Journal of Mathematical Sciences and Modelling, 5 (2) (2022) 40-47
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.1080204

Stability and Boundedness of Solutions of Nonlinear Third Order Differential Equations with Bounded Delay

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Article Info

Keywords: *Boundedness, Delay differential equations, Lyapunov functional, Stability, Third order*
2010 AMS: *34C11, 93D20*
Received: *28 February 2022*
Accepted: *4 April 2022*
Available online: *30 August 2022*

Abstract

In this paper, we investigate the boundedness and uniformly asymptotically stability of the solutions to a certain third order non-autonomous differential equations with bounded delay. By constructing a Lyapunov functional, sufficient conditions for the stability and boundedness of solutions for equations considered are obtained. We used an example to demonstrate the feasibility of our results. The results essentially improve, include, and complement the results in the literature.

1. Introduction

For years, researchers have focused on the qualitative behavior of differential equation solutions, such as stability, asymptotic stability, uniform asymptotic stability, boundedness, and uniform boundedness. In the application areas of mathematics such as physics, chemistry, biology, engineering and dynamical systems, many events are modeled with differential equations [1, 2]. The qualitative behavior of the differential equations corresponding to these models is important and finds concrete responses in the application areas [3]-[5]. Also, the rapid expansion of differential equations with lag arguments in recent years and now covers not only many physics and technology questions, but also certain areas of economics and biological sciences [6, 7]. Therefore, the qualitative behavior of the solutions of differential equations with delay arguments remains up-to-date and attracts the attention of many researchers.

The fact that the trajectory curve of a solution starting in a region does not leave this region is known as the stability of the solution [8]. A great advantage of the method known as Lyapunov's second method for determining the stability behavior of solutions of linear and nonlinear systems is to examine the stability of the solutions without any prior knowledge of the solutions [9]. The simplest form of third-order differential equations is of the form

$$x''' + ax'' + bx' + cx = 0 \tag{1.1}$$

where a, b, c are constants. In this case it is well known that all solutions tend to the trivial solution, as $t \to \infty$ provided that the Routh-Hurwitz criteria $a > 0, c > 0, ab - c > 0$ are satisfied [10]. In [10], author obtains sufficient conditions for the asymptotic stability of the trivial solution of differential equations of the form

$$x''' + f(x,x')x'' + g(x') + h(x) = 0 \tag{1.2}$$

considering the criteria Routh-Hurwitz for equation (1.1).

In [11]-[19], the authors investigated the behavior of solutions of differential equations such as asymptotic stability, global asymptotic stability, global stability, boundedness, uniform boundedness in different third-order nonlinear models using the Lyapunov method.

In [20], the author constructed some new Lyapunov functions to examine the asymptotic stability and boundedness of the solutions of non-linear delay differential equation described by

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = p(t) \tag{1.3}$$

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with $p \equiv 0$ and $p \neq 0$, respectively. In [21], the authors established sufficient conditions for the asymptotic stability and boundedness of solutions of a certain third order nonlinear non-autonomous delay differential equation described by

$$[g(x(t))x'(t)]'' + a(t)x''(t) + b(t)x'(t)) + c(t)f(x(t-r)) = p(t) \tag{1.4}$$

with $p(t) \equiv 0$ and $p(t) \neq 0$, respectively. In recent years [22]-[33], the authors obtained remarkable results by using the Lyapunov method of the behavior of the solutions of differential equations with or without delay in different third-order nonlinear models. For detailed information on the behavior of solutions of third order lagged or undelayed differential equations, references and their references can be consulted.
Inspired by the studies above, especially by expanding the scope of [21], that is, the behavior of the solutions of a new equation is examined by taking the delay variable and the coefficients as the functions of the dependent variable.
In this paper, we investigate the uniform asymtotic stability of the solutions for $p(t,x,x',x'') \equiv 0$ and additionally boundedness of solutions to the third order nonlinear differential equation with bounded delay

$$[P(x(t))x'(t)]'' + a(t)(Q(x(t))x'(t))' + b(t)(R(x(t))x'(t)) + c(t)f(x(t-r(t))) = p(t,x,x',x''). \tag{1.5}$$

