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Research Article

Inverse Sturm-Liouville problem with conformable derivative and transmission conditions

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Abstract

In this paper, we study the inverse problem for Sturm-Liouville problem with conformable fractional differential operators of order α, $0.5 < \alpha \leq 1$ and finite number of interior discontinuous conditions. For this aim first, the asymptotic formulas for solutions, eigenvalues and eigenfunctions of the problem are calculated. Then some uniqueness theorems for proposed inverse eigenvalue problem are proved. Finally, the Hald's theorem for conformable Sturm-Liouville problem is developed.

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1. Introduction

Sturm-Liouville equation is one of the most important problems in mathematics, physics and engineering. This problem arises in modeling of many systems in vibration theory, quantum mechanics, hydrodynamic and so on [6, 20, 27]. The classical Sturm-Liouville equation is a second order ordinary differential equation of the following form:

$$y'' + (\lambda - q(x))y = 0, \quad 0 < x < \pi, \tag{1.1}$$

where $q(x)$ is the potential function and λ is a parameter. For equation (1.1) two boundary conditions at end points are considered. Equation (1.1) with boundary conditions are called Sturm-Liouville problems (SLP). The value of λ for which SLP has a nontrivial solution is called an eigenvalue and the corresponding nontrivial solution $y(x)$ is called an eigenfunction. It is proved that a SLP has an infinite sequence of eigenvalues [6, 13]. The set of all eigenvalues is called spectrum. Some related problems to classical Sturm-Liouville problem can be found in [6, 13, 17, 18, 24–26, 28, 29] and references therein. In studying fractional model of SLP the integer order derivatives are replaced with a fractional derivative such as Riemann-Liouville, Caputo, Caputo-Fabrizio and Conformable derivatives. See [1, 5, 12, 21] for more details. In recent years, the study of fractional model

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of SLP grew rapidly and some results of integer order are extended for fractional case, see [10, 14, 16, 22, 30] and references therein. Asymptotic forms of eigenvalues and inverse problem for fractional SLP with Riemann-Liouville and Caputo derivatives are studied in [10, 23]. In [10, 30] the eigenvalues and eigenfunctions are approximated numerically. In [16] the authors studied conformable Sturm-Liouville problem (CSLP) with regular boundary conditions. So, the asymptotic form of eigendata and related inverse problem using trace formula and nodal points are considered. More studies on CSLP can be found in [2, 4, 7, 19]. In this paper, we consider CSLP with finite number of transmission conditions. In section 3, we find asymptotic formulas of eigendata and prove some uniqueness results related to inverse problem of CSLP using Weyl m-function. Also, we improve and develop the Hoschdat-Libermann type results for proposed problem.

2. Preliminaries

In this section, we give definition and some theorems of the conformable fractional (CF) derivative such that one can found in [1, 11]. In what follows, we always take $D_x^\alpha = D^\alpha$.

Definition 2.1. For the function $f:[0,\infty)\to\mathbb{R}$, the CF derivative of f of order $\alpha\in(0,1]$ defined by:

$$D^\alpha f(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon},$$

for all $x>0$, and

$$D^\alpha f(0) = \lim_{x\to 0^+} D^\alpha f(x).$$

If f is a differentiable function, then

$$D^\alpha f(x) = x^{1-\alpha} f'(x).$$

If $D^\alpha f(x_0)$ exists and is finite. Then the function f is α-differentiable at x_0.

Definition 2.2. The conformable integral of function f of order α is defined as:

$$J^\alpha f(x) = \int_0^x f(s)\mathrm{d}_\alpha s = \int_0^x s^{\alpha-1} f(s)\mathrm{d}s, \quad x>0.$$

where, the integrals are in Riemann setting.

Theorem 2.3. *Let f and g be arbitrary α-differentiable functions at $x>0$. Then the following results hold:*

(1) $D^\alpha(af+bg) = aD^\alpha f + bD^\alpha g, \quad \forall a,b\in\mathbb{R}$.
(2) $D^\alpha(x^p) = px^{p-\alpha}, \quad \forall p\in\mathbb{R}$.
(3) $D^\alpha(c)=0$ *for any constant* c.
(4) $D^\alpha(fg) = D^\alpha(f)g + fD^\alpha(g)$.
(5) $D^\alpha(f/g) = \dfrac{D^\alpha(f)g - fD^\alpha(g)}{g^2}$, *s.t.* $g\neq 0$.
(6) *For a continuous function* $g:[0,\infty)\to\mathbb{R}$, *we have* $D^\alpha J^\alpha g(x) = g(x)$, *for all* $x>0$.
(7) *For a differentiable function* $g:[0,\infty)\to\mathbb{R}$, *we have* $J^\alpha D^\alpha g(x) = g(x)-g(0)$, *for all* $x>0$.

Theorem 2.4. *If $g,h:[0,\infty)\to\mathbb{R}$ are an arbitrary α-differentiable functions and $f(x) = g(h(x))$, then f is α-differentiable and*

(1) $(D^\alpha f)(x) = (D^\alpha g)(h(x))(D^\alpha h)(x)h(x)^{\alpha-1}, \quad x\neq 0,\ h(x)\neq 0,$
(2) $(D^\alpha f)(0) = \lim_{x\to 0^+}(D^\alpha g)(h(x))(D^\alpha h)(x)h(x)^{\alpha-1}$.

Definition 2.5. For a real number $1 \le p < \infty$ and $\alpha > 0$, the space $L_p^\alpha(0,a)$ is defined by

$$L_p^\alpha(0,a) = \left\{ f : [0,a] \to \mathbb{R},\ \left(\int_0^a |g(t)|^p d_\alpha t \right)^{1/p} < \infty \right\}.$$

Theorem 2.6. *For two α-differentiable functions $g, h : [a,b] \to \mathbb{R}$, the α-integration by parts has the following form*

$$\int_a^b g(x) D^\alpha h(x) \mathrm{d}_\alpha x = g(x)h(x)\big|_a^b - \int_a^b h(x) D^\alpha g(x) \mathrm{d}_\alpha x. \tag{2.1}$$

