

## Estimation of subparameters by IPM method

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### ABSTRACT

In this study, a general partitioned linear model  $\mathcal{M} = \{y, X\beta, V\} = \{y, X_1\beta_1 + X_2\beta_2, V\}$  is considered to determine the best linear unbiased estimators (BLUEs) of subparameters  $X_1\beta_1$  and  $X_2\beta_2$ . Some results are given related to the BLUEs of subparameters by using the inverse partitioned matrix (IPM) method based on a generalized inverse of a symmetric block partitioned matrix which is obtained from the fundamental BLUE equation.

**Keywords:** BLUE, generalized inverse, general partitioned linear model

## IPM yöntemi ile alt parametrelerin tahmini

### ÖZ

Bu çalışmada,  $X_1\beta_1$  ve  $X_2\beta_2$  alt parametrelerinin en iyi lineer yansız tahmin edicilerini (BLUE' larını) belirlemek için bir  $\mathcal{M} = \{y, X\beta, V\} = \{y, X_1\beta_1 + X_2\beta_2, V\}$  genel parçalanmış lineer modeli ele alınmıştır. Temel BLUE denkleminde elde edilen simetrik blok parçalanmış matrisin bir genelleştirilmiş tersine dayanan parçalanmış matris tersi (IPM) yöntemi kullanılarak alt parametrelerin BLUE' ları ile ilgili bazı sonuçlar verilmiştir.

**Anahtar Kelimeler:** BLUE, genelleştirilmiş ters, genel parçalanmış lineer model

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**1. INTRODUCTION**

Consider the general partitioned linear model

$$y = X\beta + \varepsilon = X_1\beta_1 + X_2\beta_2 + \varepsilon, \tag{1}$$

where  $y \in R^{nx1}$  is an observable random vector,  $X = (X_1 : X_2) \in R^{n \times p}$  is a known matrix with  $X_1 \in R^{n \times p_1}$  and  $X_2 \in R^{n \times p_2}$ ,  $\beta = (\beta_1' : \beta_2') \in R^{p \times 1}$  is a vector of unknown parameters with  $\beta_1 \in R^{p_1 \times 1}$  and  $\beta_2 \in R^{p_2 \times 1}$ ,  $\varepsilon \in R^{nx1}$  is a random error vector. Further, the expectation  $E(y) = X\beta$  and the covariance matrix  $Cov(y) = V \in R^{n \times n}$  is a known nonnegative definite matrix. We may denoted the model (1) as a triplet

$$\mathcal{M} = \{y, X\beta, V\} = \{y, X_1\beta_1 + X_2\beta_2, V\}. \tag{2}$$

Partitioned linear models are used in the estimations of partial parameters in regression models as well as in the investigations of some submodels and reduced models associated with the original model. In this study, we consider the general partitioned linear model  $\mathcal{M}$  and we deal with the best linear unbiased estimators (BLUEs) of subparameters under this model. Our main purpose is to obtain the BLUEs of subparameters  $X_1\beta_1$  and  $X_2\beta_2$  under  $\mathcal{M}$  by using the inverse partitioned matrix (IPM) method which is introduced by Rao [1] for statistical inference in general linear models. We also investigate some consequences on the BLUEs of subparameters obtained by using IPM approach.

Under the linear models, BLUE has been investigated by many statisticians. Some valuable properties of BLUE have been obtained, e.g., [2-6]. By applying matrix rank method, some characterizations of BLUE have been given by Tian [7,8]. IPM method for the general linear model with linear restrictions has been considered by Baksalary [9].

**2. PRELIMINARIES**

The BLUE of  $X\beta$  under  $\mathcal{M}$ , denoted as  $BLUE(X\beta|\mathcal{M})$ , is defined to be an unbiased linear estimator  $Gy$  such that its covariance matrix  $Cov(Gy)$  is minimal, in the Löwner sense, among all covariance matrices  $Cov(Fy)$  such that  $Fy$  is unbiased for  $X\beta$ . It is well-known, see, e.g., [10,11], that

$Gy = BLUE(X\beta|\mathcal{M})$  if and only if  $G$  satisfies the fundamental BLUE equation

$$G(X : VQ) = (X : 0), \tag{3}$$

where  $Q = I - P_x$  with  $P_x$  is orthogonal projector onto the column space  $C(X)$ . Note that the equation (3) has a unique solution if and only if  $rank(X : V) = n$  and the observed value of  $Gy$  is unique with probability 1 if and only if  $\mathcal{M}$  is consistent, i.e.,  $y \in C(X : V) = C(X : VQ)$  holds with probability 1; see [12]. In the study, it is assumed that the model  $\mathcal{M}$  is consistent.