For convenience, we let

$$\theta_1(t) = \frac{P'(x(t))}{P^2(x(t))}x'(t),$$

$$\theta_2(t) = \frac{Q'(x(t))P(x(t)) - Q(x(t))P'(x(t))}{P^2(x(t))}x'(t),$$

and

$$\theta_3(t) = \frac{R'(x(t))P(x(t)) - R(x(t))P'(x(t))}{P^2(x(t))}x'(t).$$

We write (1.5) in the system form

$$\begin{aligned} x' &= \frac{1}{P(x)}y, \\ y' &= z, \\ z' &= -a(t)\theta_2(t)y - \frac{a(t)Q(x)}{P(x)}z - \frac{b(t)R(x(t))y}{P(x(t))} - c(t)f(x(t)) + c(t)\int_{t-r(t)}^{t} \frac{1}{P(x)}f'(x)y\,d\eta + p(t,x,y,z) \end{aligned} \tag{1.6}$$

where r is a bounded delay, $0 \leq r(t) \leq \Omega$, $r'(t) \leq \lambda$, $0 < \lambda < 1$, λ and Ω some positive constants, Ω which will be determined late, the functions a,b,c are continuously differentiable functions and the functions P,Q,R,f,p are continuous functions depending only on the arguments shown. Also derivatives $P'(x), P''(x), Q'(x), R'(x)$ and $f'(x)$ exist and are continuous, $f(0) = 0$. The continuity of the functions a,b,c,P,Q,R,f and p guarantees the existence of the solutions of equation (1.5). If the right hand side of the system (1.6) satisfies a Lipchitz condition in $x(t), y(t), z(t)$ and $x(t-r(t))$ and exists of solutions of system (1.6) , then it is unique solution of system (1.6).
Assume that there are positive constants $a_0, b_0, c_0, p_0, q_0, r_0, a_1, b_1, c_1, , p_1, q_1,$ and r_1 such that the following assumptions hold:

(A1) $0 < a_0 \leq a(t) \leq a_1$, $0 < b_0 \leq b(t) \leq b_1$ and $0 < c_0 \leq c(t) \leq c_1$ for all $t \geq 0$;
(A2) $0 < p_0 \leq P(x) \leq p_1$, $0 < q_0 \leq Q(x) \leq q_1$, and $0 < r_0 \leq R(x) \leq r_1$ for $x \in \mathbb{R}$;
(A3) $\frac{f(x)}{x} \geq \delta_0 > 0$ for $x \neq 0$ and $|f'(x)| \leq \delta_1$ for all x; and
(A4) $|p(t,x,y,z)| \leq |e(t)|$.

Using the Lyapunov's functional approach, we establish some new results on the uniformly asymtotically stability and boundedness of the solutions, which are motivated by the results of references. Our results differ from those existing in the literature (see, references and the references therein). By this way, we mean that this work offers to the literature on the subject and may be significant for researchers who study the qualitative behaviour of solutions of higher-order functional differential equations. The uniqueness and originality of the present paper can be checked based on all of the above facts.

2. Preliminaries

We also consider the functional differential equation

$$\dot{x} = f(t,x_t), \;\; x_t(\theta) = x(t+\theta), \;\; -r \leq \theta \leq 0, \;\; t \geq 0 \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $I = [0,\infty)$, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0],\mathbb{R}^n) : \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Lemma 2.1. *([5]) Let $V(t,\phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipchitz condition, $V(t,0) = 0$, and wedges W_i such that :*

(i) $W_1(\|\phi\|) \leq V(t,\phi) \leq W_2(\|\phi\|)$;
(ii) $V'_{(3)}(t,\phi) \leq -W_3(\|\phi\|)$.

Then, the zero solution of equation (2.1) is uniformly asymptotically stable.