3. Main problem and spectral properties

In this section, we define the CSLP with transmission conditions as a main problem of the paper and find some spectral properties. For $\frac{1}{2} < \alpha \le 1$ and $q \in L_1^\alpha[0,\pi]$, we consider CSLP

$$\ell_\alpha y := -D^\alpha D^\alpha y + qy = \lambda y \tag{3.1}$$

with boundary conditions

$$\begin{aligned} B_1(y) &:= D^\alpha y(0) + h\, y(0) = 0, \\ B_2(y) &:= D^\alpha y(\pi) + H\, y(\pi) = 0, \end{aligned} \tag{3.2}$$

and finite number of transmission conditions

$$\begin{aligned} U_i(y) &:= y(d_i+) - a_i y(d_i-) = 0, \\ V_i(y) &:= D^\alpha y(d_i+) - b_i D^\alpha y(d_i-) - c_i y(d_i-) = 0, \end{aligned} \tag{3.3}$$

for $i = 1, 2, \dots, m-1$ and $m \ge 2$. The parameters h, H and a_i, b_i, c_i, d_i are real numbers. We denote the problem (3.1)-(3.3) with $L_\alpha = L_\alpha(q(x); h; H; d_i)$. Consider the weighted inner product

$$\langle f, g \rangle_T := \int_0^\pi f(t)\overline{g(t)} r(t) \mathrm{d}_\alpha t,$$

where $f, g \in L_2^\alpha((0,\pi); r)$ and $r(t)$ is the weight function

$$r(t) = \begin{cases} 1, & 0 \le t < d_1, \\ \frac{1}{a_1 b_1}, & d_1 < t < d_2, \\ \vdots & \\ \frac{1}{a_1 b_1 \cdots a_{m-1} b_{m-1}}, & d_{m-1} < t \le \pi. \end{cases}$$

Note that $T := L_\alpha^2((0,\pi); r)$ is a Hilbert space with the norm $\|f\|_T = \langle f, f \rangle_T^{1/2}$. For $\alpha \in (0.5, 1]$, let $A_\alpha : T \to T$ with domain

$$\operatorname{dom}(A_\alpha) = \left\{ f \in T \;\middle|\; \begin{array}{c} f, D^\alpha f \in AC(\cup_0^{m-1} (d_i, d_{i+1})), \\ \ell_\alpha f \in L_2^\alpha(0,\pi),\ U_i(f) = V_i(f) = 0 \end{array} \right\}$$

by

$$A_\alpha f = \ell_\alpha f, \quad f \in \operatorname{dom}(A_\alpha).$$

Here $AC(\cup_0^{m-1} (d_i, d_{i+1}))$ indicates the set of all absolutely continuous functions in $\cup_0^{m-1}(d_i, d_{i+1})$. Suppose f and g are two solutions of the linear differential equation $\ell_\alpha f = \lambda f$, $\ell_\alpha g = \lambda g$ satisfying (3.3), the modified Wronskian

$$W_\alpha(f,g) = r(x)\big(f(x) D^\alpha g(x) - D^\alpha f(x) g(x)\big) \tag{3.4}$$

is constant for all $x \in [0, d_1) \cup_1^{m-2} (d_i, d_i + 1) \cup (d_{m-1}, \pi]$. Using the above formula $W_\alpha(f,g)(x) = W_\alpha(f,g)(x_0)$, for $x_0 \in [0,d) \cup (d,\pi]$. So, $W_\alpha(f,g)$ does not depend on x.

Lemma 3.1. *For $\alpha \in (0.5, 1]$, the operator A_α is self-adjoint on $L_2^\alpha((0,\pi); r)$.*

Proof. By employing twice the α-integration by part, Theorem 2.6, one can write

$$\langle \ell_\alpha f, g\rangle_T = W_\alpha(f,g)\big|_{x=\pi} - W_\alpha(f,g)\big|_{x=0} + \langle f, \ell_\alpha g\rangle_T.$$

It follows from the conditions (3.2) and (3.3),

$$W_\alpha(f,g)\big|_{x=\pi} - W_\alpha(f,g)\big|_{x=0} = 0.$$

Thus A_α is a self-adjoint operator. □

For every function $h \in \operatorname{dom}(A_\alpha)$, we will denote by h_j, $1 \le j \le m$, the restriction of h to the interval (d_{j-1}, d_j). We define $h_j(d_{j-1}) = h(d_{j-1}+0)$ and $h_j(d_j) = h(d_j - 0)$.

Let $u(x,\lambda)$ and $v(x,\lambda)$ be the solutions of (3.1) with the following initial conditions

$$u(0,\lambda) = 1, \quad D^\alpha u(0,\lambda) = -h, \tag{3.5}$$

and

$$v(\pi,\lambda) = 1, \quad D^\alpha v(\pi,\lambda) = -H, \tag{3.6}$$

and the jump conditions (3.3), respectively. In the interval $[0, d_1)$ equation (3.1) has a uniqu solution $u_1(x,\lambda)$ which satisfies conditions (3.5) and is an entire function with respect to parameter λ. Similarly in the interval $(d_{m-1}, \pi]$ Equations (3.1) has a unique entire solution $v_m(x,\lambda)$ which satisfies (3.6). The characteristic function is defined by

$$\Delta(\lambda) := W_\alpha(u(\lambda), v(\lambda)) = B_1(v(\lambda)) = -r(\pi)B_2(u(\lambda)). \tag{3.7}$$

The function $\Delta(\lambda)$ is an entire function and the eigenvalues λ_n of L_α are the zeros of $\Delta(\lambda)$. We have the following relation between the eigenfunctions $u_i(x,\lambda_n)$ and $v_i(x,\lambda_n)$ corresponding to the eigenvalues λ_n:

$$v_i(x,\lambda_n) = \beta_n u_i(x,\lambda_n), \tag{3.8}$$

from (3.5),

$$\beta_n = v(0,\lambda_n). \tag{3.9}$$

Also, we define the norming constant by

$$\gamma_n := \|u(x,\lambda_n)\|_T^{-2}. \tag{3.10}$$

In the following lemma, we find derivative of characteristic function with respect to the parameter λ.

Lemma 3.2. *All zeros λ_n of $\Delta(\lambda)$ are simple and we have*

$$\dot{\Delta}(\lambda_n) := \frac{d}{d\lambda}\Delta(\lambda_n) = -\gamma_n^{-1}\beta_n, \tag{3.11}$$

β_n and γ_n are defined in (3.9) *and* (3.10)*.*

Proof. Since

$$\ell_\alpha u(x,\lambda_n) = \lambda_n u(x,\lambda_n) \text{ and } \ell_\alpha v(x,\lambda) = \lambda v(x,\lambda),$$

using Theorem 2.3, we get

$$D^\alpha W_\alpha(v(x,\lambda), u(x,\lambda_n)) = (\lambda - \lambda_n) v(x,\lambda) u(x,\lambda_n).$$

Using conformable integrating of order α over $[0,\pi]$, (3.2) and (3.3), we obtain

$$(\lambda - \lambda_n)\int_0^\pi v(t,\lambda)u(t,\lambda_n)r(t)\mathrm{d}_\alpha t = W_\alpha(v(x,\lambda), u(x,\lambda_n))|_0^\pi = -\Delta(\lambda).$$

For $\lambda \to \lambda_n$, and using $\Delta(\lambda_n) = 0$, we get

$$\int_0^\pi v(t,\lambda_n)u(t,\lambda_n)r(t)\mathrm{d}_\alpha t = -\dot{\Delta}(\lambda_n),$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$. Using (3.8) and (3.10), we arrive at (3.11). □

Finally, we have the following simple unitary transformation for the problem (3.1)-(3.3).

Remark 3.3. Without losing of generality of the problem (3.1)-(3.3), by [25, Lemma 2.3], we can take $a_i b_i = 1$, for $i = 1, 2, \dots, m$.