The corresponding condition for  $Ay$  to be BLUE of an estimable parametric function  $K\beta$  is  $A(X : VQ) = (K : 0)$ . Recall that a parametric function  $K\beta$  is estimable under  $\mathcal{M}$  if and only if  $C(K') \subseteq C(X')$  and in particular,  $X_1\beta_1$  and  $X_2\beta_2$  is estimable under  $\mathcal{M}$  if and only if  $C(X_1) \cap C(X_2) = \{0\}$ ; see [13,14].

The fundamental BLUE equation given in (3) equivalently expressed as follows.  $Gy = BLUE(X\beta|\mathcal{M})$  if and only if there exists a matrix  $L \in R^{p \times n}$  such that  $G$  is solution to,

$$\begin{pmatrix} V & X \\ X' & 0 \end{pmatrix} \begin{pmatrix} G' \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ X' \end{pmatrix}, \text{ i.e., } Z \begin{pmatrix} G' \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ X' \end{pmatrix}. \tag{4}$$

Partitioned matrices and their generalized inverses play an important role in the concept of linear models. According to Rao [1], the problem of inference from a linear model can be completely solved when one has obtained an arbitrary generalized inverse of the partitioned matrix  $Z$ . This approach based on the numerical evaluation of an inverse of the partitioned matrix  $Z$  is known as the IPM method, see [1-15].

Let the matrix  $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}$  be an arbitrary generalized inverse of  $Z$ , i.e.,  $C$  is any matrix satisfying the equation  $ZCZ = Z$ , where  $C_1 \in R^{n \times n}$  and  $C_2 \in R^{n \times p}$ . Then one solution to the (consistent) equation (4) is

$$\begin{pmatrix} G' \\ L \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \begin{pmatrix} 0 \\ X' \end{pmatrix} = \begin{pmatrix} C_2 X' \\ -C_4 X' \end{pmatrix}. \tag{5}$$

Therefore, we see that  $BLUE(X\beta|\mathcal{M}) = XC_2'y = XC_3'y$ , which is one representation for the BLUE of  $X\beta$  under  $\mathcal{M}$ . If we let  $C$  vary through all generalized inverses of  $Z$  we obtain all solutions to (4) and thereby all representations  $Gy$  for the BLUE of  $X\beta$  under  $\mathcal{M}$ . As further reference for submatrices  $C_i, i=1,2,3,4$ , and their statistical applications, see [16-23].

### 3. SOME RESULTS ON A GENERALIZED INVERSE OF $Z$

Some explicit algebraic expression for the submatrices of  $C$  was obtained in [15, Theorem 2.3]. The purpose of this section is to extend this theorem to  $3 \times 3$  symmetric block partitioned matrix to obtain the BLUEs of subparameters and their properties.

Let  $D \in \{Z^-\}$ , expressed as

$$D = \begin{pmatrix} D_0 & D_1 & D_2 \\ E_1 & -F_1 & -F_2 \\ E_2 & -F_3 & -F_4 \end{pmatrix} = \begin{pmatrix} V & X_1 & X_2 \\ X_1' & 0 & 0 \\ X_2' & 0 & 0 \end{pmatrix}^-, \tag{6}$$

where  $D_0 \in R^{n \times n}$ ,  $D_1 \in R^{n \times p_1}$ ,  $D_2 \in R^{n \times p_2}$ ,  $E_1, E_2, F_1, F_2, F_3, F_4$  are conformable matrices and  $\{Z^-\}$  stands for the set of all generalized inverse of  $Z$ . In the following theorem, we collect some properties related to the submatrices of  $D$  given in (6).

**Theorem 1.** Let  $V, X_1, X_2, D_0, D_1, D_2, E_1, E_2, F_1, F_2, F_3, F_4$  be defined as before and let  $C(X_1) \cap C(X_2) = \{0\}$ . Then the following hold:

$$(i) \begin{pmatrix} V & X_1 & X_2 \\ X_1' & 0 & 0 \\ X_2' & 0 & 0 \end{pmatrix}^- = \begin{pmatrix} D_0' & E_1' & E_2' \\ D_1' & -F_1' & -F_3' \\ D_2' & -F_2' & -F_4' \end{pmatrix}$$

is another choice of a generalized inverse.

$$(ii) VD_0X_1 + X_1E_1X_1 + X_2E_2X_1 = X_1, \quad X_1'D_0X_1 = 0, \quad X_2'D_0X_1 = 0.$$

$$(iii) VD_0X_2 + X_1E_1X_2 + X_2E_2X_2 = X_2, \quad X_1'D_0X_2 = 0, \quad X_2'D_0X_2 = 0.$$

$$(iv) VD_0V + X_1E_1V + X_2E_2V = V, \quad X_1'D_0V = 0, \quad X_2'D_0V = 0.$$

$$(v) VD_1X_1' = X_1F_1X_1' + X_2F_3X_1', \quad X_1D_1'X_1 = X_1, \quad X_1D_1'X_2 = 0.$$

$$(vi) VD_2X_2' = X_1F_2X_2' + X_2F_4X_2', \quad X_2D_2'X_2 = X_2, \quad X_2D_2'X_1 = 0.$$

