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Some Notes on Geodesics of Vertical Rescaled Berger Deformation Metric in Tangent Bundle

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Abstract. In this paper, we study the geodesics on the tangent bundle TM with respect to the vertical rescaled Berger deformation metric over an anti-paraKähler manifold (M, φ, g). In this case, we establish the necessary and sufficient conditions under which a curve be geodesic with respect to this. Finally, we also present certain examples of geodesic.

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1. Introduction

On the tangent bundle of a Riemannian manifold one can define natural Riemannian metrics. Their construction make use of the Levi-Civita parallelization. Among them, the so called Sasaki metric [9] is of particular interest. That is why the geometry of tangent bundle equipped with the Sasaki metric has been studied by many authors such as Yano, K., Ishihara, S. [13], Dombrowski, P. [3], Salimov, A. A., Gezer, A., Akbulut, K. [6] etc. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on tangent bundle. This is the reason why they have attempted to search for different metrics on the tangent bundle which are different deformations of the Sasaki metric. Musso, E., Tricerri, F. has introduced the notion of Cheeger-Gromoll metric [5], this metric has been studied also by many authors (see [4, 8, 10]). In this direction, Yampolsky, A. [11] propose a Berger-type deformed Sasaki metric on tangent bundle over a Kählerian manifold, which was studied by Altunbas, M. and collaborators in [1]. The study of the Berger-type deformed Sasaki metric on the tangent bundle or on the cotangent are not limited to those mentioned above. We also refer to new studies by Zagane, A. among which we [15–20]. .

In previous work [19], we proposed a “Vertical rescaled Berger deformation metric on the tangent bundle”. Also in [20], we presented “A study of harmonic sections of tangent bundles with vertically rescaled Berger-type deformed Sasaki metric”, and as a supplement to these works, in this paper, we give some geodesic properties for the Vertical rescaled Berger deformation metric, then we establish necessary and sufficient conditions under which a curve be a geodesic with respect to this metric (Theorem 4.5 and Theorem 4.6). As well as, when it is the natural lift and horizontal lift is geodesic (Corollary 4.7 and Corollary 4.8). Finally, we also mention special cases (Corollary 4.13, Theorem 4.14, Corollary 4.15, Theorem 4.16 and Corollary 4.17).

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2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im^1_0(M)$ be the module over $C^\infty(M)$ of C^∞ vector fields on M.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned}\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad \xi^i \in \mathbb{R}\},\\ \mathcal{H}_{(x,u)} &= \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)}; \quad \xi^i \in \mathbb{R}\},\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Note that the map $X \to X^H$ is an isomorphism between the vector spaces T_xM and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^V$ is an isomorphism between the vector spaces T_xM and $\mathcal{V}_{(x,u)}$. Obviously, each tangent vector $Z \in T_{(x,u)}TM$ can be written in the form $Z = X^H + Y^V$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$\begin{aligned}X^V &= X^i \frac{\partial}{\partial y^i},\\ X^H &= X^i \frac{\delta}{\delta x^i} = X^i\{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}.\end{aligned}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

If U is a local vector field constant on each fiber T_xM, i.e ($U = u = u^i \frac{\partial}{\partial x^i}$), the vertical lift U^V is called the canonical vertical vector field or Liouville vector field on TM.

Lemma 2.1 ([3, 13]). *Let (M^m, g) be a Riemannian manifold. The bracket operation of vertical and horizontal vector fields is given by the formulas*

(1) $[X^H, Y^H] = [X, Y]^H - (R(X, Y)u)^V$,
(2) $[X^H, Y^V] = (\nabla_X Y)^V$,
(3) $[X^V, Y^V] = 0$,

for all vector fields $X, Y \in \Im^1_0(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^m, g).

3. Vertical Rescaled Berger Deformation Metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X, Y),$$

or equivalently (purity condition), (B-metric) [7]

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for all $X, Y \in \Im^1_0(M)$.

If (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M^{2m}, φ, g) is said to be an almost anti-paraHermitian manifold (an almost B-manifold) [7]. Moreover, (M^{2m}, φ, g) is said to be anti-paraKähler manifold (B-manifold) [7] if φ is parallel with respect to the Levi-Civita connection ∇ of g i.e ($\nabla\varphi = 0$).

As is well known, the anti-paraKähler condition ($\nabla\varphi = 0$) is equivalent to paraholomorphicity of the anti-paraHermitian metric g, that is, $(\phi_\varphi g) = 0$, where ϕ_φ is the Tachibana operator [12].