3.1. Asymptotic formulas for eigendata

In this subsection, we study the asymptotic forms of solutions and eigenvalues. For these purposes, we prove some Lemmas and Theorems as follows.

Theorem 3.4. *Let $\lambda = \rho^2$ and $\tau := |\mathrm{Im}\rho|$. For CSLP (3.1)-(3.3) as $|\lambda| \to \infty$, the asymptotic forms of solutions and the characteristic function formula are in the following forms:*

$$u(x,\lambda) = \begin{cases} \cos\left(\frac{\rho}{\alpha}x^\alpha\right) + O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0 \le x < d_1, \\ \alpha_1 \cos\left(\frac{\rho}{\alpha}x^\alpha\right) + \alpha_1' \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right) + O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_1 < x < d_2, \\ \alpha_1\alpha_2 \cos\rho\left(\frac{\rho}{\alpha}x^\alpha\right) + \alpha_1'\alpha_2 \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right) + \alpha_1\alpha_2' \cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_2^\alpha)\right) & \\ \quad +\alpha_1'\alpha_2' \cos\left(\frac{\rho}{\alpha}(x^\alpha + 2d_1^\alpha - 2d_2^\alpha)\right) + O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_2 < x < d_3, \\ \quad \vdots & \\ \alpha_1\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}x^\alpha\right) + & \\ \quad +\alpha_1'\alpha_2\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right) + \cdots & \\ \quad +\alpha_1\alpha_2\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_{m-1}^\alpha)\right) + & \\ \quad +\alpha_1'\alpha_2'\alpha_3 ... \alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha + 2d_1^\alpha - 2d_2^\alpha)\right) + \cdots & \\ \quad +\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha + 2d_j^\alpha - 2d_k^\alpha)\right) & \\ \quad +\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha - 2d_j^\alpha + 2d_k^\alpha - 2d_s^\alpha)\right) + \cdots & \\ \quad +\alpha_1'\alpha_2'\dots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha + 2(-1)^{m-1}d_1^\alpha + 2(-1)^{m-2}d_2^\alpha - 2d_m^\alpha)\right) & \\ \quad +O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), \quad d_{m-1} < x \le \pi, & \end{cases} \tag{3.12}$$

$$D_x^\alpha u(x,\lambda) = \begin{cases} \rho\left[-\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right] + O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0 \le x < d_1, \\ \rho\left[-\alpha_1\sin\left(\frac{\rho}{\alpha}x^\alpha\right) - \alpha_1'\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right)\right] + O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_1 < x < d_2, \\ \rho\left[-\alpha_1\alpha_2\sin\rho\left(\frac{\rho}{\alpha}x^\alpha\right) - \alpha_1'\alpha_2\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right) - \alpha_1\alpha_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_2^\alpha)\right)\right. & \\ \quad \left.-\alpha_1'\alpha_2'\sin\left(\frac{\rho}{\alpha}(x^\alpha + 2d_1^\alpha - 2d_2^\alpha)\right)\right] + O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_2 < x < d_3, \\ \quad \vdots & \\ \rho\left[-\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}x^\alpha\right)\right. & \\ \quad -\alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_1^\alpha)\right) + \cdots & \\ \quad -\alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_{m-1}^\alpha)\right) + & \\ \quad -\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha + 2d_1^\alpha - 2d_2^\alpha)\right) + \cdots & \\ \quad -\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha + 2d_j^\alpha - 2d_k^\alpha)\right) & \\ \quad -\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(x^\alpha - 2d_j^\alpha + 2d_k^\alpha - 2d_s^\alpha)\right) + \cdots & \\ \quad \left.+ -\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(x^\alpha + 2(-1)^{m-1}d_1^\alpha + 2(-1)^{m-2}d_2^\alpha - 2d_m^\alpha)\right)\right] & \\ \quad +O\left(\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), \quad d_{m-1} < x \le \pi, & \end{cases} \tag{3.13}$$

where

$$\alpha_i = \frac{a_i + b_i}{2} \quad \text{and} \quad \alpha_i' = \frac{a_i - b_i}{2}, \quad i = 1, 2, \dots, m-1. \tag{3.14}$$

Also the similar asymptotic forms hold for the solutions $v, \tilde{v}$ and $\tilde{u}$. Moreover, we have

$$\begin{aligned} \Delta(\lambda) =& \rho w(\pi)\left[\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}\pi^\alpha\right) + \alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha - 2d_1^\alpha)\right) + \cdots\right. \\ &+ \alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^\alpha - 2d_{m-1}^\alpha)\right) + \alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha + 2d_1^\alpha - 2d_2^\alpha)\right) \\ &+ \cdots + \alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha + 2d_j^\alpha - 2d_k^\alpha)\right) \\ &+ \alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_s'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^\alpha - 2d_j^\alpha + 2d_k^\alpha - 2d_s^\alpha)\right) + \cdots \end{aligned}$$

$$+\alpha_1'\alpha_2'\ldots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\right)\Bigg]$$
$$+O\left(\exp\left(\frac{\tau}{\alpha}\pi^\alpha\right)\right). \tag{3.15}$$

Proof. Let $\mathcal{S}(x,\lambda)$ and $\mathcal{C}(x,\lambda)$ be the solutions of (3.1) and (3.3) with the following conditions

$$\mathcal{S}(0,\lambda)=0,\ D^\alpha\mathcal{S}(0,\lambda)=1,\ \mathcal{C}(0,\lambda)=1,\ \text{and}\ D^\alpha\mathcal{C}(0,\lambda)=0. \tag{3.16}$$

Using (3.3) for $\mathcal{C}(x,\lambda)$, we obtain

$$\mathcal{C}(x,\lambda)=\begin{cases}\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\rho}\exp\frac{|\tau|}{\alpha}x^\alpha\right), & 0\le x<d_1,\\ a_1\mathcal{C}_1(d_1,\lambda)\cos\left(\frac{\rho}{\alpha}(x^\alpha-d_1^\alpha)\right)+\frac{b_1}{\rho}\mathcal{C}_1'(d_1,\lambda)\sin\left(\frac{\rho}{\alpha}(x^\alpha-d_1^\alpha)\right) & \\ \qquad +O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}(x^\alpha-d_1^\alpha)\right), & d_1<x<d_2,\\ a_2\mathcal{C}_2(d_2,\lambda)\cos\left(\frac{\rho}{\alpha}(x^\alpha-d_2^\alpha)\right)+\frac{b_2}{\rho}\mathcal{C}_2'(d_2,\lambda)\sin\left(\frac{\rho}{\alpha}(x^\alpha-d_2^\alpha)\right) & \\ \qquad +O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}(x^\alpha-d_2^\alpha)\right), & d_2<x<d_3,\\ \vdots & \\ a_{m-1}\mathcal{C}_{m-1}(d_{m-1},\lambda)\cos\left(\frac{\rho}{\alpha}(x^\alpha-d_{m-1}^\alpha)\right)+ & \\ \quad +\frac{b_{m-1}}{\rho}\mathcal{C}_{m-1}'(d_{m-1},\lambda)\sin\left(\frac{\rho}{\alpha}(x^\alpha-d_{m-1}^\alpha)\right)+ & \\ \quad +O\left(\frac{1}{\rho}\exp\frac{\tau}{\alpha}(x^\alpha-d_{m-1}^\alpha)\right), & d_{m-1}<x\le\pi.\end{cases}$$

