında Bishop Çatısına Dayalı Bitişik Yaklaşımı

Öz

Bitişik yaklaşım genellikle krank-rocker bağlantılarının kuplör eğrilerini ve uzay hareketinde katı cisimlerin geometrisini incelemek için kullanılır. Bu makalede, Bishop çatısına dayalı bir uzay eğrisi ve bir regle yüzey arasındaki bitişik yaklaşım sunulmaktadır. Ayrıca, Tip-1 ve Tip-2 Bishop çatılarının bileşenleri kullanılarak bir regle yüzey ifade edilmiştir. Ayrıca, bir regle yüzeye bitişik bir eğri için Bishop çatısına bağlı sabit nokta koşulları belirlenmiştir. Son olarak, regle yüzey ile onun bitişik eğrisi arasındaki ilişkiyi göstermek için dört örnek sunulmuştur.

Anahtar Kelimeler: Tip-1 Bishop çatısı, Tip-2 Bishop çatısı, Serret-Frenet çatısı, bitişik eğri, regle yüzey.

1. Introduction

The Serret-Frenet frame is formed for the curve which is differentiable nondegenerate curves. Since, the curvature may vanish at some points on the curve when the Serret- Frenet frame is used, Bishop defined a new frame for a space curve and called it the Bishop frame (parallel transport frame). Bishop frame can be defined even if the curve has a vanishing second derivative in $\mathbb{E}^3$. The advantages of the Bishop frame and relationship between the Bishop and the Serret- Frenet frame in $\mathbb{E}^3$ can be found in [1] and [2].

The adjoint approach is widely studied on the properties of a spatial curve or a surface with a spatial curve or surface [3, 4], such as the properties of an involute and evolute of a curve or Bertrand curves [5]. Also, some researchers have been used the adjoint approach in mechanical engineering [6, 7, 8, 9].

In this study, we present the ruled surface by using the components of the Type-1 and Type-2 Bishop frames. Moreover, we express the generator trihedron of this ruled surface and determine the fixed point FP conditions for a curve that adjoint to a ruled surface based on the Bishop frame.

2. Preliminaries

A spatial curve $\boldsymbol{\alpha}: I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ in $\mathbb{E}^3$ is called a unit speed curve if $\|\boldsymbol{\alpha}'(s)\| = 1$, where s is the arc length parameter of this curve. Serret-Frenet frame of the curve $\boldsymbol{\alpha}(s)$ in $\mathbb{E}^3$ parameterized by arc length parameter s is given with

$$\boldsymbol{\alpha}'(s) = \boldsymbol{T}, \quad \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|} = \boldsymbol{N} \ and \quad \boldsymbol{T} \times \boldsymbol{N} = \boldsymbol{B}, \tag{1}$$

where the unit vectors $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$ are called the unit tangent, unit principal normal, and unit binormal vectors, respectively. Also, the invariants $\kappa = \kappa(s) = \|\boldsymbol{T}'(s)\|$ and $\tau = \tau(s) = \|\boldsymbol{B}'(s)\|$ of this curve are the curvature and the torsion at the point s_0, respectively. The derivative formulas of the Serret-Frenet frame are written as

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}. \tag{2}$$

The Bishop frame [1] is based on parallel fields. Parallel transport of the Serret-Frenet frame along the curve $\boldsymbol{\alpha}$ can be defined as a parallel transporting each component of the frame. The derivative formulas of the Type-1 Bishop frame are expressed with the following equation.

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{M_1}' \\ \boldsymbol{M_2}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{M_1} \\ \boldsymbol{M_2} \end{bmatrix}, \tag{3}$$

where $\{\boldsymbol{T}, \boldsymbol{M_1}, \boldsymbol{M_2}\}$ is the Type-1 Bishop frame, and k_1 and k_2 are called the first and second Bishop curvature, respectively [1]. Moreover, the relationship between the Serret-Frenet frame and Type-1 Bishop frame can be expressed as

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{T} \\ \boldsymbol{N} &= \cos\varphi\, \boldsymbol{M_1} + \sin\varphi\, \boldsymbol{M_2} \\ \boldsymbol{B} &= -\sin\varphi\, \boldsymbol{M_1} + \cos\varphi\, \boldsymbol{M_2}, \end{aligned} \tag{4}$$

where $\varphi(s) = arctan\left(\frac{k_2}{k_1}\right)$, $\tau(s) = \left(\frac{d\varphi}{ds}\right)$ and $\kappa(s) = \sqrt{{k_1}^2 + {k_2}^2}$. Also, Type-1 Bishop curvatures are defined with

$$k_1 = \kappa\cos\varphi\,, \quad k_2 = \kappa\sin\varphi\,.$$

The Type-2 Bishop frame formulas are given below:

$$\begin{bmatrix} \boldsymbol{N_1}' \\ \boldsymbol{N_2}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\mu_1 \\ 0 & 0 & -\mu_2 \\ \mu_1 & \mu_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{N_1} \\ \boldsymbol{N_2} \\ \boldsymbol{B} \end{bmatrix}, \tag{5}$$

in which $\{\boldsymbol{N_1}, \boldsymbol{N_2}, \boldsymbol{B}\}$ is the Type-2 Bishop frame, μ_1 and μ_2 are called the first and second Bishop curvature, respectively [1]. Also, we can express the relationship between the Serret-Frenet frame and Type-2 Bishop frame as

$$\begin{aligned} \boldsymbol{T} &= \sin\phi\, \boldsymbol{N_1} - \cos\phi\, \boldsymbol{N_2} \\ \boldsymbol{N} &= \cos\phi\, \boldsymbol{N_1} + \sin\phi\, \boldsymbol{N_2} \\ \boldsymbol{B} &= \boldsymbol{B}, \end{aligned} \tag{6}$$

where $\phi(s) = arctan\left(\frac{\mu_2}{\mu_1}\right)$, $\tau(s) = \sqrt{{\mu_1}^2 + {\mu_2}^2}$. Additionally, Type-2 Bishop curvatures are defined by

$$\mu_1 = -\tau\cos\phi\,, \quad \mu_2 = -\tau\sin\phi\,.$$

2.1. Ruled Surface

A ruled surface is a surface generated by moving a line in space. A ruled surface is shown as;

$$\boldsymbol{R}(u, t) = \boldsymbol{\alpha}(u) + t\boldsymbol{l}(u) \tag{7}$$

where $\boldsymbol{\alpha}(u)$ is the directrix curve and the unit vector $\boldsymbol{l}(u)$ is the direction vector of the generator or the ruling of the ruled surface (see Figure (1)).