It is well known that, if (M^{2m}, φ, g) is an anti-paraKähler manifold, the Riemannian curvature tensor is pure [7], and

$$\begin{cases} R(\varphi Y, Z) & = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) & = R(Y, Z), \end{cases}$$

for all $Y, Z \in \Im_0^1(M)$.

Definition 3.1 ([19]). Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. Define a vertical rescaled Berger deformation metric noted $\tilde{g}$ on TM, by

$$\begin{aligned} \tilde{g}(X^H, Y^H) &= g(X, Y), \\ \tilde{g}(X^H, Y^V) &= 0, \\ \tilde{g}(X^V, Y^V) &= f[g(X, Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)], \end{aligned}$$

for all $X, Y \in \Im_0^1(M)$, where δ is some constant [1, 11] and f is called twisting function.

Remark 3.2.

1. If $f = 1$ and $\delta = 0$, $\tilde{g}$ is the Sasaki metric [9],
2. If $f = 1$, $\tilde{g}$ is the Berger-type deformed Sasaki metric [1],
3. If $\delta = 0$, $\tilde{g}$ is the vertical rescaled metric [2],
4. $\tilde{g}(X^V, \varphi U^V) = (1 + \delta^2 r^2) f g(X, \varphi u)$ and $r^2 = g(u, u)$, for any $X \in \Im_0^1(M)$.

In the following, we consider $\lambda = 1 + \delta^2 r^2$ and $r^2 = g(u, u)$.

Theorem 3.3 ([19]). *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric, then we have the following formulas.*

$$\begin{aligned} 1.\, \widetilde{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H - \frac{1}{2}(R(X, Y)u)^V, \\ 2.\, \widetilde{\nabla}_{X^H} Y^V &= (\nabla_X Y)^V + \frac{1}{2f} X(f) Y^V + \frac{f}{2}(R(u, Y)X)^H, \\ 3.\, \widetilde{\nabla}_{X^V} Y^H &= \frac{1}{2f} Y(f) X^V + \frac{f}{2}(R(u, X)Y)^H, \\ 4.\, \widetilde{\nabla}_{X^V} Y^V &= \frac{-1}{2f}\tilde{g}(X^V, Y^V)(grad\, f)^H + \frac{\delta^2}{\lambda} g(X, \varphi Y)(\varphi U)^V, \end{aligned}$$

for all vector fields $X, Y \in \Im_0^1(M)$, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^{2m}, φ, g).

4. Geodesics of Vertical Rescaled Berger Deformation Metric

Let (M^m, g) be a Riemannian manifold and $x : I \to M$ be a curve on M. We define a curve $C : I \to TM$ by for all $t \in I, C(t) = (x(t), y(t))$ where $y(t) \in T_{x(t)}M$ i.e. $y(t)$ is a vector field along $x(t)$.

Definition 4.1 ([13]). Let (M^m, g) be a Riemannian manifold and $x(t)$ be a curve on M. The curve $C(t) = (x(t), \dot{x}(t))$ is called the natural lift of curve $x(t)$.

Definition 4.2 ([13]). Let (M^m, g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M^m, g). A curve $C(t) = (x(t), y(t))$ is said to be a horizontal lift of the curve $x(t)$ if and only if $\nabla_{\dot{x}} y = 0$.

Lemma 4.3 ([14]). *Let (M^m, g) be a Riemannian manifold. If $X, Y \in \Im_0^1(M)$ are vector fields on M and $(x, u) \in TM$ such that $Y_x = u$, then we have:*

$$d_x Y(X_x) = X^H_{(x,u)} + (\nabla_X Y)^V_{(x,u)}.$$

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, y^j)$ be the induced chart on TM, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x = u$, then

$$\begin{aligned} d_xY(X_x) &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} + X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)} \\ &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,u)} - X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)} + X^i(x)Y^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,u)} + X^i(x)\frac{\partial Y^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,u)} \\ &= [X^i(\frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij})\frac{\partial}{\partial y^k}]_{(x,u)} + [X^i(\frac{\partial Y^k}{\partial x^i} + Y^j\Gamma^k_{ij})\frac{\partial}{\partial y^k}]_{(x,u)} \\ &= X^H_{(x,u)} + [X^i(\frac{\partial Y^k}{\partial x^i} + Y^j\Gamma^k_{ij})\frac{\partial}{\partial x^k}]^V_{(x,u)} \\ &= X^H_{(x,u)} + (\nabla_X Y)^V_{(x,u)}. \end{aligned}$$