3. The Main Results

Theorem 3.1. *In addition to the basic assumptions imposed on the functions a, b, c, P, Q, R and e suppose that there are positive constants $\delta_0, \delta_1, \eta_1$ and η_2 such that the following conditions are satisfied:*

(i) $\frac{p_1\delta_1}{r_0} < d < a_0q_0$;

(ii) $c(t) \le b(t)$ *and* $b'(t) \le c'(t) \le 0$ *for* $t \in [0,\infty)$;

(iii) $\frac{1}{2}da'(t)Q(x) - b_0(dr_0 - p_1\delta_1) \le -\varepsilon < 0$;

(iv) $\int\limits_{-\infty}^{\infty} (|P'(u)| + |Q'(u)| + |R'(u)|)du \le \eta_1 < \infty$; *and*

(v) $\int\limits_{0}^{\infty} |e(s)|\, ds \le \eta_2 < \infty$.

Then any solution $x(t)$ equation (1.5) are bounded and trival solution of equation (1.5) for $p(t,x,x',x'') \equiv 0$ is uniformly asymtotically stability, if

$$\Omega < \frac{2p_0}{p_1c_1\delta_1} \min\left\{ \frac{\varepsilon(1-\lambda)p_0}{p_1\left(p_0 + d(2-\lambda)\right)},\ (a_0q_0 - d) \right\}. \tag{3.1}$$

Proof. To prove the theorem, we define a Lyapunov functional

$$W = W(t,x,y,z) = exp\left(\frac{-1}{\eta}\int_0^t \gamma(s)ds\right)V, \tag{3.2}$$

where

$$\gamma(t) = |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|, \tag{3.3}$$

and

$$V = V(t,x,y,z) = dc(t)F(x) + c(t)f(x)y + \frac{b(t)R(x)}{2P(x)}y^2 + \frac{1}{2}z^2 + \frac{1}{2}\frac{da(t)Q(x)}{P^2(x)}y^2 + \frac{d}{P(x)}yz + \sigma\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds \tag{3.4}$$

with $F(x) = \int_0^x f(s)ds$, and η are positive constants that will be determined at a later point of the proof. We can write V as

$$V = dc(t)F(x) + M(x,y) - \frac{c^2(t)P(x)f^2(x)}{2b(t)R(x)} + \frac{1}{2}z^2 + \frac{d}{P(x)}yz + \frac{da(t)Q(x)}{2P^2(x)}y^2 + \sigma\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds$$

where

$$M(x,y) = \frac{b(t)R(x)}{2P(x)}\left\{y + \frac{c(t)f(x)P(x)}{b(t)R(x)}\right\}^2 \ge 0.$$

Note that

$$\frac{1}{2}f^2(x) = \int_0^x f(u)f'(u)du \le \int_0^x f(u)\delta_1 du$$

and

$$\sigma\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds \ge 0.$$

From conditions (A1)–(A3) and (ii), we have

$$-\frac{c^2(t)P(x)f^2(x)}{2b(t)R(x)} \ge -\frac{c(t)}{b(t)}\frac{c(t)p_1}{r_0}\frac{f^2(x)}{2} \ge -\frac{c(t)p_1}{r_0}\delta_1\int_0^x f(u)du.$$

Hence,

$$V \ \ \ge \ \ dc(t)\int_0^x\left(1 - \frac{p_1\delta_1}{dr_0}\right)f(u)du + \frac{1}{2}\left(z + \frac{d}{P(x)}y\right)^2 + \frac{d(a_0q_0-d)}{2P^2(x)}y^2 \ge \frac{\delta_4\delta_0}{2}x^2 + \frac{1}{2}\left(z + \frac{d}{P(x)}y\right)^2 + \frac{d(a_0q_0-d)}{2P^2(x)}y^2,$$

where $\delta_4 = dc_0\left(1 - \frac{p_1\delta_1}{dr_0}\right) > 0$ by (i). So we can find a constant $d_0 > 0$ small enough, such that

$$V \ge d_0(x^2 + y^2 + z^2). \tag{3.5}$$

It is clear that $V(t,x,y,z) \geq 0$ and $V(t,0,0,0) = 0$ if and only if $x^2 = y^2 = z^2 = 0$. Now contidions (A2) and (iv) imply