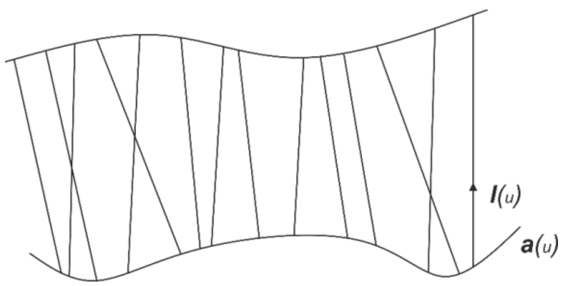

Figure 1: A ruled surface

Definition 1: The point of a ruling of a ruled surface at which the tangent plane of the ruling is perpendicular to the limit position of the tangent plane is called the striction point of the ruling. The set of all striction points forms a curve called the striction curve of the ruled surface. The striction curve can be presented with the following equation [10]:

$$\boldsymbol{b}(u)=\boldsymbol{a}(u)-\frac{\left(\frac{d\boldsymbol{a}}{du}.\frac{d\boldsymbol{l}}{du}\right)}{\left(\frac{d\boldsymbol{l}}{du}\right)^2}.\boldsymbol{l}(u). \quad (8)$$

Theorem 1: The directrix curve is the striction curve of the ruled surface $\boldsymbol{R}(u,t)$ if and only if

$$\frac{d\boldsymbol{a}}{du}.\frac{d\boldsymbol{l}}{du}=0, \quad (9)$$

in which $\boldsymbol{l}(u)$ and $\boldsymbol{a}(u)$ are the generator and the directrix curve of $\boldsymbol{R}(u,t)$ [11].

Since, a directrix curve can be any curve if the curve intersects with all the rulings, the ruled surface $\boldsymbol{R}(u,t)$ can be rewritten by considering the striction curve with

$$\boldsymbol{R}(u,t)=\boldsymbol{b}(u)+t\boldsymbol{l}(u). \quad (10)$$

On the other hand, the trihedron that is located on the striction curve is called a generator trihedron. We can present this trihedron as $\{\boldsymbol{b};\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$ and the differential formulas of this trihedron are defined as

$$\begin{cases}\frac{d\boldsymbol{b}}{d\sigma}=\lambda\boldsymbol{x_1}+\eta\boldsymbol{x_3}\\ \frac{d\boldsymbol{x_1}}{d\sigma}=\boldsymbol{x_2}\\ \frac{d\boldsymbol{x_2}}{d\sigma}=-\boldsymbol{x_1}+\rho\boldsymbol{x_3}\\ \frac{d\boldsymbol{x_3}}{d\sigma}=-\rho\boldsymbol{x_2}\end{cases} \quad (11)$$

where λ, η and ρ are called the construction parameters and they are a ruled surface's kinematic invariants. Also, σ is the spherical image curve's arc length and the relationship between σ and u can be expressed with

$$d\sigma=\left|\frac{d\boldsymbol{l}}{du}\right|du. \quad (12)$$

The geometrical meaning of these kinematic invariants is that:

i) λ is the angle between the striction curve's tangent vector and the generator and can be determined as

$$\lambda=\frac{d\boldsymbol{b}}{d\sigma}.\boldsymbol{x_1}=\frac{\frac{d\boldsymbol{b}}{du}}{\left|\frac{d\boldsymbol{l}}{du}\right|}.\boldsymbol{l}, \quad (13)$$

ii) η is the distribution parameter and can be determined as

$$\eta=\frac{d\boldsymbol{b}}{d\sigma}.\boldsymbol{x_3}=\frac{\left(\frac{d\boldsymbol{b}}{du},\boldsymbol{l},\frac{d\boldsymbol{l}}{du}\right)}{\left|\frac{d\boldsymbol{l}}{du}\right|^2}, \quad (14)$$

iii) ρ is the spherical image curve's geodesic curvature and can be determined as [11]

$$\rho=\left(\boldsymbol{l},\frac{d\boldsymbol{l}}{d\sigma},\frac{d^2\boldsymbol{l}}{d\sigma^2}\right)=\frac{\left(\boldsymbol{l},\frac{d\boldsymbol{l}}{du},\frac{d^2\boldsymbol{l}}{du^2}\right)}{\left|\frac{d\boldsymbol{l}}{du}\right|^3}. \quad (15)$$

3. A Spatial Curve Adjoining A Ruled Surface Based On Bishop Frame

We examined a spatial curve adjoining a spatial curve according to the Type-1 and Type-2 Bishop frames in [12]. Now, we will examine a spatial curve adjoining a ruled surface according to the Type-1 and Type-2 Bishop frames. For this purpose, let's take a ruled surface $\boldsymbol{R(s,t)}$ in the fixed coordinate system $\{\boldsymbol{O};\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ with

$$\boldsymbol{R(s,t)}=\boldsymbol{a(s)}+\boldsymbol{tl(s)} \quad (16)$$

where $\boldsymbol{s}$ is the arc length parameter. A point $\boldsymbol{P}$ that does not belong to the ruled surface $\boldsymbol{R(s,t)}$ traces a curve $\boldsymbol{r}^*$ in the same fixed coordinate system $\{\boldsymbol{O};\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$. Consequently, each position of the point $\boldsymbol{P}$ along the curve $\boldsymbol{r}^*$ adjoint to the generator of the ruled surface $\boldsymbol{R(s,t)}$. The curve $\boldsymbol{r}^*$ is called an adjoint curve and the surface $\boldsymbol{R(s,t)}$ is called an original surface [11] (see Figure (2)).

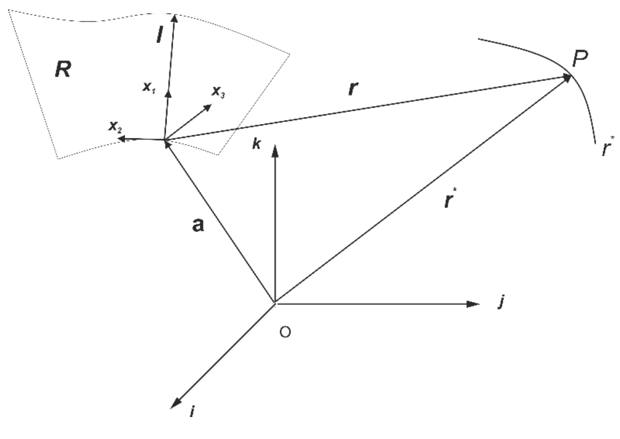

Figure 2: The spatial curve $\boldsymbol{r}^*$ adjoint to the ruled surface $\boldsymbol{R(s,t)}$.

Let us express this curve adjoint to the ruled surface $\boldsymbol{R(s,t)}$ by using Type-1 and Type-2 Bishop frame in the following subsections.

3.1. Type-I Bishop Frame $\{\boldsymbol{a};\boldsymbol{T},\boldsymbol{M_1},\boldsymbol{M_2}\}$

3.1.1. The Generator is T

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit tangent vector $\boldsymbol{T}$ the curve $\boldsymbol{a}(s)$ according to the arc length parameter s as

$$\boldsymbol{R}(s,t)=\boldsymbol{a}(s)+t\boldsymbol{T}(s). \tag{17}$$

On the other hand, since $\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{T}}{ds}=\boldsymbol{T}.(k_1\boldsymbol{M_1}+k_2\boldsymbol{M_2})=0$ according to Theorem 1, the directrix curve $\boldsymbol{a}(s)$ can be taken as a striction curve. Therefore, we can express the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ as

$$\boldsymbol{x_1}=\boldsymbol{T}\quad \boldsymbol{x_2}=\frac{k_1\boldsymbol{M_1}+k_2\boldsymbol{M_2}}{\sqrt{{k_1}^2+{k_2}^2}}$$
$$\boldsymbol{x_3}=\frac{k_1\boldsymbol{M_2}-k_2\boldsymbol{M_1}}{\sqrt{{k_1}^2+{k_2}^2}}. \tag{18}$$