So, we insert the k'th statement into the $(k+1)$'th statement to get

$$\mathcal{C}(x,\lambda)=\begin{cases}\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & 0\le x<d_1,\\ \alpha_1\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+\alpha_1'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_1<x<d_2,\\ \alpha_1\alpha_2\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+\alpha_1'\alpha_2\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+\alpha_1\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_2^\alpha)\right) & \\ \quad +\alpha_1'\alpha_2'\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)+O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_2<x<d_3,\\ \vdots & \\ \alpha_1\alpha_2\ldots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}x^\alpha\right)+ & \\ \quad +\alpha_1'\alpha_2\ldots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_1^\alpha)\right)+\cdots & \\ \quad +\alpha_1\alpha_2\ldots\alpha_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_{m-1}^\alpha)\right)+ & \\ \quad +\alpha_1'\alpha_2'\alpha_3\ldots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_1^\alpha-2d_2^\alpha)\right)+\cdots & \\ \quad +\alpha_1\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha+2d_j^\alpha-2d_k^\alpha)\right) & \\ \quad +\alpha_1\ldots\alpha_j'\ldots\alpha_k'\ldots\alpha_s'\ldots\alpha_{m-1}\cos\left(\frac{\rho}{\alpha}(x^\alpha-2d_j^\alpha+2d_k^\alpha-2d_s^\alpha)\right)+\cdots & \\ \quad +\alpha_1'\alpha_2'\ldots b_{m-1}'\cos\left(\frac{\rho}{\alpha}(x^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\right) & \\ \quad +O\left(\frac{1}{\rho}\exp\left(\frac{\tau}{\alpha}x^\alpha\right)\right), & d_{m-1}<x\le\pi,\end{cases}$$

where α_i and α_i' are defined in (3.14) and $j<k<s$, $j,k,s=1,2,\ldots,m-1$. Similarly, we can obtain the asymptotic formula for $\mathcal{S}(x,\lambda)$. Applying the Definition 2.1, we calculate the asymptotic form of $D^\alpha\mathcal{S}(x,\lambda)$ and $D^\alpha\mathcal{C}(x,\lambda)$. This completes the proof by using $u(x,\lambda)=\mathcal{C}(x,\lambda)+h\,\mathcal{S}(x,\lambda)$. □

Using Theorem 3.4 and Definition 2.1, we find

$$\begin{aligned}
|u(x,\lambda)| &= O\left(\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right),\\
|D^{\alpha}u(x,\lambda)| &= |x^{1-\alpha}u'(x,\lambda)| = O\left(|\rho|\exp\left(\frac{\tau}{\alpha}x^{\alpha}\right)\right),\ 0\le x\le \pi
\end{aligned}\tag{3.17}$$

By changing x to $\pi - x$ and using the jump conditions (3.3) and Definition 2.1, we calculate the asymptotic forms of $v(x,\lambda)$ and $D^{\alpha}v(x,\lambda)$. Specially,

$$\begin{aligned}
|v(x,\lambda)| &= O\left(\exp\left(\frac{\tau}{\alpha}(\pi-x)^{\alpha}\right)\right),\\
|D^{\alpha}v(x,\lambda)| &= |x^{1-\alpha}v'(x,\lambda)| = O\left(|\rho|\exp\left(\frac{\tau}{\alpha}(\pi-x)^{\alpha}\right)\right),\quad 0\le x\le \pi.
\end{aligned}\tag{3.18}$$

Define

$$\begin{aligned}
\Delta_{\circ}(\lambda) :=&\rho r(\pi)\Big[\alpha_1\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}\pi^{\alpha}\right)+\alpha_1'\alpha_2\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_1^{\alpha})\right)+\cdots\\
&+\alpha_1\alpha_2\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_{m-1}^{\alpha})\right)+\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2d_1^{\alpha}-2d_2^{\alpha})\right)\\
&+\cdots+\alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2d_i^{\alpha}-2d_j^{\alpha})\right)\\
&+\alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}-2d_i^{\alpha}+2d_j^{\alpha}-2d_k^{\alpha})\right)+\cdots\\
&+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\sin\left(\frac{\rho}{\alpha}(\pi^{\alpha}+2(-1)^{m-1}d_1^{\alpha}+2(-1)^{m-2}d_2^{\alpha}-2d_m^{\alpha})\right)\Big].
\end{aligned}\tag{3.19}$$

Let $\lambda_n=\rho_n^2$ and $\lambda_n^{\circ}=(\rho_n^{\circ})^2$ be the zeros of the functions (3.15) and (3.19), respectively, then

$$\rho_n=\rho_n^{\circ}+o(1),\qquad n\to\infty.\tag{3.20}$$

The roots of $\Delta_{\circ}(\lambda)$ are

$$\rho_n^{\circ}=\alpha\pi^{1-\alpha}n+\vartheta_n,$$

where $\sup_n \vartheta_n<M$ for all $n\in\mathbb{N}$. As a result of Valiron's theorem ([15, Thm. 13.4]) and (3.15), we obtain the following asymptotic form.

Theorem 3.5. *Let $\lambda_n=\rho_n^2$ be the eigenvalues of the problem L_{α}, then we have the following asymptotic formula*

$$\rho_n=\alpha\pi^{1-\alpha}n+O(1)\tag{3.21}$$

as $n\to\infty$.

Lemma 3.6. *The characteristic function $\Delta(\lambda)$ can be written in terms of eigenvalues λ_n, parameters a_i,b_i in (3.3) and the order α as follows*

$$\Delta(\lambda)=C\prod_{n=1}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n^{\circ}},\tag{3.22}$$

where $C=-\lambda_0\Omega\prod\limits_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n^{\circ}}$ and

$$\begin{aligned}
\Omega =&\frac{r(\pi)}{\alpha}\Big[\alpha_1\alpha_2\dots\alpha_{m-1}\pi^{\alpha}+\alpha_1'\alpha_2\dots\alpha_{m-1}(\pi^{\alpha}-2d_1^{\alpha})+\cdots\\
&+\alpha_1\alpha_2\dots\alpha_{m-1}'(\pi^{\alpha}-2d_{m-1}^{\alpha})+\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}(\pi^{\alpha}+2d_1^{\alpha}-2d_2^{\alpha})\\
&+\cdots+\alpha_1\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}(\pi^{\alpha}+2d_j^{\alpha}-2d_k^{\alpha})
\end{aligned}$$

$$+\alpha_1\dots\alpha'_j\dots\alpha'_k\dots\alpha'_s\dots\alpha_{m-1}(\pi^\alpha-2d_j^\alpha+2d_k^\alpha-2d_s^\alpha)+\cdots$$
$$+\alpha'_1\alpha'_2\dots\alpha'_{m-1}(\pi^\alpha+2(-1)^{m-1}d_1^\alpha+2(-1)^{m-2}d_2^\alpha-2d_m^\alpha)\Big].$$