$$\begin{aligned}
\int_0^t \gamma(s)ds &\leq (1+r_1+q_1)\int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|P'(u)|}{P^2(u)}du + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|R'(u)|+|Q'(u)|}{P^2(u)}du \\
&\leq \frac{(1+r_1+q_1)}{p_0^2}\int_{\alpha_1(t)}^{\alpha_2(t)} |P'(u)|du + \frac{1}{p_0}\int_{\alpha_1(t)}^{\alpha_2(t)} (|R'(u)|+|Q'(u)|)du \\
&\leq N < \infty, \qquad (3.6)
\end{aligned}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$ and $\alpha_2(t) = \max\{x(0), x(t)\}$. Hence,

$$W \geq D_0(x^2+y^2+z^2) \qquad (3.7)$$

for some $D_0 > 0$. Also, from (A1)–(A3), it is not difficult to see that

$$W \leq D_1(x^2+y^2+z^2), \qquad (3.8)$$

for all x, y, and z.
From (3.7), and (3.8), it is easy to see that $W(t,x,y,z) = 0$ if and only if $x^2+y^2+z^2 = 0$ for all $t \geq 0$, and $W(t,x,y,z) > 0$ if $x^2+y^2+z^2 \neq 0$. Now, we illustrate that $\dot{W}$ is a negative definite function. The derivative of the function V along any solution $(x(t), y(t), z(t))$ of system (1.6), with respect to t is after rearranging

$$\begin{aligned}
\frac{d}{dt}V(t) &= \left[\frac{da'(t)Q(x)+2c(t)P(x)f'(x)-2db(t)R(x)}{2P^2(x)}\right]y^2 \\
&+V_1(t)+V_2(t)+\frac{1}{P(x)}(d-a(t)Q(x))z^2+\sigma r(t)y^2(t)-\sigma(1-r'(t))\int_{t-r(t)}^{t} y^2(\eta)d\eta \\
&+c(t)\left(\frac{d}{P(x)}y+z\right)\int_{t-r(t)}^{t}\frac{1}{P(x)}f'(x)yd\eta+\left(\frac{d}{P(x)}y+z\right)p(t,x,y,z)
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= dc'(t)F(x)+c'(t)yf(x)+\frac{b'(t)R(x)}{2P(x)}y^2 \\
V_2 &= -d\theta_1(t)\left(yz+\frac{a(t)Q(x)}{2P(x)}y^2\right)+\frac{b(t)}{2}\theta_3(t)y^2-a(t)\theta_2(t)\left(yz+\frac{d}{2P(x)}y^2\right).
\end{aligned}$$

By regarding conditions (A2), (A3), and (ii), we have the following

$$da'(t)Q(x)+2c(t)P(x)f'(x)-2db(t)R(x) \leq da'(t)Q(x)+2c(t)P(x)\delta_1-2db(t)r_0 \leq da'(t)Q(x)+2b(t)(p_1\delta_1-dr_0).$$

From (A1), (A2), (i) and (iii), and using the inequality $2ab \leq a^2+b^2$, we can rearrange

$$\begin{aligned}
V'(t) &\leq V_1(t)+V_2(t)-\left(\frac{\varepsilon}{p_1^2}-\sigma r(t)-\frac{d\delta_1 c_1 r(t)}{2p_0^2}\right)y^2-\left(\frac{1}{p_1}(a_0q_0-d)-\frac{\delta_1 c_1 r(t)}{2p_0}\right)z^2 \\
&+\left(\frac{\delta_1 c_1 p_0+d\delta_1 c_1}{2p_0^2}-\sigma(1-\lambda)\right)\int_{t-r(t)}^{t} y^2(\eta)d\eta+\left(\frac{d}{P(x)}|y|+|z|\right)|p(t,x,y,z)| \qquad (3.9)
\end{aligned}$$

By choosing $\sigma = \frac{\delta_1 c_1 p_0+d\delta_1 c_1}{2p_0^2(1-\lambda)}$, we have

$$\begin{aligned}
V'(t) &\leq -\left(\frac{\varepsilon}{p_1^2}-\frac{\delta_1 c_1 p_0+d\delta_1 c_1(2-\lambda)}{2p_0^2(1-\lambda)}\Omega\right)y^2-\left(\frac{1}{p_1}(a_0q_0-d)-\frac{\delta_1 c_1}{2p_0}\Omega\right)z^2 \\
&+V_1(t)+V_2(t)+\left(\frac{d}{P(x)}|y|+|z|\right)|p(t,x,y,z)|. \qquad (3.10)
\end{aligned}$$