The relationship between σ and s is $\frac{d\sigma}{ds}=\sqrt{{k_1}^2+{k_2}^2}$. The differential formulas of the generator trihedron are given below:

$$\begin{cases}\dfrac{d\boldsymbol{a}}{d\sigma}=\lambda\boldsymbol{x_1}\\[2mm] \dfrac{d\boldsymbol{x_1}}{d\sigma}=\boldsymbol{x_2}\\[2mm] \dfrac{d\boldsymbol{x_2}}{d\sigma}=-\boldsymbol{x_1}+\rho\boldsymbol{x_3}\\[2mm] \dfrac{d\boldsymbol{x_3}}{d\sigma}=-\rho\boldsymbol{x_2}\end{cases} \tag{19}$$

in which the kinematic invariants are determined as follows:

$$\lambda=\frac{1}{\sqrt{{k_1}^2+{k_2}^2}},\quad \eta=0,\quad \rho=\frac{k_1k_2'-k_2k_1'}{\left({k_1}^2+{k_2}^2\right)^{3/2}}. \tag{20}$$

where $k_1'=\frac{dk_1}{ds}$ and $k_2'=\frac{dk_2}{ds}$.

Remark 1: It should be noted that the kinematic invariants have a relationship with the torsion of the directrix curve $\boldsymbol{a}(s)$, since $\sqrt{{k_1}^2+{k_2}^2}=\kappa$.

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be written as

$$\boldsymbol{r}^*=\boldsymbol{a}+\boldsymbol{r}=\boldsymbol{a}+u_1\boldsymbol{x_1}+u_2\boldsymbol{x_2}+u_3\boldsymbol{x_3}, \tag{21}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{a};\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$ and $\boldsymbol{a}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas in (19), we can express the first derivative of the equation (21) wrt. the arc length σ with the following equation.

$$\begin{aligned}\frac{d\boldsymbol{r}^*}{d\sigma}&=\Sigma_1\boldsymbol{x_1}+\Sigma_2\boldsymbol{x_2}+\Sigma_3\boldsymbol{x_3}\\ \Sigma_1&=\frac{du_1}{d\sigma}-u_2+\frac{1}{\sqrt{{k_1}^2+{k_2}^2}}\\ \Sigma_2&=\frac{du_2}{d\sigma}+u_1-u_3\frac{k_1k_2'-k_2k_1'}{\left({k_1}^2+{k_2}^2\right)^{3/2}}\\ \Sigma_3&=\frac{du_3}{d\sigma}+u_2\frac{k_1k_2'-k_2k_1'}{\left({k_1}^2+{k_2}^2\right)^{3/2}}\end{aligned} \tag{22}$$

where Σ_i, $(i=1,2,3)$ are the rates of change of the absolute coordinates of the point P in $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, and $\frac{du_i}{d\sigma}$, $(i=1,2,3)$ are the relative coordinates' rates of change of the point P in $\{\boldsymbol{a};\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$. On the other hand, if the point P is a fixed point in $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, then it will be true that $\frac{d\boldsymbol{r}^*}{d\sigma}=0$. Hence, we can write the equation (22) as

$$\begin{aligned}\Sigma_1&=\frac{du_1}{d\sigma}-u_2+\frac{1}{\sqrt{{k_1}^2+{k_2}^2}}=0\\ \Sigma_2&=\frac{du_2}{d\sigma}+u_1-u_3\frac{k_1k_2'-k_2k_1'}{\left({k_1}^2+{k_2}^2\right)^{3/2}}=0\\ \Sigma_3&=\frac{du_3}{d\sigma}+u_2\frac{k_1k_2'-k_2k_1'}{\left({k_1}^2+{k_2}^2\right)^{3/2}}=0.\end{aligned} \tag{23}$$

For a curve that adjoint to a ruled surface, the equations in (23) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{T}$.

3.1.2. The Generator is $\boldsymbol{M_1}$

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{M_1}$ of the curve $\boldsymbol{a}(s)$ according to the arc length parameter s with

$$\boldsymbol{R}(s,t)=\boldsymbol{a}(s)+t\boldsymbol{M_1}. \tag{24}$$

Also, since $\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{M_1}}{ds}=-k_1\neq 0$ according to Theorem 1, the directrix curve $\boldsymbol{a}(s)$ can not be taken as a striction curve. Consequently, the striction curve can be found with

$$\boldsymbol{b}(s) = \boldsymbol{a}(s) - \frac{\left(\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{M_1}}{ds}\right)}{\left(\frac{d\boldsymbol{M_1}}{ds}\right)^2}\boldsymbol{M_1}(s) = \boldsymbol{a}(s) + \frac{1}{k_1}\boldsymbol{M_1}(s). \tag{25}$$

Then, the ruled surface in (24) can be rewritten as

$$\boldsymbol{R}(s,t) = \boldsymbol{b}(s) + t\boldsymbol{M_1}. \tag{26}$$

Therefore, the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ can be expressed by

$$\boldsymbol{x_1} = \boldsymbol{M_1}, \quad \boldsymbol{x_2} = \frac{\frac{d\boldsymbol{M_1}}{ds}}{\left|\frac{d\boldsymbol{M_1}}{ds}\right|} = -\boldsymbol{T}, \quad \boldsymbol{x_3} = \boldsymbol{M_2}\,. \tag{27}$$

The relationship between σ and s is $\frac{d\sigma}{ds} = k_1$. The differential formulas of the generator trihedron are given with the following equation.

$$\begin{cases} \frac{d\boldsymbol{b}}{d\sigma} = \lambda \boldsymbol{x_1} \\ \frac{d\boldsymbol{x_1}}{d\sigma} = \boldsymbol{x_2} \\ \frac{d\boldsymbol{x_2}}{d\sigma} = -\boldsymbol{x_1} + \rho\boldsymbol{x_3} \\ \frac{d\boldsymbol{x_3}}{d\sigma} = -\rho\boldsymbol{x_2} \end{cases} \tag{28}$$

in which the kinematic invariants are shown below:

$$\lambda = \frac{k_1'}{{k_1}^3}, \quad \eta = 0, \quad and \quad \rho = -\frac{k_2}{k_1}. \tag{29}$$

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be written as

$$\boldsymbol{r}^* = \boldsymbol{b} + \boldsymbol{r} = \boldsymbol{b} + u_1\boldsymbol{x_1} + u_2\boldsymbol{x_2} + u_3\boldsymbol{x_3}, \tag{30}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{b}; \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}\}$ and $\boldsymbol{b}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas (28), we can express the first derivative of the equation (30) wrt. the arc length σ with the equation below.

$$\begin{aligned} \frac{d\boldsymbol{r}^*}{d\sigma} &= \Sigma_1\boldsymbol{x_1} + \Sigma_2\boldsymbol{x_2} + \Sigma_3\boldsymbol{x_3} \\ \Sigma_1 &= \frac{du_1}{d\sigma} - u_2 + \frac{k_1'}{{k_1}^3} \\ \Sigma_2 &= \frac{du_2}{d\sigma} + u_1 + u_3\frac{k_2}{k_1} \\ \Sigma_3 &= \frac{du_3}{d\sigma} - u_2\frac{k_2}{k_1} \end{aligned} \tag{31}$$

On the other hand, if the point P is a fixed point in $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, then it will be true that $\frac{d\boldsymbol{r}^*}{d\sigma} = 0$. Hence, we can write the equation (31) as

$$\begin{aligned} \Sigma_1 &= \frac{du_1}{d\sigma} - u_2 + \frac{k_1'}{{k_1}^3} = 0 \\ \Sigma_2 &= \frac{du_2}{d\sigma} + u_1 + u_3\frac{k_2}{k_1} = 0 \\ \Sigma_3 &= \frac{du_3}{d\sigma} - u_2\frac{k_2}{k_1} = 0. \end{aligned} \tag{32}$$

For a curve that adjoint to a ruled surface, the equations in (32) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{M_1}$.