**Proof:** The result (i) is proved by taking transposes of either side of (6). We observe that the equations

$$Va + X_1b + X_2c = X_1d, \quad X_1'a = 0, \quad X_2'a = 0 \tag{7}$$

are solvable for any  $d$ , in which case  $a = D_0X_1d$ ,  $b = E_1X_1d$ ,  $c = E_2X_1d$  is a solution. Substituting this solution in (7) and omitting  $d$ , we have (ii). To prove (iii), we can write the equations

$$Va + X_1b + X_2c = X_2d, \quad X_1'a = 0, \quad X_2'a = 0 \tag{8}$$

which are solvable for any  $d$ . Then  $a = D_0X_2d$ ,  $b = E_1X_2d$ ,  $c = E_2X_2d$  is a solution. Substituting this solution in (8) and omitting  $d$ , we have (iii). To prove (iv), the equations which are solvable for any  $d$

$$Va + X_1b + X_2c = Vd, \quad X_1'a = 0, \quad X_2'a = 0 \tag{9}$$

are considered. In this case, one solution is  $a = D_0Vd$ ,  $b = E_1Vd$ ,  $c = E_2Vd$ . If we substitute this solution in (9) and omit  $d$ , we have (iv). In view of the assumption  $C(X_1) \cap C(X_2) = \{0\}$ , we can consider the equations

$$Va + X_1b + X_2c = 0, \quad X_1'a = X_1'd_1, \quad X_2'a = 0 \tag{10}$$

and

$$Va + X_1b + X_2c = 0, \quad X_1'a = 0, \quad X_2'a = X_2'd_2 \tag{11}$$

for the proof of (v) and (vi), respectively, see [18, Theorem 7.4.8]. In this case  $a = D_1X_1'd_1$ ,  $b = -F_1X_1'd_1$ ,  $c = -F_3X_1'd_1$  is a solution for (10) and  $a = D_2X_2'd_2$ ,  $b = -F_2X_2'd_2$ ,  $c = -F_4X_2'd_2$  is a solution for (11). Substituting these solutions into corresponding equations and omitting  $d_1$  and  $d_2$ , we obtain the required results.

**4. IPM METHOD FOR SUBPARAMETERS**

The fundamental BLUE equation given in (4) can be accordingly written for  $Ay$  being the BLUE of estimable  $K\beta$ , that is,  $Ay = BLUE(K\beta|\mathcal{M})$  if and only if there exists a matrix  $L \in R^{p \times n}$  such that  $A$  is solution to

$$Z \begin{pmatrix} A' \\ L \end{pmatrix} = \begin{pmatrix} 0 \\ K' \end{pmatrix}. \tag{12}$$

Now, assumed that  $X_1\beta_1$  and  $X_2\beta_2$  are estimable under  $\mathcal{M}$ . If we take  $K = (X_1 : 0)$  and  $K = (0 : X_2)$ , respectively, from equation (12), we get the BLUE equations of subparameters  $X_1\beta_1$  and  $X_2\beta_2$ . There exist  $L_1 \in R^{p_1 \times n}$ ,  $L_2 \in R^{p_2 \times n}$ ,  $L_3 \in R^{p_1 \times n}$ ,  $L_4 \in R^{p_2 \times n}$  such that  $G_1$  and  $G_2$  are solution to following the equations, respectively,

$$G_1y = BLUE(X_1\beta_1|\mathcal{M}) \Leftrightarrow \begin{pmatrix} V & X_1 & X_2 \\ X_1' & 0 & 0 \\ X_2' & 0 & 0 \end{pmatrix} \begin{pmatrix} G_1' \\ L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 0 \\ X_1' \\ 0 \end{pmatrix} \tag{13}$$

and

$$G_2y = BLUE(X_2\beta_2|\mathcal{M}) \Leftrightarrow \begin{pmatrix} V & X_1 & X_2 \\ X_1' & 0 & 0 \\ X_2' & 0 & 0 \end{pmatrix} \begin{pmatrix} G_2' \\ L_3 \\ L_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ X_2' \end{pmatrix}. \tag{14}$$

Therefore, the following theorem can be given to determine the BLUE of subparameters by the IPM method.