We claim that $V_1(t) \leq 0$. To show this we distinguish two cases. If $c'(t) = 0$, then $V_1 = \frac{b'(t)R(x)}{2P(x)}y^2 \leq 0$.
If $c'(t) < 0$, then we can write

$$\begin{aligned}
V_1(t) &= dc'(t)\left[F(x)+\frac{1}{d}yf(x)+\frac{b'(t)R(x)}{2dP(x)c'(t)}y^2\right] \\
&= dc'(t)\left[F(x)+\frac{b'(t)R(x)}{2dP(x)c'(t)}\left\{y+\frac{c'(t)P(x)f(x)}{b'(t)R(x)}\right\}^2-\frac{c'(t)P(x)f^2(x)}{2db'(t)R(x)}\right],
\end{aligned}$$

from which condition (ii) implies

$$\begin{aligned} V_1(t) &\leq dc'(t)\int_0^x\left(1-\frac{P(x)f'(u)}{dR(x)}\right)f(u)du \\ &\leq dc'(t)\int_0^x\left(1-\frac{p_1\delta_1}{dr_0}\right)f(u)du \\ &\leq c'(t)\frac{\delta_4}{c_0}F(x)\leq 0. \end{aligned}$$

Combining the two cases, we have $V_1(t)\leq 0$.
Using the inequality $2ab\leq a^2+b^2$, we obtain the estimate

$$\begin{aligned} V_2 &\leq \left[\frac{d}{2}|\theta_1(t)|\left(1+\frac{a_1q_1}{p_0}\right)+\frac{a_1}{2}|\theta_2(t)|\left(1+\frac{d}{p_0}\right)\right](y^2+z^2)+\frac{b_1}{2}|\theta_3(t)|y^2 \\ &\leq k_1(|\theta_1(t)|+|\theta_2(t)|+|\theta_3(t)|)(y^2+z^2), \end{aligned}$$

where $k_1=\max\left\{\frac{d}{2}\left(1+\frac{a_1q_1}{p_0}\right),\frac{a_1}{2}\left(1+\frac{d}{p_0}\right),\frac{b_1}{2}\right\}$. Using these estimates for V_1 and V_2 in (3.10), we obtain

$$V'(t)\leq -D_2\left(y^2+z^2\right)+k_1(|\theta_1(t)|+|\theta_2(t)|+|\theta_3(t)|)(y^2+z^2)+\left(\frac{d}{P(x)}y+z\right)p(t,x,y,z) \tag{3.11}$$

where $D_2=\min\left\{\frac{\varepsilon}{p_1^2}-\frac{\delta_1c_1p_0+d\delta_1c_1(2-\lambda)}{2p_0^2(1-\lambda)}\Omega,\ \frac{1}{p_1}(a_0q_0-d)-\frac{\delta_1c_1}{2p_0}\Omega\right\}$.
From (A4), (3.5), (3.6), (3.7), (3.11) and the Cauchy Schwartz inequality, we get

$$\begin{aligned} \dot{W}_{(2)} &= \left(\dot{V}_{(2)}-\frac{1}{\eta}\gamma(t)V\right)exp\left(\frac{-1}{\eta}\int_0^t\gamma(s)ds\right) \\ &\leq \left(-D_2\left(y^2+z^2\right)+\left(\frac{d}{P(x)}y+z\right)p(t,x,y,z)\right)exp\left(\frac{-1}{\eta}\int_0^t\gamma(s)ds\right) \\ &\leq \left(\frac{d}{p_0}|y|+|z|\right)|p(t,x,y,z)| \\ &\leq D_3\left(2+y^2+z^2\right)|e(t)| \\ &\leq D_3\left(2+\frac{1}{D_0}W\right)|e(t)| \\ &\leq 2D_3|e(t)|+\frac{D_3}{D_0}W|e(t)|, \end{aligned} \tag{3.12}$$

where $D_3=\max\left\{\frac{d}{p_0},1\right\}$, $\eta=\frac{d_0}{k_1}$. Using the Gronwall inequality and the condition (v) and integrating inequalty (3.12) from 0 to t, we have