3.1.3. The Generator is $\boldsymbol{M_2}$

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{M_2}$ of the curve $\boldsymbol{a}(s)$ according to the arc length parameter s with

$$\boldsymbol{R}(s,t) = \boldsymbol{a}(s) + t\boldsymbol{M_2}. \tag{33}$$

Also, since $\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{M_2}}{ds} = -k_2 \neq 0$ according to Theorem 1, the directrix curve $\boldsymbol{a}(s)$ can not be taken as a striction curve. Consequently, the striction curve can be obtained as

$$\boldsymbol{b}(s) = \boldsymbol{a}(s) - \frac{\left(\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{M_2}}{ds}\right)}{\left(\frac{d\boldsymbol{M_2}}{ds}\right)^2}\boldsymbol{M_2}(s) \quad \boldsymbol{a}(s) + \frac{1}{k_2}\boldsymbol{M_2}(s). \tag{34}$$

Then, the ruled surface in (33) can be rewritten as

$$\boldsymbol{R}(s,t) = \boldsymbol{b}(s) + t\boldsymbol{M_2}. \tag{35}$$

Therefore, we can express the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ as

$$\boldsymbol{x_1} = \boldsymbol{M_2}, \quad \boldsymbol{x_2} = -\boldsymbol{T}, \quad \boldsymbol{x_3} = \boldsymbol{x_1} \times \boldsymbol{x_2} = -\boldsymbol{M_1}\,. \tag{36}$$

The relationship between σ and s is $\frac{d\sigma}{ds} = k_2$. The differential formulas of the generator trihedron are

$$\begin{cases} \dfrac{d\boldsymbol{b}}{d\sigma} = \lambda \boldsymbol{x_1} \\ \dfrac{d\boldsymbol{x_1}}{d\sigma} = \boldsymbol{x_2} \\ \dfrac{d\boldsymbol{x_2}}{d\sigma} = -\boldsymbol{x_1} + \rho \boldsymbol{x_3} \\ \dfrac{d\boldsymbol{x_3}}{d\sigma} = -\rho \boldsymbol{x_2} \end{cases} \tag{37}$$

in which the kinematic invariants are expressed as follows:

$$\lambda = -\frac{k_2'}{{k_2}^3}, \quad \eta = 0, \quad and \quad \rho = \frac{k_1}{k_2}. \tag{38}$$

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be written as

$$\boldsymbol{r}^* = \boldsymbol{b} + \boldsymbol{r} = \boldsymbol{b} + u_1 \boldsymbol{x_1} + u_2 \boldsymbol{x_2} + u_3 \boldsymbol{x_3}, \tag{39}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{b}; \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}\}$ and $\boldsymbol{b}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas (37), we can express the first derivative of the equation (39) wrt. the arc length σ with the following equation.

$$\begin{aligned} \frac{d\boldsymbol{r}^*}{d\sigma} &= \Sigma_1 \boldsymbol{x_1} + \Sigma_2 \boldsymbol{x_2} + \Sigma_3 \boldsymbol{x_3} \\ \Sigma_1 &= \frac{du_1}{d\sigma} - u_2 - \frac{k_2'}{{k_2}^3} \\ \Sigma_2 &= \frac{du_2}{d\sigma} + u_1 - u_3 \frac{k_1}{k_2} \\ \Sigma_3 &= \frac{du_3}{d\sigma} + u_2 \frac{k_1}{k_2} \end{aligned} \tag{40}$$

If the point P is a fixed point in $\{O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, then it will be true that $\frac{dr^*}{d\sigma} = 0$. Hence, we can write the equation (40) as

$$\begin{aligned} \Sigma_1 &= \frac{du_1}{d\sigma} - u_2 - \frac{k_2'}{{k_2}^3} = 0 \\ \Sigma_2 &= \frac{du_2}{d\sigma} + u_1 - u_3 \frac{k_1}{k_2} = 0 \\ \Sigma_3 &= \frac{du_3}{d\sigma} + u_2 \frac{k_1}{k_2} = 0. \end{aligned} \tag{41}$$

For a curve that adjoint to a ruled surface, the equations in (41) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{M_2}$.

3.2. Type-II Bishop Frame $\{\boldsymbol{a}; \boldsymbol{N_1}, \boldsymbol{N_2}, \mathbf{B}\}$

3.2.1. The Generator is $\boldsymbol{N_1}$

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector N_1 of the curve $\boldsymbol{a}(s)$ according to the arc length parameter s with

$$\boldsymbol{R}(s,t) = \boldsymbol{a}(s) + tN_1(s). \tag{42}$$

On the other hand, since $\frac{d\boldsymbol{a}}{ds} . \frac{dN_1}{ds} = \boldsymbol{T}.(-\mu_1 \boldsymbol{B}) = 0$ according to Theorem 1, the directrix curve $\boldsymbol{a}(s)$ can be taken as a striction curve. Therefore, we can express the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ as

$$\boldsymbol{x_1} = \boldsymbol{N_1}, \quad \boldsymbol{x_2} = -\boldsymbol{B}, \quad \boldsymbol{x_3} = \boldsymbol{x_1} \times \boldsymbol{x_2} = \boldsymbol{N_2}\,. \tag{43}$$

The relationship between σ and s is $\frac{d\sigma}{ds} = \mu_1$. The differential formulas of the generator trihedron are given below:

$$\begin{cases} \dfrac{d\boldsymbol{a}}{d\sigma} = \lambda \boldsymbol{x_1} + \eta \boldsymbol{x_3} \\ \dfrac{d\boldsymbol{x_1}}{d\sigma} = \boldsymbol{x_2} \\ \dfrac{d\boldsymbol{x_2}}{d\sigma} = -\boldsymbol{x_1} + \rho \boldsymbol{x_3} \\ \dfrac{d\boldsymbol{x_3}}{d\sigma} = -\rho \boldsymbol{x_2} \end{cases} \tag{44}$$

in which the kinematic invariants can be written as follows:

$$\lambda = \frac{1}{\mu_1} \sin\phi\,, \quad \eta = -\frac{1}{\mu_1} \cos\phi\,, \quad \rho = -\frac{\mu_2}{\mu_1}. \tag{45}$$