□

Lemma 4.4 ([14]). *Let (M^m, g) be a Riemannian manifold and ∇ denotes the Levi-Civita connection of (M^m, g). If $x(t)$ is a curve on M and $C(t) = (x(t), y(t))$ is a curve on TM, then*

$$\dot{C} = \dot{x}^H + (\nabla_{\dot{x}}y)^V.$$

Proof. Locally, if $Y \in \Gamma(TM)$ is a vector field such $Y(x(t)) = y(t)$, then we have

$$\dot{C}(t) = dC(t) = dY(x(t)).$$

Using Lemma 4.3, we obtain

$$\dot{C}(t) = dY(x(t)) = \dot{x}^H + (\nabla_{\dot{x}}y)^V.$$

□

Theorem 4.5. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. If $C(t) = (x(t), y(t))$ is curve on TM, then*

$$\begin{aligned} \widetilde{\nabla}_{\dot{C}}\dot{C} &= [\nabla_{\dot{x}}\dot{x} + fR(y, \nabla_{\dot{x}}y)\dot{x} - \frac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2)grad\, f]^H \\ &\quad +[\nabla_{\dot{x}}\nabla_{\dot{x}}y + \frac{1}{f}\dot{x}(f)\nabla_{\dot{x}}y + \frac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y]^V. \end{aligned}$$

Proof. Using Lemma 4.4 we obtain

$$\begin{aligned} \widetilde{\nabla}_{\dot{C}}\dot{C} &= \widetilde{\nabla}_{[\dot{x}^H + (\nabla_{\dot{x}}y)^V]}[\dot{x}^H + (\nabla_{\dot{x}}y)^V] \\ &= \widetilde{\nabla}_{\dot{x}^H}\dot{x}^H + \widetilde{\nabla}_{\dot{x}^H}(\nabla_{\dot{x}}y)^V + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^V}\dot{x}^H + \widetilde{\nabla}_{(\nabla_{\dot{x}}y)^V}(\nabla_{\dot{x}}y)^V \\ &= (\nabla_{\dot{x}}\dot{x})^H - \frac{1}{2}(R(\dot{x}, \dot{x})y)^V + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{1}{2f}\dot{x}(f)(\nabla_{\dot{x}}y)^V + \frac{f}{2}(R(y, \nabla_{\dot{x}}y)\dot{x})^H + \frac{1}{2f}\dot{x}(f)(\nabla_{\dot{x}}y)^V \\ &\quad +\frac{f}{2}(R(y, \nabla_{\dot{x}}y)\dot{x})^H - \frac{1}{2f}\tilde{g}((\nabla_{\dot{x}}y)^V, (\nabla_{\dot{x}}y)^V)(grad\, f)^H + \frac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)(\varphi y)^V \\ &= (\nabla_{\dot{x}}\dot{x})^H + f(R(y, \nabla_{\dot{x}}y)\dot{x})^H - \frac{1}{2f}\tilde{g}((\nabla_{\dot{x}}y)^V, (\nabla_{\dot{x}}y)^V)(grad\, f)^H + (\nabla_{\dot{x}}\nabla_{\dot{x}}y)^V + \frac{1}{f}\dot{x}(f)(\nabla_{\dot{x}}y)^V \\ &\quad +\frac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)(\varphi y)^V \\ &= [\nabla_{\dot{x}}\dot{x} + fR(y, \nabla_{\dot{x}}y)\dot{x} - \frac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2)grad\, f]^H \\ &\quad +[\nabla_{\dot{x}}\nabla_{\dot{x}}y + \frac{1}{f}\dot{x}(f)\nabla_{\dot{x}}y + \frac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y]^V. \end{aligned}$$

□

Theorem 4.6. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. If $C(t) = (x(t), y(t))$ is curve on TM, then $C(t)$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} = \dfrac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2) grad\, f - fR(y, \nabla_{\dot{x}}y)\dot{x} \\ \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -\dfrac{1}{f}\dot{x}(f)\nabla_{\dot{x}}y - \dfrac{\delta^2}{\lambda} g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y \end{cases} \tag{4.1}$$

Proof. The statement is a direct consequence of Theorem 4.5 and definition of geodesic. □

Using Theorem 4.6 we deduce:

Corollary 4.7. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. The natural lift $C(t) = (x(t), \dot{x}(t))$ of any geodesic $x(t)$ on (M^{2m}, φ, g) is a geodesic on TM.*

Corollary 4.8. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. If $C(t) = (x(t), y(t))$ be a horizontal lift of the curve $x(t)$. Then $C(t)$ is a geodesic on TM if and only if $x(t)$ is a geodesic on (M^{2m}, φ, g).*

Remark 4.9. If $x(t)$ is a geodesic on (M^{2m}, φ, g) locally we have:

$$\nabla_{\dot{x}}\dot{x} = 0 \quad \Leftrightarrow \quad \frac{d^2x^h}{dt^2} + \sum_{i,j=1}^{2m} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h = 0, \quad h = \overline{1, 2m}$$

If $C(t) = (x(t), y(t))$ is a horizontal lift of the curve $x(t)$, locally we have:

$$\nabla_{\dot{x}}y = 0 \quad \Leftrightarrow \quad \frac{dy^h}{dt} + \sum_{i,j=1}^{2m} \frac{dx^j}{dt} y^i \Gamma_{ij}^h = 0, \quad h = \overline{1, 2m}.$$

Remark 4.10. Using the Remark 4.9 we can costruct an infinity of examples of geodesics on $(TM, \tilde{g})$.

Example 4.11. Let $(\mathbb{R}^2, \varphi, g)$ be an anti-paraKähler manifold such that

$$g = e^{2x^1}(dx^1)^2 + e^{2x^2}(dx^2)^2,$$

and

$$\varphi\frac{\partial}{\partial x^1} = \frac{e^{x^1}}{e^{x^2}}\frac{\partial}{\partial x^2} \quad , \quad \varphi\frac{\partial}{\partial x^2} = \frac{e^{x^2}}{e^{x^1}}\frac{\partial}{\partial x^1}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

The geodesics $x(t) = (x^1(t), x^2(t))$ such that $x(0) = (a, b) \in \mathbb{R}^2$, $\dot{x}(0) = (v, w) \in \mathbb{R}^2$ satisfy the system of equations,

$$\frac{d^2x^h}{dt^2} + \sum_{i,j=1}^{2} \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{d^2x^1}{dt^2} + (\dfrac{dx^1}{dt})^2 = 0 \\ \\ \dfrac{d^2x^2}{dt^2} + (\dfrac{dx^2}{dt})^2 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x^1(t) = a + \ln(1 + vt) \\ \\ x^2(t) = b + \ln(1 + wt). \end{cases}$$

Hence $\dot{x}(t) = \dfrac{v}{1 + vt}\dfrac{\partial}{\partial x^1} + \dfrac{w}{1 + wt}\dfrac{\partial}{\partial x^2}$ and $x(t) = (a + \ln(1 + vt), b + \ln(1 + wt))$.

1) From Corollary 4.7, the natural lift $C(t) = (x(t), \dot{x}(t))$ is a geodesic on $T\mathbb{R}^2$.

2) If $C(t) = (x(t), y(t))$ is horizontal lift of the curve $x(t)$ i.e $\nabla_{\dot{x}}y = 0$ then,

$$\frac{dy^h}{dt} + \sum_{i,j=1}^{2m} \frac{dx^j}{dt} y^i \Gamma_{ij}^h = 0 \Leftrightarrow \begin{cases} \dfrac{dy^1}{dt} + \dfrac{dx^1}{dt} y^1 = 0 \\ \\ \dfrac{dy^2}{dt} + \dfrac{dx^2}{dt} y^2 = 0 \end{cases} \Leftrightarrow \begin{cases} y_1(t) = \dfrac{k_1}{1 + vt} \\ \\ y_2(t) = \dfrac{k_2}{1 + wt}. \end{cases}$$

Hence $y(t) = \frac{k_1}{1+vt}\frac{\partial}{\partial x^1} + \frac{k_2}{1+wt}\frac{\partial}{\partial x^2}$, where $k_1, k_2 \in \mathbb{R}$.
From Corollary 4.8, the horizontal lift $C(t) = (x(t), y(t))$ is a geodesic on $T\mathbb{R}^2$.