Proof. By the Hadamard's factorization theorem [3, P. 289], the characteristic function $\Delta(\lambda)$ can be written in the form

$$\Delta(\lambda)=C\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right)=\frac{C(\lambda-\lambda_0)}{\lambda_0}\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right). \tag{3.23}$$

Using the Hadamard's factorization [15, Sec. 4.2] for the function $\Delta_\circ(\lambda)$ defined in (3.19), we obtain the infinite product

$$\Delta_\circ(\lambda)=\Omega\lambda\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n^\circ}\right),$$

Then

$$\frac{\Delta(\lambda)}{\Delta_\circ(\lambda)}=\frac{C(\lambda_0-\lambda)}{\lambda_0\Omega\lambda}\prod_{n=1}^{\infty}\frac{\lambda_n^\circ}{\lambda_n}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_n-\lambda_n^\circ}{\lambda_n^\circ-\lambda}\right).$$

Taking (3.15) and (3.20) into account, we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\Delta_\circ(\lambda)}=1,\qquad \lim_{\lambda\to-\infty}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_n-\lambda_n^\circ}{\lambda_n^\circ-\lambda}\right)=1$$

and hence

$$C=-\lambda_0\Omega\prod_{n=1}^{\infty}\frac{\lambda_n}{\lambda_n^\circ}.$$

Substituting this into (3.23), we arrive at (3.22). □

Example 3.7. Consider the following CSLP with one jump condition

$$\begin{aligned}&-D^\alpha D^\alpha y=\lambda y\\&D^\alpha y(0)=0,\quad y(\pi)=0,\\&y(\frac{\pi}{4}+)-2y(\frac{\pi}{4}-)=0,\quad D^\alpha y(\frac{\pi}{4}+)-\frac{1}{2}D^\alpha y(\frac{\pi}{4}-)=0.\end{aligned}\tag{3.24}$$

The characteristic function and eigenfunctions are

$$\Delta(\lambda)=\frac{5}{4}\cos(\frac{\rho}{\alpha}\pi^\alpha)+\frac{3}{5}\cos(\frac{\rho}{\alpha}(\pi^\alpha-2(\frac{\pi}{4})^\alpha)),$$

$$y_{n,\alpha}(x)=\begin{cases}\cos(\frac{\rho_n}{\alpha}x^\alpha), & 0\le x<\frac{\pi}{4},\\ \frac{5}{4}\cos(\frac{\rho_n}{\alpha}x^\alpha)+\frac{3}{5}\cos(\frac{\rho_n}{\alpha}(x^\alpha-2(\frac{\pi}{4})^\alpha)), & \frac{\pi}{4}\le x\le\pi.\end{cases}$$

The eigenvalues and eigenfunctions are presented in Table 1 and Figure 1. We use the **fzero** function in **MATLAB R2015a** to compute the zeros $\rho_{n,\alpha}$ of the function $\Delta(\lambda)$.

Example 3.8. We consider a CSLP with two jump conditions

$$\begin{aligned}&-D^\alpha D^\alpha y=\lambda y\\&D^\alpha y(0)=0,\quad y(\pi)=0,\\&y(\frac{\pi}{4}+)-2y(\frac{\pi}{4}-)=0,\quad D^\alpha y(\frac{\pi}{4}+)-\frac{1}{2}D^\alpha y(\frac{\pi}{4}-)=0,\\&y(\frac{7\pi}{10}+)-3y(\frac{7\pi}{10}-)=0,\quad D^\alpha y(\frac{7\pi}{10}+)-\frac{1}{3}D^\alpha y(\frac{7\pi}{10}-)=0.\end{aligned}\tag{3.25}$$

The characteristic function and eigenfunctions are

$$\Delta(\lambda)=\frac{25}{12}\cos(\frac{\rho}{\alpha}\pi^\alpha)+\frac{5}{4}\cos(\frac{\rho}{\alpha}(\pi^\alpha-2(\frac{\pi}{4})^\alpha)+\frac{5}{3}\cos(\frac{\rho}{\alpha}(\pi^\alpha-2(\frac{7\pi}{10})^\alpha)$$

n	$\rho_{n,\alpha}$ $\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	$\zeta_{n,\alpha}$ $\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
1	0.666	0.656	0.636	0.612	0.67	0.65	0.63	0.61
2	1.388	1.517	1.621	1.675	0.70	0.75	0.80	0.84
3	2.413	2.354	2.317	2.324	0.81	0.78	0.77	0.77
4	3.643	3.665	3.537	3.388	0.92	0.91	0.88	0.85
10	9.246	9.683	9.382	0.6615	0.94	0.96	0.93	0.97
15	14.317	14.398	14.781	14.324	0.97	0.95	0.98	0.95
20	19.384	19.712	19.532	19.388	0.98	0.98	0.97	0.97
25	24.379	24.723	24.759	24.612	0.99	0.98	0.98	0.98
30	29.277	29.467	29.815	29.675	0.99	0.98	0.98	0.99
35	34.120	34.860	34.728	34.324	0.99	0.99	0.98	0.98

Table 1. Eigenvalues and asymptotic results for Example 3.7.

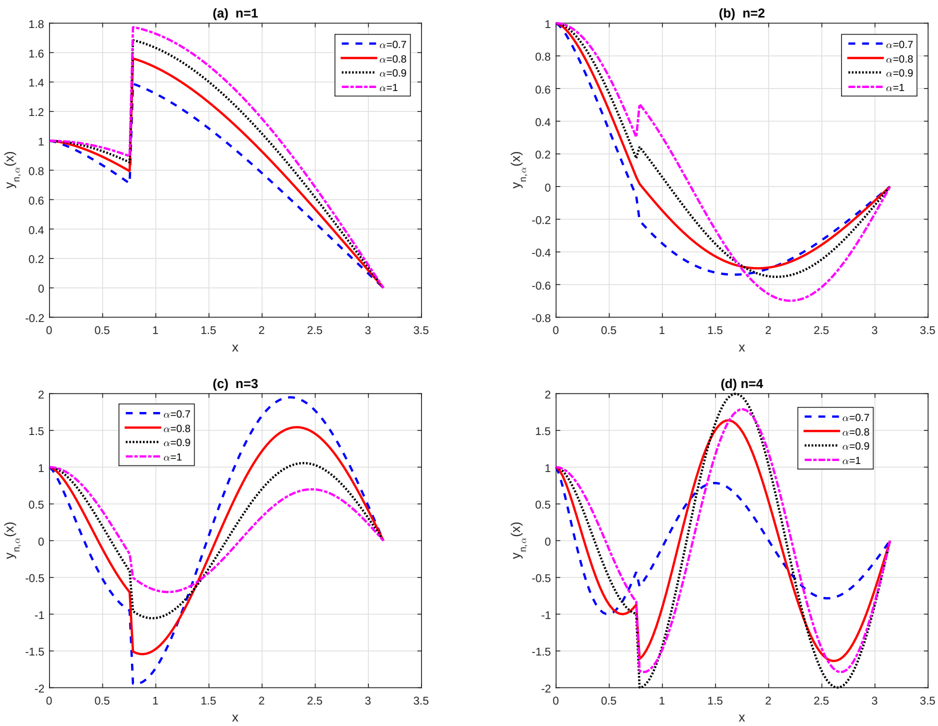

Figure 1. Eigenfunctions of Example 3.7 for different values of n and α.