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be expressed with

$$\boldsymbol{r}^* = \boldsymbol{a} + \boldsymbol{r} = \boldsymbol{a} + u_1 \boldsymbol{x_1} + u_2 \boldsymbol{x_2} + u_3 \boldsymbol{x_3}, \tag{46}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{a}; \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}\}$ and $\boldsymbol{a}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas (44), we can express the first derivative of the equation (46) wrt. the arc length σ as follows:

$$\begin{aligned}\frac{d\boldsymbol{r}^*}{d\sigma} &= \Psi_1\boldsymbol{x_1}+\Psi_2\boldsymbol{x_2}+\Psi_3\boldsymbol{x_3}\\ \Psi_1 &= \frac{du_1}{d\sigma}-u_2+\frac{1}{\mu_1}\sin\phi\\ \Psi_2 &= \frac{du_2}{d\sigma}+u_1-u_3\frac{\mu_2}{\mu_1}\\ \Psi_3 &= \frac{du_3}{d\sigma}-u_2\frac{\mu_2}{\mu_1}-\frac{1}{\mu_1}\cos\phi\end{aligned} \tag{47}$$

However, if the point P is a fixed point in $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, then it will be true that $\frac{dr^*}{d\sigma}=0$. Hence, we can write the equation (47) as

$$\begin{aligned}\Psi_1 &= \frac{du_1}{d\sigma}-u_2+\frac{1}{\mu_1}\sin\phi &= 0\\ \Psi_2 &= \frac{du_2}{d\sigma}+u_1-u_3\frac{\mu_2}{\mu_1} &= 0\\ \Psi_3 &= \frac{du_3}{d\sigma}-u_2\frac{\mu_2}{\mu_1}-\frac{1}{\mu_1}\cos\phi &= 0.\end{aligned} \tag{48}$$

For a curve that adjoint to a ruled surface, the equations in (48) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{N_1}$.

3.2.2. *The Generator is* $\boldsymbol{N_2}$

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector N_2 of the curve $\boldsymbol{a}(s)$ according to the arc length parameter s with

$$\boldsymbol{R}(s,t)=\boldsymbol{a}(s)+tN_2(s). \tag{49}$$

On the other hand, since $\frac{d\boldsymbol{a}}{ds}.\frac{dN_2}{ds}=\boldsymbol{T}.(-\mu_2\boldsymbol{B})=0$ according to Theorem 1, the directrix curve $\boldsymbol{a}(s)$ can be taken as a striction curve. Therefore, we can express the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ as

$$\boldsymbol{x_1}=\boldsymbol{N_2},\quad \boldsymbol{x_2}-\boldsymbol{B},\quad \boldsymbol{x_3}=\boldsymbol{x_1}\times\boldsymbol{x_2}=-\boldsymbol{N_1}\,. \tag{50}$$

The relationship between σ and s is $\frac{d\sigma}{ds}=\mu_2$. The differential formulas of the generator trihedron are:

$$\begin{cases}\frac{d\boldsymbol{a}}{d\sigma}=\lambda\boldsymbol{x_1}+\eta\boldsymbol{x_3}\\ \frac{d\boldsymbol{x_1}}{d\sigma}=\boldsymbol{x_2}\\ \frac{d\boldsymbol{x_2}}{d\sigma}=-\boldsymbol{x_1}+\rho\boldsymbol{x_3}\\ \frac{d\boldsymbol{x_3}}{d\sigma}=-\rho\boldsymbol{x_2}\end{cases} \tag{51}$$

in which the kinematic invariants are given as follows:

$$\lambda=-\frac{1}{\mu_2}\cos\phi,\quad \eta=-\frac{1}{\mu_2}\sin\phi,\quad \rho=\frac{\mu_1}{\mu_2}. \tag{52}$$

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be written as

$$\boldsymbol{r}^*=\boldsymbol{a}+\boldsymbol{r}=\boldsymbol{a}+u_1\boldsymbol{x_1}+u_2\boldsymbol{x_2}+u_3\boldsymbol{x_3}, \tag{53}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{a};\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$ and $\boldsymbol{a}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas (51), we can express the first derivative of the equation (53) wrt. the arc length σ with the following equation.

$$\begin{aligned}\frac{d\boldsymbol{r}^*}{d\sigma} &= \Psi_1\boldsymbol{x_1}+\Psi_2\boldsymbol{x_2}+\Psi_3\boldsymbol{x_3}\\ \Psi_1 &= \frac{du_1}{d\sigma}-u_2-\frac{1}{\mu_2}\cos\phi\\ \Psi_2 &= \frac{du_2}{d\sigma}+u_1-u_3\frac{\mu_1}{\mu_2}\\ \Psi_3 &= \frac{du_3}{d\sigma}+u_2\frac{\mu_1}{\mu_2}-\frac{1}{\mu_2}\sin\phi\end{aligned} \tag{54}$$

On the other hand, if the point P is a fixed point in $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, then it will be true that $\frac{dr^*}{d\sigma}=0$. Hence, we can write the equation (54) as

$$\begin{aligned}\Psi_1 &= \frac{du_1}{d\sigma}-u_2-\frac{1}{\mu_2}\cos\phi &= 0\\ \Psi_2 &= \frac{du_2}{d\sigma}+u_1-u_3\frac{\mu_1}{\mu_2} &= 0\\ \Psi_3 &= \frac{du_3}{d\sigma}+u_2\frac{\mu_1}{\mu_2}-\frac{1}{\mu_2}\sin\phi &= 0.\end{aligned} \tag{55}$$

For a curve that adjoint to a ruled surface, the equations in (55) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{N_2}$.

3.2.3. The Generator is $\boldsymbol{B}$

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{B}$ of the curve $\boldsymbol{a}(s)$ according to the arc length parameter s by

$$\boldsymbol{R}(s,t) = \boldsymbol{a}(s) + t\boldsymbol{B}(s). \tag{56}$$

Also, since $\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{B}}{ds} = \boldsymbol{T}.(\mu_1\boldsymbol{N_1} + \mu_2\boldsymbol{N_2})$ according to Theorem 1, we can express the striction curve as

$$\boldsymbol{b}(s) = \quad \boldsymbol{a}(s) - \frac{\left(\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{B}}{ds}\right)}{\left(\frac{d\boldsymbol{B}}{ds}\right)^2}\boldsymbol{B}(s).$$

In the above equation, since

$$\frac{\left(\frac{d\boldsymbol{a}}{ds}.\frac{d\boldsymbol{B}}{ds}\right)}{\left(\frac{d\boldsymbol{B}}{ds}\right)^2}\boldsymbol{B}(s) = \frac{\boldsymbol{T}.(\mu_1\boldsymbol{N_1} + \mu_2\boldsymbol{N_2})}{\left(\frac{d\boldsymbol{B}}{ds}\right)^2}\boldsymbol{B}(s) \neq 0,$$

we can not take the curve $\boldsymbol{a}(s)$ as a striction curve. Therefore, the striction curve is

$$\boldsymbol{b}(s) = \boldsymbol{a}(s) - \frac{\mu_1\sin\phi - \mu_2\cos\phi}{\mu_1^2 + \mu_2^2}\boldsymbol{B}(s).$$

On the other hand, the generator trihedron of the ruled surface $\boldsymbol{R}(s,t)$ is

$$\boldsymbol{x_1} = \boldsymbol{B}, \quad \boldsymbol{x_2} = \frac{\mu_1\boldsymbol{N_1} + \mu_2\boldsymbol{N_2}}{\sqrt{\mu_1^2+\mu_2^2}}, \quad \boldsymbol{x_3} = \frac{\mu_1\boldsymbol{N_2} - \mu_2\boldsymbol{N_1}}{\sqrt{\mu_1^2+\mu_2^2}}. \tag{57}$$