Example 4.12. Let $\left(]0,+\infty[^2, g, \varphi\right)$ be an anti-paraKähler manifold such that

$$g = x^2dx^2 + y^2dy^2,$$

and

$$\varphi\frac{\partial}{\partial x} = \frac{x}{y}\frac{\partial}{\partial y} \quad , \quad \varphi\frac{\partial}{\partial y} = \frac{y}{x}\frac{\partial}{\partial x}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma^1_{11} = \frac{1}{x}\ ,\ \ \Gamma^2_{22} = \frac{1}{y}.$$

With similar calculations in the previous example, we find the curve

$$\gamma(t) = \left(\sqrt{2a\xi t + a^2}, \sqrt{2b\eta t + b^2}\right)$$

is geodesics on $]0,+\infty[^2$, where $\gamma(0) = (a, b)$, $\gamma'(0) = (\xi, \eta) \in \mathbb{R}^2$, then, the natural lift $C_1(t) = (\gamma(t), \gamma'(t))$ is a geodesic on $T]0,+\infty[^2$.
The curve $C_2(t) = (\gamma(t), u(t))$ such that

$$u(t) = \frac{k_1}{\sqrt{2a\xi t + a^2}}\partial_x + \frac{k_2}{\sqrt{2b\eta t + b^2}}\partial_y,$$

where $k_1, k_2 \in \mathbb{R}$ is a horizontal lift of the curve $\gamma(t)$, hence it is a geodesic on $T]0,+\infty[^2$.

Corollary 4.13. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and f be a constant. Then the curve $C(t) = (x(t), y(t))$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} = -fR(y, \nabla_{\dot{x}}y)\dot{x} \\ \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -\dfrac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y. \end{cases}$$

Proof. The statement is a direct consequence of Theorem 4.6. □

Theorem 4.14. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and $x(t)$ be a geodesic on (M^{2m}, φ, g). If the curve $C(t) = (x(t), y(t))$ is a geodesic on TM such $\nabla_{\dot{x}}y \neq 0$, then f is a constant along the curve $x(t)$.*

Proof. Let $x(t)$ be a geodesic on M, then $\nabla_{\dot{x}}\dot{x} = 0$. Using the first equation of formula (4.1) we obtain

$$\begin{aligned} g(\nabla_{\dot{x}}\dot{x}, \dot{x}) = 0 \quad &\Rightarrow \quad \frac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2)g(grad\, f, \dot{x}) - g(R(y, \nabla_{\dot{x}}y)\dot{x}, \dot{x}) = 0 \\ &\Rightarrow \quad \frac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2)\dot{x}(f) = 0 \\ &\Rightarrow \quad \dot{x}(f) = 0. \end{aligned}$$

□

Corollary 4.15. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. If the curve $C(t) = (x(t), y(t))$ is a geodesic on TM such f is a constant along the curve $x(t)$, then*

$$\begin{cases} g(\nabla_{\dot{x}}\dot{x}, \dot{x}) = 0 \\ \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -\dfrac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y. \end{cases}$$

Proof. The proof follows directly from Theorem 4.14. □

Theorem 4.16. *Let (M^{2m}, φ, g) be a flat anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the curve $C(t) = (x(t), y(t))$ is a geodesic on TM if and only if*

$$\begin{cases} \nabla_{\dot{x}}\dot{x} = \dfrac{1}{2}(g(\nabla_{\dot{x}}y, \nabla_{\dot{x}}y) + \delta^2 g(\nabla_{\dot{x}}y, \varphi y)^2) grad\, f \\ \\ \nabla_{\dot{x}}\nabla_{\dot{x}}y = -\dfrac{1}{f}\dot{x}(f)\nabla_{\dot{x}}y - \dfrac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y. \end{cases}$$

Proof. The statement is a direct consequence of Theorem 4.5. □

Corollary 4.17. *Let (M^{2m}, φ, g) be a flat anti-paraKähler manifold, $(TM, \tilde{g})$ its tangent bundle equipped with the vertical rescaled Berger deformation metric and f be a constant. Then the curve $C(t) = (x(t), y(t))$ is a geodesic on TM if and only if $x(t)$ is a geodesic on (M^{2m}, φ, g) and*

$$\nabla_{\dot{x}}\nabla_{\dot{x}}y = -\frac{\delta^2}{\lambda}g(\nabla_{\dot{x}}y, \varphi\nabla_{\dot{x}}y)\varphi y.$$

Proof. The statement is a direct consequence of Theorem 4.16. □

Conclusion

In this work, we studied the geodesics on the tangent bundle with respect to the vertical rescaled Berger deformation metric and we gave the necessary and sufficient conditions under which a curve be geodesic respect to this metric. Also, we studied certain examples of geodesic.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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