$$\begin{aligned} W &\leq W(0,x(0),y(0),z(0))+2D_3\eta_2+\frac{D_3}{D_0}\int_0^t W(s,x(s),y(s),z(s))|e(s)|ds \\ &\leq (W(0,x(0),y(0),z(0))+2D_3\eta_2)\,exp\left(\frac{D_3}{D_0}\int_0^t|e(s)|ds\right) \\ &\leq (W(0,x(0),y(0),z(0))+2D_3\eta_2)\,exp\left(\frac{D_3}{D_0}\eta_2\right)=K_1<\infty \end{aligned} \tag{3.13}$$

Because of inequalities (3.7) and (3.13), we write

$$\left(x^2+y^2+z^2\right)\leq\frac{1}{D_0}W\leq K_2, \tag{3.14}$$

where $K_2=\frac{K_1}{D_0}$. Clearly (3.14) imlies that

$$|x(t)|\leq\sqrt{K_2},\ \ |y(t)|\leq\sqrt{K_2},\ \ |z(t)|\leq\sqrt{K_2}\text{ for all }t\geq 0.$$

That is

$$|x(t)|\leq\sqrt{K_2},\ \ \left|x'(t)\right|\leq\sqrt{K_2},\ \ \left|x''(t)\right|\leq\sqrt{K_2}\ \text{ for all }\ t\geq 0 \tag{3.15}$$

which completes the proof boundedness solutions of equation (1.5).
Now we show that the solutions of equation (1.5) for $p(t,x,x',x'')\equiv 0$ is uniformly asymtotically stability. The inequality (3.12) can write as

$$\begin{aligned} \dot{W}_{(2)} &= \left(\dot{V}_{(2)}-\frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds} \\ &\leq -D_2\left(y^2+z^2\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds} \\ &\leq -\mu(y^2+z^2), \end{aligned}$$

where $\mu=D_2e^{-\frac{N}{\eta}}$. It can also be observed that the unique solution of system (1.6) for which $\dot{W_{(2)}}(t,x,y,z,w)=0$ is the solution $x=y=z=0$.
Due to the the above discussion, the trival solution of the equation system (1.6) is uniformly asymptotically stable. □

Example 3.2. *We consider the following third order non-autonomous nonlinear differential equation with delay*

$$
\begin{aligned}
&\left[\left(\frac{x\cos x}{7(1+x^6)}+2\right)x'\right]''+\left(\frac{e^{-t}\cos t}{4}+\frac{1}{2}\right)\left(\left(\frac{x^2\sin x}{7(1+x^6)}+3\right)x'\right)'+\left(\frac{1}{2+t^6}+1\right)\left(\frac{x}{2(e^{2x}+e^{-2x})}+\frac{21}{10}\right)x'\\
&+\frac{1}{50}\left(\frac{1}{3+t^6}+\frac{1}{4}\right)\left(x(t-\frac{1}{e^t+100})+\frac{x(t-\frac{1}{e^t+100})}{1+x^6(t-\frac{1}{e^t+100})}\right)\\
&=\frac{2\sin t}{t^2+1+x^2+(x'x'')^2}
\end{aligned}
\tag{3.16}
$$

where $P(x)=\frac{x\cos x}{7(1+x^6)}+2$, $Q(x)=\frac{x^2\sin x}{7(1+x^6)}+3$, $R(x)=\frac{x}{2(e^{2x}+e^{-2x})}+\frac{21}{10}$, $f(x)=x+\frac{x}{1+x^6}$, $r(t)=\frac{1}{e^t+100}$, $a(t)=\frac{e^{-t}\cos t}{4}+\frac{1}{2}$, $b(t)=\frac{1}{2+t^6}+1$, $c(t)=\frac{1}{3+t^6}+\frac{1}{4}$, $p(t)=\frac{2\sin t}{t^2+1+x^2+(x'x'')^2}$.
It is easy to see that $p_0=1$, $p_1=3$, $q_0=1$, $q_1=3$, $r_0=2$, $r_1=\frac{7}{3}$, $a_0=0.25$, $a_1=0.75$, $b_0=1$, $b_1=1.5$, $c_0=0.25$, $c_1=\frac{7}{12}$