The relationship between σ and s is $\frac{d\sigma}{ds} = \sqrt{\mu_1^2+\mu_2^2}$. The differential formulas of the generator trihedron are:

$$\begin{cases} \frac{d\boldsymbol{a}}{d\sigma} = \lambda\boldsymbol{x_1} \\ \frac{d\boldsymbol{x_1}}{d\sigma} = \boldsymbol{x_2} \\ \frac{d\boldsymbol{x_2}}{d\sigma} = -\boldsymbol{x_1} + \rho\boldsymbol{x_3} \\ \frac{d\boldsymbol{x_3}}{d\sigma} = -\rho\boldsymbol{x_2} \end{cases} \tag{58}$$

in which the kinematic invariants are given as follows:

$$\begin{aligned} \lambda = & \frac{(\mu_1' + \mu_2)\sin\phi + (\mu_1 - \mu_2')\cos\phi}{(\mu_1^2+\mu_2^2)^{3/2}} \\ & - \frac{2(\mu_1^2\mu_1'\sin\phi - \mu_2^2\mu_2'\cos\phi)}{(\mu_1^2+\mu_2^2)^{5/2}}, \\ \eta = 0 & \\ \rho = & \frac{\mu_1\mu_2' - \mu_2\mu_1'}{(\mu_1^2+\mu_2^2)^{3/2}} \end{aligned} \tag{59}$$

Remark 2: It should be noted that the kinematic invariants have a relationship with the torsion of the directrix curve $\boldsymbol{a}(s)$, since $\sqrt{\mu_1^2+\mu_2^2} = \tau$.

The adjoint curve $\boldsymbol{r}^*$ of the ruled surface $\boldsymbol{R}(s,t)$ can be written as

$$\boldsymbol{r}^* = \boldsymbol{a} + \boldsymbol{r} = \boldsymbol{a} + u_1\boldsymbol{x_1} + u_2\boldsymbol{x_2} + u_3\boldsymbol{x_3}, \tag{60}$$

where u_1, u_2, and u_3 are the coordinates of the point P in $\{\boldsymbol{a};\boldsymbol{x_1},\boldsymbol{x_2},\boldsymbol{x_3}\}$ and $\boldsymbol{a}$ is the striction curve of $\boldsymbol{R}(s,t)$. Based on the generator trihedron formulas (58), we can express the first derivative of the equation (60) wrt. the arc length σ with the following equation.

$$\begin{aligned} \frac{d\boldsymbol{r}^*}{d\sigma} &= \Psi_1\boldsymbol{x_1} + \Psi_2\boldsymbol{x_2} + \Psi_3\boldsymbol{x_3} \\ \Psi_1 &= \frac{du_1}{d\sigma} - u_2 \\ \Psi_2 &= \frac{du_2}{d\sigma} + u_1 - u_3\frac{\mu_1\mu_2' - \mu_2\mu_1'}{(\mu_1^2+\mu_2^2)^{3/2}} \\ \Psi_3 &= \frac{du_3}{d\sigma} + u_2\frac{\mu_1\mu_2' - \mu_2\mu_1'}{(\mu_1^2+\mu_2^2)^{3/2}} + \frac{\mu_1\sin\phi + \mu_2\cos\phi}{\mu_1^2+\mu_2^2} \end{aligned} \tag{61}$$

If the point P is a fixed point in $\{O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$, then it will be true that $\frac{d\boldsymbol{r}^*}{d\sigma} = 0$. Hence, we can write the equation (54) as

$$\begin{aligned} \Psi_1 &= \frac{du_1}{d\sigma} - u_2 &= 0 \\ \Psi_2 &= \frac{du_2}{d\sigma} + u_1 - u_3\frac{\mu_1\mu_2' - \mu_2\mu_1'}{(\mu_1^2+\mu_2^2)^{3/2}} &= 0 \\ \Psi_3 &= \frac{du_3}{d\sigma} + u_2\frac{\mu_1\mu_2' - \mu_2\mu_1'}{(\mu_1^2+\mu_2^2)^{3/2}} + \frac{\mu_1\sin\phi + \mu_2\cos\phi}{\mu_1^2+\mu_2^2} &= 0. \end{aligned} \tag{62}$$

For a curve that adjoint to a ruled surface, the equations in (62) can be called as the fixed point conditions according to the Type-1 Bishop frame with the generator $\boldsymbol{B}$.

4. Examples

In this section, first, we present the Type-1 and Type 2 curvatures, and kinematic invariants and thereafter illustrate four examples of ruled surfaces and their adjoint curves.

Let us take a unit speed curve in E^3 by

$$\boldsymbol{a}(s) = \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right), \quad \|\boldsymbol{a}'(s)\| = 1 \tag{63}$$

The curve $\boldsymbol{a} = \boldsymbol{a}(s)$ can be seen in Figure (3).

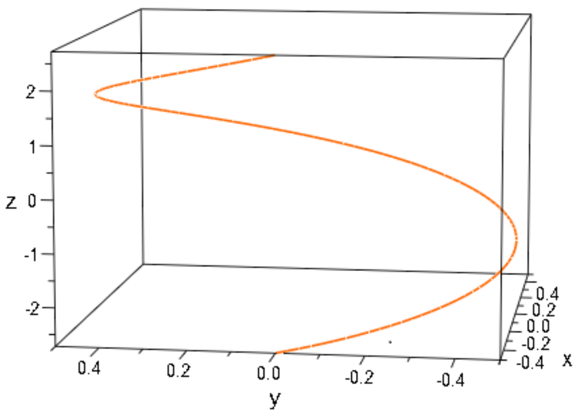

Figure 3: Directrix curve $\alpha = \alpha(s)$.

One can calculate its Frenet–Serret frame components as follows:

$$\begin{aligned} \boldsymbol{T} &= \left(-\tfrac{1}{2}\sin s, \tfrac{1}{2}\cos s, \tfrac{\sqrt{3}}{2}\right) & \\ \boldsymbol{N} &= (-\cos s, -\sin s, 0) & \kappa = \tfrac{1}{2} \\ \boldsymbol{B} &= \left(\tfrac{\sqrt{3}}{2}\sin s, -\tfrac{\sqrt{3}}{2}\cos s, \tfrac{1}{2}\right) & \tau = \tfrac{\sqrt{3}}{2}. \end{aligned} \quad (64)$$

Next, we can find the Type-1 Bishop trihedra of $\boldsymbol{\alpha}(s)$ by using the equations (4) and $\varphi(s) = \int_0^s \frac{\sqrt{3}}{2} ds = \frac{\sqrt{3}}{2} s$ as:

$$\boldsymbol{T} = \left(-\frac{1}{2}\sin s, \frac{1}{2}\cos s, \frac{\sqrt{3}}{2}\right) \quad (65)$$

$$\begin{aligned} \boldsymbol{M_1} = \Big(&-\cos\frac{\sqrt{3}}{2}s.\cos s - \frac{\sqrt{3}}{4}\sin\frac{\sqrt{3}}{2}s.\sin s, \\ &-\cos\frac{\sqrt{3}}{2}s.\sin s + \frac{\sqrt{3}}{4}\sin\frac{\sqrt{3}}{2}s.\cos s, \\ &-\frac{1}{2}\sin s\Big) \end{aligned} \quad (66)$$

$$\begin{aligned} \boldsymbol{M_2} = \Big(&-\sin\frac{\sqrt{3}}{2}s.\cos s + \frac{\sqrt{3}}{4}\cos\frac{\sqrt{3}}{2}s.\sin s, \\ &-\sin\frac{\sqrt{3}}{2}s.\sin s - \frac{\sqrt{3}}{4}\cos\frac{\sqrt{3}}{2}s.\cos s, \\ &\frac{1}{2}\cos s\Big) \end{aligned} \quad (67)$$