$$+\cos(\frac{\rho}{\alpha}(\pi^\alpha+2(\frac{\pi}{4})^\alpha-2(\frac{7\pi}{10})^\alpha),$$

$$y_{n,\alpha}(x)=\begin{cases}\cos(\frac{\rho_n}{\alpha}x^\alpha), & 0\le x<\frac{\pi}{4},\\ \frac{5}{4}\cos(\frac{\rho_n}{\alpha}x^\alpha)+\frac{5}{3}\cos(\frac{\rho_n}{\alpha}(x^\alpha-2(\frac{\pi}{4})^\alpha), & \frac{\pi}{4}\le x\le\frac{7\pi}{10},\\ \frac{25}{12}\cos(\frac{\rho_n}{\alpha}\pi^\alpha)+\frac{5}{4}\cos(\frac{\rho_n}{\alpha}(\pi^\alpha-2(\frac{\pi}{4})^\alpha)+\frac{5}{3}\cos(\frac{\rho_n}{\alpha}(\pi^\alpha-2(\frac{7\pi}{10})^\alpha) & \\ \qquad+\cos(\frac{\rho_n}{\alpha}(\pi^\alpha+2(\frac{\pi}{4})^\alpha-2(\frac{7\pi}{10})^\alpha), & \frac{7\pi}{10}\le x\le\pi.\end{cases}$$

The eigenvalues and eigenfunctions are presented in Table 2 and Figure 2.

We compared the eigenvalues with first term of asymptotic form (3.21) as $\zeta_{n,\alpha}=\frac{\rho_{n,\alpha}}{n\alpha\pi^{1-\alpha}}$. The eigenvalues and ratios $\zeta_{n,\alpha}$ are presented in Tables 1 and 2. According to asymptotic form (3.21), the values of $\zeta_{n,\alpha}$ must tend to one, that hold for results of $\zeta_{n,\alpha}$ in Tables 1

	$\rho_{n,\alpha}$				$\zeta_{n,\alpha}$			
n	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
1	0.8592	0.8858	0.8907	0.8810	0.87	0.88	0.88	0.88
2	1.5686	1.6744	1.7300	1.6697	0.79	0.83	0.86	0.83
3	2.2871	2.1361	2.0634	2.1247	0.77	0.71	0.68	0.71
4	3.4614	3.5151	3.4746	3.4144	0.88	0.87	0.86	0.85
10	9.2889	9.9688	9.2520	9.5834	0.94	0.99	0.92	0.96
15	14.4103	14.2520	14.8143	14.5870	0.97	0.94	0.98	0.97
20	19.5312	19.6282	19.7697	19.1189	0.99	0.98	0.98	0.96
25	24.4888	24.8875	24.3889	24.8760	0.99	0.99	0.97	0.99
30	29.1105	29.7962	29.8781	29.5834	0.98	0.99	0.99	0.99
35	33.7523	34.6472	34.8633	33.7604	0.98	0.98	0.99	0.96

Table 2. Eigenvalues and asymptotic results for example 3.1.

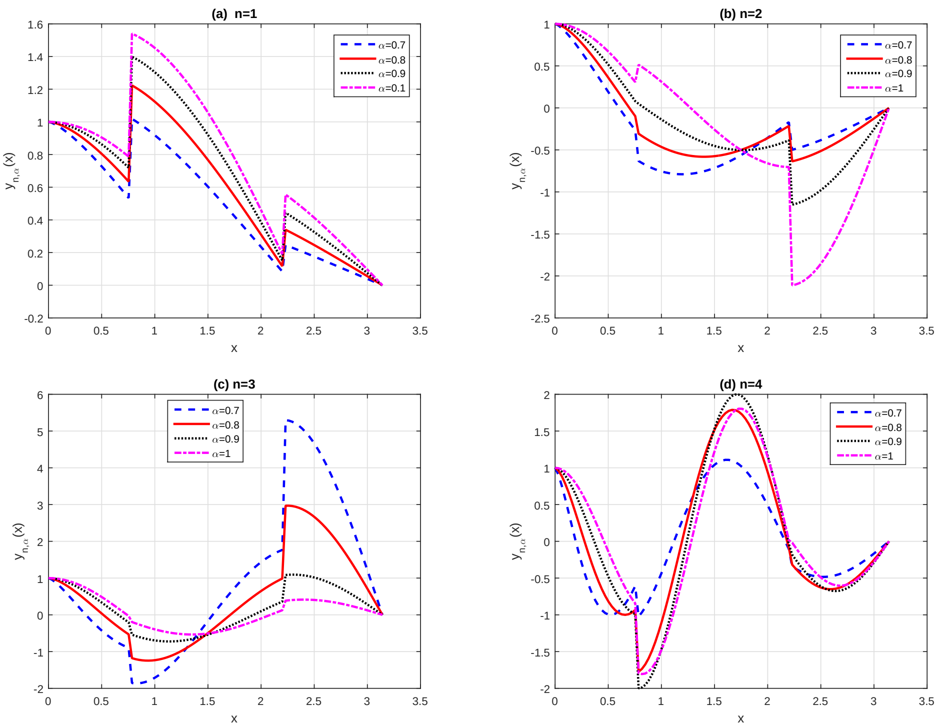

Figure 2. Eigenfunctions of Example 3.8 for different values of n and α.

and 2. The first four eigenfunctions for different values of α are plotted in Figures 1 and 2. It is well known that, the nth eigenfunction of classical Sturm-Liouville problem defined on $[0,\pi]$, has $(n-1)$ zero in interval $(0,\pi)$. The graphs in Figures 1 and 2 indicate that this result hold also for CSLP with jump conditions.