**Theorem 2.** Consider the general partitioned linear model  $\mathcal{M}$  and the matrix  $D$  given in (6). Suppose that  $C(X_1) \cap C(X_2) = \{0\}$ . Then

$$BLUE(X_1\beta_1|\mathcal{M}) = X_1D_1'y = X_1E_1y \tag{15}$$

and

$$BLUE(X_2\beta_2|\mathcal{M}) = X_2D_2'y = X_2E_2y.$$

**Proof:** The general solution of the matrix equation given in (13) is

$$\begin{pmatrix} G_1' \\ L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} D_0 & D_1 & D_2 \\ E_1 & -F_1 & -F_2 \\ E_2 & -F_3 & -F_4 \end{pmatrix} \begin{pmatrix} 0 \\ X_1' \\ 0 \end{pmatrix} + \left[ I - \begin{pmatrix} D_0 & D_1 & D_2 \\ E_1 & -F_1 & -F_2 \\ E_2 & -F_3 & -F_4 \end{pmatrix} \begin{pmatrix} V & X_1 & X_2 \\ X_1' & 0 & 0 \\ X_2' & 0 & 0 \end{pmatrix} \right] U$$

and thereby we get

$$G_1y = X_1D_1'y + U_1'(I - VD_0' - X_1D_1' - X_2D_2')y + U_2'(-X_1'D_0')y + U_3'(-X_2'D_0')y.$$

Here  $y$  can be written as  $y = X_1L_1 + X_2L_2 + VQL_3$  for some  $L_1, L_2$  and  $L_3$  since the model  $\mathcal{M}$  is assumed to be consistent. From Theorem 1, we see that

$$U_1'(I - VD_0' - X_1D_1' - X_2D_2')(X_1L_1 + X_2L_2 + VQL_3) = 0, \\ U_2'(-X_1'D_0')(X_1L_1 + X_2L_2 + VQL_3) = 0, \\ U_3'(-X_2'D_0')y = U_3'(-X_2'D_0')(X_1L_1 + X_2L_2 + VQL_3) = 0.$$

Moreover, according to Theorem 1 (i), we can replace  $D_1'$  by  $E_1$ . Therefore,  $BLUE(X_1\beta_1|\mathcal{M}) = X_1D_1'y = X_1E_1y$  is obtained.  $BLUE(X_2\beta_2|\mathcal{M}) = X_2D_2'y = X_2E_2y$  is obtained by similar way

The following results are easily obtained from Theorem 1 (v) and (vi) under  $\mathcal{M}$ .

$$E(BLUE(X_1\beta_1|\mathcal{M})) = X_1\beta_1 \tag{16}$$

and

$$E(BLUE(X_2\beta_2|\mathcal{M})) = X_2\beta_2,$$

$$Cov(BLUE(X_1\beta_1|\mathcal{M})) = X_1F_1X_1' \tag{17}$$

and

$$Cov(BLUE(X_2\beta_2|\mathcal{M})) = X_2F_4X_2',$$

$$Cov(BLUE(X_1\beta_1|\mathcal{M}), BLUE(X_2\beta_2|\mathcal{M})) = X_1F_2X_2' = X_1F_3X_2'. \tag{18}$$

**5. ADDITIVE DECOMPOSITION OF THE BLUES OF SUBPARAMETERS**

The purpose of this section is to give some additive properties of BLUEs of subparameters under  $\mathcal{M}$ .

**Theorem 3.** Consider the model  $\mathcal{M}$  and assume that  $X_1\beta_1$  and  $X_2\beta_2$  are estimable under  $\mathcal{M}$ .  $\{BLUE(X_1\beta_1|\mathcal{M})+BLUE(X_2\beta_2|\mathcal{M})\}$  is always BLUE for  $X\beta$  under  $\mathcal{M}$ .

**Proof:** Let  $BLUE(X_1\beta_1|\mathcal{M})$  and  $BLUE(X_2\beta_2|\mathcal{M})$  be given as in (15). Then we can write

$$BLUE(X_1\beta_1|\mathcal{M})+BLUE(X_2\beta_2|\mathcal{M}) \\ = (X_1D'_1+X_2D'_2)y.$$

According to fundamental BLUE equation and from Theorem 1 (v) and (vi), we see that  $(X_1D'_1+X_2D'_2)(X_1:X_2:VQ)=(X_1:X_2:0)$  for all  $y \in C(X:VQ)$ . Therefore the required result is obtained.

The following results are easily obtained from Theorem 1 (iv) and (16)-(18).

$$E(BLUE(X_1\beta_1|\mathcal{M})+BLUE(X_2\beta_2|\mathcal{M}))=X\beta, \\ Cov(BLUE(X_1\beta_1|\mathcal{M})+BLUE(X_2\beta_2|\mathcal{M})) \\ = X_1F_1X'_1+X_2F_4X'_2+X_1F_2X'_2+X_2F_3X'_1.$$

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