Also, the Type-1 Bishop curvatures can be determined with

$$k_1 = \frac{1}{2}\cos\frac{\sqrt{3}}{2}s \quad and \quad k_2 = \frac{1}{2}\sin\frac{\sqrt{3}}{2}s \quad (68)$$

Moreover, we can find the Type-2 Bishop trihedra of $\boldsymbol{\alpha}(s)$ by using the equations (6) and $\phi(s) = \int_0^s \frac{1}{2} ds = \frac{1}{2} s$ as:

$$\begin{aligned} \boldsymbol{N_1} = \Big(&-\cos\frac{1}{2}s.\cos s - \frac{1}{2}\sin\frac{1}{2}s.\sin s, \\ &-\cos\frac{1}{2}s.\sin s + \frac{1}{2}\sin\frac{1}{2}s.\cos s, \\ &\frac{\sqrt{3}}{2}\sin\frac{1}{2}s\Big) \end{aligned} \quad (69)$$

$$\begin{aligned} \boldsymbol{N_2} = \Big(&\tfrac{1}{2}\cos\tfrac{1}{2}s.\sin s - \sin\tfrac{1}{2}s.\cos s, \\ &-\tfrac{1}{2}\cos\tfrac{1}{2}s.\cos s + \sin\tfrac{1}{2}s.\sin s, \\ &-\tfrac{\sqrt{3}}{2}\cos\tfrac{1}{2}s\Big) \end{aligned} \quad (70)$$

$$\boldsymbol{B} = \left(\frac{\sqrt{3}}{2}\sin s, -\frac{\sqrt{3}}{2}\cos s, \frac{1}{2}\right). \quad (71)$$

Also, the Type-2 Bishop curvatures can be determined with

$$\mu_1 = -\frac{\sqrt{3}}{2}\cos\frac{1}{2}s, \quad \mu_2 = -\frac{\sqrt{3}}{2}\sin\frac{1}{2}s. \quad (72)$$

4.1. Example 1:

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit tangent vector $\boldsymbol{T}$ in (65) the curve $\boldsymbol{\alpha}(s)$ in (63) according to the arc length parameter s as follows:

$$\begin{aligned} \boldsymbol{R}(s,t) = &\left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right) \\ &+t\left(-\frac{1}{2}\sin s, \frac{1}{2}\cos s, \frac{\sqrt{3}}{2}\right). \end{aligned} \quad (73)$$

Additionally, the kinematic invariants are $\lambda = 2$, $\eta = 0$ and $\rho = \sqrt{3}$. This ruled surface' adjoint curve can be written by using the equations (65), (66), and (67) as

$$\begin{aligned} \boldsymbol{r}^* &= \boldsymbol{\alpha} + \boldsymbol{r} \\ &= \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right) + u_1\boldsymbol{T} \\ &\quad + \frac{(u_2+u_3)k_1}{\sqrt{{k_1}^2+{k_2}^2}}\boldsymbol{M_1} + \frac{(u_2-u_3)k_2}{\sqrt{{k_1}^2+{k_2}^2}}\boldsymbol{M_2}. \end{aligned} \quad (74)$$

One can see the relationship between this ruled surface in (73) and its adjoint curve below from Figure (4).

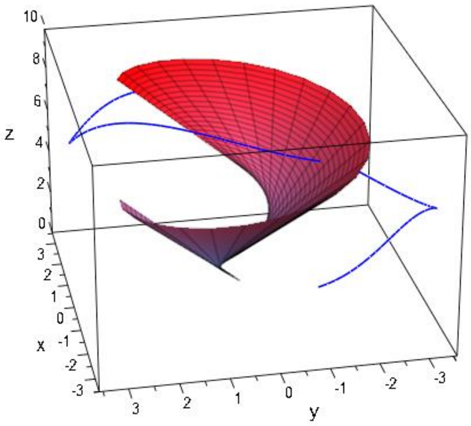

Figure 4: The ruled surface $R(s,t)$ in (73) with the generator T and its adjoint curve in (74).

4.2. Example 2:

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{M_1}$ in (66) the curve $\boldsymbol{a}(s)$ in (63) according to the arc length parameter s as follows:

$$\begin{aligned} \boldsymbol{R}(s,t) \;=\; & \left(\frac{1}{2}\cos s\,,\frac{1}{2}\sin s\,,\frac{\sqrt{3}}{2}s\right) \\ & +t\left(-\cos\frac{\sqrt{3}}{2}s\,.\cos s-\frac{\sqrt{3}}{4}\sin\frac{\sqrt{3}}{2}s.\sin s\,,\right. \\ & -\cos\frac{\sqrt{3}}{2}s\,.\sin s+\frac{\sqrt{3}}{4}\sin\frac{\sqrt{3}}{2}s.\cos s\,, \\ & \left.-\frac{1}{2}\sin s\right). \end{aligned} \tag{75}$$

Additionally, the kinematic invariants are $\lambda=-2\sqrt{3}\frac{\sin\frac{\sqrt{3}}{2}s}{\left(\cos\frac{\sqrt{3}}{2}s\right)^3}$, $\eta=0$ and $\rho=-\tan\frac{\sqrt{3}}{2}s$. This ruled surface's adjoint curve can be written by using the equations (65), (66), and (67) as

$$\begin{aligned} \boldsymbol{r}^* \;=\; & \boldsymbol{b}+\boldsymbol{r} \\ =\; & \left(\frac{1}{2}\cos s\,,\frac{1}{2}\sin s\,,\frac{\sqrt{3}}{2}s\right)-u_2\boldsymbol{T} \\ & +\left(u_1+2\frac{1}{\cos\frac{\sqrt{3}}{2}s}\right)\boldsymbol{M_1}+u_3\boldsymbol{M_2}. \end{aligned} \tag{76}$$

One can see the relationship between this ruled surface in (75) and its adjoint curve (76) from Figure (5).

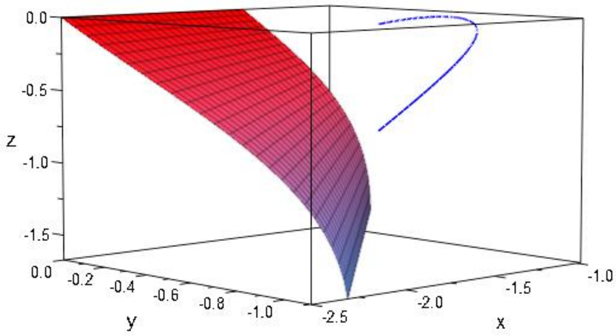

Figure 5: The ruled surface $R(s,t)$ in (75) with the generator M_1 and its adjoint curve in (76).