3.2. Uniqueness results

In this section, we propose three inverse problems corresponding to CSLP (3.1)-(3.3). First, we state and prove uniqueness results using Weyl M-function. Second, we consider one spectrum and corresponding norming constants. Finally, we study the inverse problem using two spectra corresponding to two different set of boundary conditions. Define the

Weyl M-function as

$$M(\lambda) = -\frac{v(0,\lambda)}{\Delta(\lambda)}. \tag{3.26}$$

Using (3.5) and (3.18), we find

$$M(\lambda) = \frac{1}{\sqrt{-\lambda}} + O(\lambda^{-1}), \quad \lambda \notin \mathbb{R}^+. \tag{3.27}$$

Using $\mathcal{S}(x,\lambda)$ and $u(x,\lambda)$ in (3.5) and (3.16), $W_\alpha(u(\lambda),\mathcal{S}(\lambda)) = 1$ and substituting $v(x,\lambda)$, we obtain

$$\Phi(x,\lambda) := \frac{v(x,\lambda)}{\Delta(\lambda)} = \mathcal{S}(x,\lambda) - M(\lambda)u(x,\lambda). \tag{3.28}$$

The function $\Phi(x,\lambda)$ is called the Weyl solution for L_α. Clearly

$$W_\alpha(u(\lambda),\Phi(\lambda)) = 1. \tag{3.29}$$

Lemma 3.9. *The function $M(\lambda)$ is a meromorphic function and satisfies*

$$\mathrm{Im}(M(\lambda)) = \mathrm{Im}(\lambda)\|\Phi(\lambda)\|_T^2$$

and we have

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\lambda_n - \lambda}, \tag{3.30}$$

where that

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{1+|\lambda_n|^\gamma} < \infty, \qquad \forall\gamma > \frac{1}{2}.$$

Proof. Given two solutions $u(x)$, $v(x)$ of $\ell_\alpha u = \lambda u$, $\ell_\alpha v = \hat{\lambda} v$, respectively. Using (3.4) and from the straightforward calculations, we get

$$(\hat{\lambda} - \lambda)\int_0^x u(t)v(t)r(t)\mathrm{d}_\alpha t = W_\alpha(u,v)\big|_x - W_\alpha(u,v)\big|_0.$$

Now choose $u(x) = \Phi(x,\lambda)$ and $v(x) = \overline{\Phi(x,\lambda)} = \Phi(x,\overline{\lambda})$, then

$$W_\alpha(\Phi(x,\lambda),\overline{\Phi(x,\lambda)})|_{x=\pi} - 2\mathrm{Im}(M(\lambda)) = -2\mathrm{Im}(\lambda)\int_0^\pi |\Phi(x,\lambda)|^2 r(x) d_\alpha x,$$

and observe that $W_\alpha(\Phi(x,\lambda),\overline{\Phi(x,\lambda)})|_{x=\pi}$ vanishes as $x=\pi$. Thus $M(\lambda)$ can be represented as follows

$$M(\lambda) = \int_{\mathbb{R}} \frac{d\rho(t)}{\lambda_n - t},$$

where ρ is a Borel measure satisfies

$$\int_{\mathbb{R}} \frac{d\rho(t)}{1+|\lambda|^\gamma}, \qquad \forall\gamma > \frac{1}{2},$$

see [27] for more details. Note that by (3.26) the function $M(\lambda)$ is meromorphic. It yields that ρ is a pure point measurement that is supported at the poles with masses given by negative residues. Therefore, the result from Lemma 3.2 is obtained. □

We are now ready to state the uniqueness results of the problems (3.1)-(3.3). For this aim, we denote the problem of the form L_α and parameters $\tilde{h}$, $\tilde{H}$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{d}_i$ with $\tilde{L}_\alpha$. We denote by η and $\tilde{\eta}$ an object related to L_α and $\tilde{L}_\alpha$, respectively.

Theorem 3.10. *If the M-functions $M(\lambda) = \tilde{M}(\lambda)$ and $r(x) = \tilde{r}(x)$ then $L_\alpha = \tilde{L}_\alpha$. That is, the Weyl M-function and weight function determined the problem L_α, uniquely.*

Proof. It follows from (3.18) and (3.28) that

$$|\Phi(x,\lambda)| \le C|\rho|^{-1}\exp\left(\frac{-\tau}{\alpha}x^\alpha\right), \qquad |D^\alpha\Phi(x,\lambda)| \le C\exp\left(\frac{-\tau}{\alpha}x^\alpha\right), \tag{3.31}$$

as $\lambda \to \infty$ along each beam except the positive real axis. We introduce a new matrix $T(x,\lambda)$ as follows

$$T(x,\lambda)\begin{pmatrix}\tilde{u}(x,\lambda) & \tilde{\Phi}(x,\lambda)\\ D^\alpha\tilde{u}(x,\lambda) & D^\alpha\tilde{\Phi}(x,\lambda)\end{pmatrix} = \begin{pmatrix}u(x,\lambda) & \Phi(x,\lambda)\\ D^\alpha u(x,\lambda) & D^\alpha\Phi(x,\lambda)\end{pmatrix}.$$

Using (3.29), we find

$$\begin{pmatrix}T_{11}(x,\lambda) & T_{12}(x,\lambda)\\ T_{21}(x,\lambda) & T_{22}(x,\lambda)\end{pmatrix} = \begin{pmatrix}uD^\alpha\tilde{\Phi} - D^\alpha\tilde{u}\Phi & \tilde{u}\Phi - u\tilde{\Phi}\\ D^\alpha uD^\alpha\tilde{\Phi} - D^\alpha\tilde{u}D^\alpha\Phi & \tilde{u}D^\alpha\Phi - D^\alpha u\tilde{\Phi}\end{pmatrix} \tag{3.32}$$

and

$$\begin{pmatrix}u(x,\lambda)\\ \Phi(x,\lambda)\end{pmatrix} = \begin{pmatrix}T_{11}(x,\lambda)\tilde{u}(x,\lambda) + T_{12}(x,\lambda)D^\alpha\tilde{u}(x,\lambda)\\ T_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + T_{12}(x,\lambda)D^\alpha\tilde{\Phi}(x,\lambda)\end{pmatrix}. \tag{3.33}$$

The entries $T_{jk}(x,\lambda)$, $j,k = 1,2$ are meromorphic functions with respect to λ and have simple poles λ_n and $\tilde{\lambda}_n$. Moreover, if $M(\lambda) = \tilde{M}(\lambda)$ then using the Eqs. (3.28) and (3.32) we see that the functions $T_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire of order $1/2$ with respect to the λ. Using (3.31) we get

$$|T_{11}(x,\lambda)| \le C, \qquad |T_{12}(x,\lambda)| \le \frac{C}{|\rho|} \tag{3.34}$$

for $\lambda \notin \mathbb{R}^+$. Moreover, by assumptions these functions grow with order $s < 1/2$. So, we can apply the Phragmén-Lindelöf theorem (e.g., [15, Sect. 6.1]) the two half-planes bounded by the imaginary axis. This indicates that the functions T_{11} and T_{12} are bounded on $\mathbb{C}$ and by Liouville's theorem, they are constant. The function $T_{12} \to 0$ as $\rho \to \infty$, hence this function must be equal to zero. So, we obtain

$$T_{11}(x,\lambda) = \mathrm{A}(x),\ T_{12}(x,\lambda) = 0.$$

Applying (3.33), we have

$$u(x,\lambda) = \mathrm{A}(x)\tilde{u}(x,\lambda),\ \Phi(x,\lambda) = \mathrm{A}(x)\tilde{\Phi}(x,\lambda). \tag{3.35}$$