$$\delta_0=\frac{1}{50}\le\frac{f(x)}{x}=\frac{1}{50}\left(1+\frac{1}{1+x^2}\right)\quad for\quad x\neq 0,\quad |f'(x)|\le\frac{1}{25}=\delta_1,$$

$$\frac{p_1\delta_1}{r_0}=\frac{3}{50}<d<\frac{1}{4}=a_0q_0,$$

and

$$\frac{1}{2}da'(t)Q(x)-b_0(dr_0-p_1\delta_1)\le-\frac{d}{8}+\frac{3}{25}<0\quad for\quad d=\frac{1}{10}.$$

Also we have

$$\int_0^{+\infty}|a'(t)|dt=\int_0^{+\infty}\left|\frac{-e^{-t}\cos t-e^{-t}\sin t}{4}\right|dt\le\int_0^{+\infty}\frac{2}{4}e^{-t}dt=\frac{1}{2},$$

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|P'(u)|du &= \frac{1}{7}\int_{-\infty}^{+\infty}\left|\frac{(\cos u-u\sin u)(1+u^6)-6u^6\cos u}{(1+u^6)^2}\right|du\\
&= \frac{1}{7}\int_{-\infty}^{+\infty}\left|\frac{\cos u}{1+u^6}-\frac{u\sin u}{1+u^6}-\frac{6u^6\cos u}{(1+u^6)^2}\right|du\\
&\le \frac{1}{7}\int_{-\infty}^{+\infty}\left[\frac{7}{1+u^6}+\frac{u^2}{1+u^6}\right]du\\
&= \frac{5}{7}\pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|Q'(u)|du &= \frac{1}{7}\int_{-\infty}^{+\infty}\left|\frac{(2u\sin u+u^2\cos u)(1+u^6)-6u^7\sin u}{(1+u^6)^2}\right|du\\
&= \frac{1}{7}\int_{-\infty}^{+\infty}\left|\frac{2u\sin u}{1+u^6}+\frac{u^2\cos u}{1+u^6}-\frac{6u^7\sin u}{(1+u^6)^2}\right|du\\
&\le \frac{1}{7}\int_{-\infty}^{+\infty}\left[\frac{3u^2}{1+u^6}+\frac{6u^8}{(1+u^6)^2}\right]du\\
&= \frac{2}{7}\pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{+\infty}|R'(u)|du &= \frac{1}{2}\int_{-\infty}^{+\infty}\left|\frac{(e^{2u}+e^{-2u})-2u(e^{2u}-e^{-2u})}{(e^{2u}+e^{-2u})^2}\right|du\\
&= \frac{1}{2}\int_0^{+\infty}\left[\frac{1}{e^{2u}+e^{-2u}}+2u\frac{e^{2u}-e^{-2u}}{(e^{2u}+e^{-2u})^2}\right]du+\frac{1}{2}\int_{-\infty}^{0}\left[\frac{1}{e^{2u}+e^{-2u}}+2u\frac{e^{2u}-e^{-2u}}{(e^{2u}+e^{-2u})^2}\right]du\\
&= \frac{\pi}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^{+\infty}|p(t,x,x',x'')|dt &\le \int_0^{+\infty}|\frac{2\sin t}{t^2+1+x^2+(x'x'')^2}|dt\\
&\le \int_0^{+\infty}\left|\frac{2\sin t}{t^2+1}\right|dt\\
&\le \int_0^{+\infty}\frac{2}{t^2+1}dt\\
&= \pi
\end{aligned}
$$

As a result, all of Theorem assumptions hold, indicating that any solution $x(t)$ equation (3.16) are bounded and trival solution of equation (3.16) for $p(t,x,x',x'')\equiv 0$ is uniformly asymtotically stability.

4. Conclusion

For the asymptotic stability of solutions of a class of nonlinear differential equation systems with bounded delay is obtained new sufficient conditions using a theorem presented in this paper. Since the special cases of our equation are the studies done in the literature, our results include the present results. The effectiveness of the theorem is demonstrated using an example.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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