4.3. Example 3:

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{N_1}$ in (69) the curve $\boldsymbol{a}(s)$ in (63) according to the arc length parameter s as follows:

$$\begin{aligned} \boldsymbol{R}(s,t) \;=\; & \left(\frac{1}{2}\cos s\,,\frac{1}{2}\sin s\,,\frac{\sqrt{3}}{2}s\right) \\ & +t\left(-\cos\frac{1}{2}s\,.\cos s-\frac{1}{2}\sin\frac{1}{2}s.\sin s\,,\right. \\ & -\cos\frac{1}{2}s\,.\sin s+\frac{1}{2}\sin\frac{1}{2}s.\cos s\,, \\ & \left.\frac{\sqrt{3}}{2}\sin\frac{1}{2}s\right). \end{aligned} \tag{77}$$

Additionally, the kinematic invariants are $\lambda=-\frac{2}{\sqrt{3}}\tan\frac{1}{2}s$, $\eta=\frac{2}{\sqrt{3}}$ and $\rho=-\tan\frac{1}{2}s$. This ruled surface' adjoint curve can be written by using the equations (69), (70), and (71) as

$$\begin{aligned} \boldsymbol{r}^* \;=\; & \boldsymbol{a}+\boldsymbol{r} \\ =\; & \left(\frac{1}{2}\cos s\,,\frac{1}{2}\sin s\,,\frac{\sqrt{3}}{2}s\right)+u_1\boldsymbol{N_1} \\ & -u_2\boldsymbol{N_2}+u_3\boldsymbol{B}. \end{aligned} \tag{78}$$

One can see the relationship between this ruled surface in (77) and its adjoint curve (78) from Figure (6).

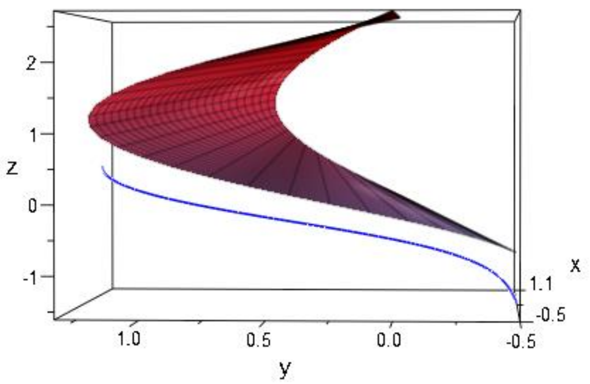

Figure 6: The ruled surface $R(s,t)$ in (77) with the generator N_1 and its adjoint curve in (78).

4.4. Example 4:

The ruled surface $\boldsymbol{R}(s,t)$ in (16) can be written by taking the generator as the unit vector $\boldsymbol{N_2}$ in (70) the curve $\boldsymbol{a}(s)$ in (63) according to the arc length parameter s as follows:

$$\begin{aligned}\boldsymbol{R}(s,t) &= \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right) \\ &+t\left(\frac{1}{2}\cos\frac{1}{2}s.\sin s - \sin\frac{1}{2}s.\cos s,\right. \\ &-\frac{1}{2}\cos\frac{1}{2}s.\cos s + \sin\frac{1}{2}s.\sin s, \\ &\left.-\frac{\sqrt{3}}{2}\cos\frac{1}{2}s\right).\end{aligned} \tag{79}$$

Additionally, the kinematic invariants are $\lambda = \frac{2}{\sqrt{3}}\cot\frac{1}{2}s$, $\eta = \frac{2}{\sqrt{3}}$ and $\rho = \cot\frac{1}{2}s$. This ruled surface' adjoint curve can be written by using the equations (69), (70), and (71) as

$$\begin{aligned}\boldsymbol{r}^* &= \boldsymbol{a} + \boldsymbol{r} \\ &= \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{\sqrt{3}}{2}s\right) + u_1\boldsymbol{N_2} \\ &-u_2\boldsymbol{B} - u_3\boldsymbol{N_1}.\end{aligned} \tag{80}$$

One can see the relationship between this ruled surface in (79) and its adjoint curve (80) from Figure (7).

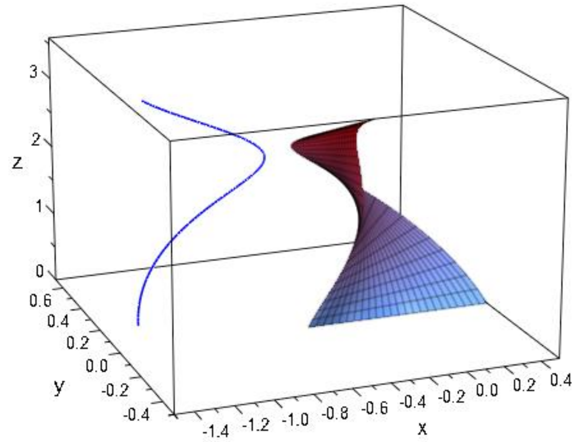

Figure 7: The ruled surface $R(s,t)$ in (79) with the generator N_2 and its adjoint curve in (80).

Remark 3: The ruled surface $\boldsymbol{R}(s,t)$ in (16) with the generators $\boldsymbol{M_2}$ and $\boldsymbol{B}$ with their adjoint curves can be expressed similarly.

5. Conclusion

In this paper, we examined the adjoint approach between a curve and a ruled surface based on Type-1 and Type-2 Bishop frames. First of all, we expressed the ruled surface and its generator trihedron by using the components of the Type-1 and Type-2 Bishop frames. Also, we determined the fixed point conditions for a curve adjoint to a ruled surface according to the Bishop frame.

6. References

Bishop, L.R. (1975). There is a more than one way to frame a curve, Amer. Math. Monthly. Vol 82, Issue 3, 246-251.

Hanson, A. J. and Ma, H.H. (1995). Parallel Transport Approach to Curve Framing, Tech. Math. Rep. 425, Indiana University Computer Science Department.

Gray, A. (1996). Modern Differential Geometry of Curves and Surfaces with Mathematica, CRC Press, Inc.

Lipschutz, M. (1969). Schaum's Outline of Differential Geometry, Schaum's Outlines.

Lucas, P., Ortega-Yagües, J.A. (2012). Bertrand curves in the three-dimensional sphere, Journal of Geometry and Physics, 62, 1903-1914.

Sasaki, S. (1955). Differential Geometry (in Japanese), Kyolitsu Press.

Wang, D. and Xiao, D.Z. (1993). Distribution of coupler curves for crank-rocker linkages, Mechanism and Machine Theory, 28, 671-684.

Wang, D., Liu, J. and Xiao, D.Z. (1997). Kinematic differential geometry of a rigid body in spatial motion-I. A new adjoint approach and instantaneous properties of a point trajectory in spatial kinematics, Mechanism and Machine Theory, 32, 419-432.

Wang, D., Liu, J. and Xiao, D.Z. (1997). Kinematic differential geometry of a rigid body in spatial motion-III. Distribution of characteristic lines in the moving body in spatial motion, Mechanism and Machine Theory, 32, 445-457.

Goetz, A. (1970). Introduction to differential geometry, Addison-Wesley series in mathematics, Intermediate Mathematics Series; Addison Wesley Pub. Co.

Wang, D. and Wang, W. (2015). Kinematic Differential Geometry and Saddle Synthesis of Linkages, Wiley, 1 edition.

Bulut, V. (2018). A Spatial Curve Adjoining Another Spatial Curve Based on Bishop Frame, International Conference on Mathematical Advances and Applications (ICOMAA2018), Istanbul.