It follows from (3.7), $W_\alpha(u(\lambda),\Phi(\lambda)) = W_\alpha(\tilde{u}(\lambda),\tilde{\Phi}(\lambda)) = 1$ and so we deduce $\mathrm{A}(x) = \frac{\tilde{r}(x)}{r(x)} = 1$, that is, $u(x,\lambda) = \tilde{u}(x,\lambda)$, $\Phi(x,\lambda) = \tilde{\Phi}(x,\lambda)$, and $v(x,\lambda) = \tilde{v}(x,\lambda)$. Therefore from (3.1), (3.3), (3.7), and (3.6) we calculate $q(x) = \tilde{q}(x)$, a.e. on $[0,\pi]$ and $a_i = \tilde{a}_i$, $b_i = \tilde{b}_i$, $c_i = \tilde{c}_i$, $d_i = \tilde{d}_i$, $h = \tilde{h}$ and $H = \tilde{H}$. Consequently $L_\alpha = \tilde{L}_\alpha$. □

Corollary 3.11. *If $\lambda_n = \tilde{\lambda}_n$ and $\gamma_n = \tilde{\gamma}_n$, for $n = 0,1,2,...,$ and $r(x) = \tilde{r}(x)$ then $L_\alpha = \tilde{L}_\alpha$.*

Proof. According to (3.30), the specification of the $M(\lambda)$ is equivalent to the specification of the $\{\lambda_n,\gamma_n\}_{n\ge 0}$ (spectral data). Then $M(\lambda) = \tilde{M}(\lambda)$. □

Finally, we consider the CSLP L_α^k such that in the problem (3.1)-(3.3) we replaced the boundary condition $L_1(y)$ by

$$L_1'(y) = \begin{cases} D^\alpha y(0) + k\,y(0) = 0, & k \in \mathbb{R},\\ y(0) = 0, & k = \infty.\end{cases}$$

The eigenvalues of the problem L_α^k are denoted $\{\mu_n\}_{n\ge 0}$.

Corollary 3.12. *If $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \tilde{\mu}_n$ for $n = 0,1,2,\dots,$ and $r(x) = \tilde{r}(x)$ for $k \ne h$, then $L_\alpha = \tilde{L}_\alpha$.*

Proof. First we verify the case $k = \infty$. By virtue of (3.26) the poles and zeros of $M(\lambda)$ coincide with the spectra of L_α^k and L_α, respectively. Thus the function $M(\lambda)$ can be constructed using two spectra λ_n and μ_n, uniquely. In the case $k \in \mathbb{R}$, the Weyl function defined in (3.26) and (3.27) is replaced with $M(\lambda) + (k-h)^{-1}$. Since $h = \tilde{h}$ and $k = \tilde{k}$, it follows $L_\alpha = \tilde{L}_\alpha$. □

Now, in the following theorem we extend the Hochstadt-Lieberman [9] and Hald's [8] Theorems for the conformable Sturm-Liouville problem. The case $\alpha = 1$ yields to the Theorem 4.5 in [25].

Theorem 3.13. *If $\lambda_n = \tilde{\lambda}_n$, $r(x) = \tilde{r}(x)$, $L_1 = \tilde{L}_1$, $q(x) = \tilde{q}(x)$ for a.e. $x < \frac{\pi}{2}$ and $U_i = \tilde{U}_i$, $V_i = \tilde{V}_i$ for all i with $d_i < \frac{\pi}{2}$, then $L_\alpha = \tilde{L}_\alpha$.*

Proof. Using the Hadamard factorization theorem, we get

$$W_\alpha(\tilde{u}, \tilde{v}) = K\, W_\alpha(u, v) \tag{3.36}$$

for some constant K, which can be determined from the asymptotic form of (3.15) as $\lambda \to \infty$:

$$K = \prod_{i:d_i \geq \frac{\pi}{2}} \frac{\tilde{\alpha}_i}{\alpha_i} \neq 0.$$

By assumptions, we conclude that $u = \tilde{u}$ for $x < \frac{\pi}{2}$. Applying in (3.36), we obtain the following fractional differential equation

$$\frac{D^\alpha u}{u} = \frac{D^\alpha(\tilde{v} - Kv)}{(\tilde{v} - Kv)}, \quad x < \frac{\pi}{2}.$$

Taking $\alpha-$integration, we get

$$\tilde{v}(x, \lambda) = K\, v(x, \lambda) + F(\lambda) u(x, \lambda), \quad x < \frac{\pi}{2}.$$

The function $F(\lambda)$ is entire and the order of it is at most $\frac{1}{2}$. Since by asymptotic forms in Theorem 3.4 u, v and $\tilde{v}$ are entire functions, the limit of $F(\lambda)$ for $x \to \frac{\pi}{2}-$ is

$$F(\lambda) = \frac{\tilde{v}(\frac{\pi}{2}-, \lambda) - Kv(\frac{\pi}{2}-, \lambda)}{u(\frac{\pi}{2}-, \lambda)} = K \frac{v(\frac{\pi}{2}-, \lambda)}{u(\frac{\pi}{2}-, \lambda)} \left(\frac{\tilde{v}(\frac{\pi}{2}-, \lambda)}{Kv(\frac{\pi}{2}-, \lambda)} - 1 \right).$$

Using the asymptotic forms of the expression in parenthesis, we see that this is bounded and vanishes for along every ray different from the positive real axis, while the first part of $F(\lambda)$ is bounded by the asymptotic form of (3.12) for u and the analogous for v. The Phragmén-Lindelöf Theorem concludes that $F(\lambda)$ must be identically zero. Thus we have

$$\tilde{v}(x, \lambda) = Kv(x, \lambda). \tag{3.37}$$

We rewrite the equation (3.28) for $v, \triangle(\lambda)$ and $\tilde{v}, \tilde{\triangle}(\lambda)$ as follows

$$\frac{v(x, \lambda)}{\Delta(\lambda)} = \mathcal{S}(x, \lambda) - M(\lambda) u(x, \lambda), \qquad \frac{\tilde{v}(x, \lambda)}{\tilde{\Delta}(\lambda)} = \tilde{\mathcal{S}}(x, \lambda) - \tilde{M}(\lambda) \tilde{u}(x, \lambda). \tag{3.38}$$

By assumptions, for $x < \frac{\pi}{2}$ we have $\mathcal{S} = \tilde{\mathcal{S}}, u = \tilde{u}$ and by (3.36) $\tilde{\triangle}(\lambda) = K \triangle(\lambda)$. Substituting this relations and (3.37) in (3.38), we get that $M(\lambda) = \tilde{M}(\lambda)$ and the proof is completed using Theorem 3.10. □

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