ON CURVES AND SURFACES OF *AW(k)* TYPE

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ABSTRACT

In the present study we consider curves and surfaces of *AW(k)* (k =1, 2 or 3) type. We also give related examples of curves and surfaces satisfying *AW(k)* type conditions.
Keywords: Frenet curve, curves and surfaces of *AW(k)* type.

ÖZET

Bu çalışmada, *AW(k)* (k=1, 2 yada 3) tipindeki eğri ve yüzeyler gözönüne alındı. *AW(k)* şartını sağlayan eğri ve yüzeylere örnekler verildi.
Anahtar Kelimeler: Frenet eğrisi, *AW(k)* tipinde eğri ve yüzey.

1- INTRODUCTION

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional connected Riemannian manifold M into an m-dimensional Riemannian manifold $\widetilde{M}$. Letters X, Y and Z (resp.ζ, μ and ξ) vector fields tangent (resp. normal) to M. We denote the tangent bundle of M (resp. $\widetilde{M}$) by TM (resp. $T\widetilde{M}$) , unit tangent bundle of M by UM and the normal bundle by $T^{\perp}M$. Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita connections of $\widetilde{M}$ and M , resp. Then the Gauss formula is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad (1)$$

where h denotes the second fundamental form. The Weingarten formula is given by

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \quad (2)$$

where A denotes the shape operator and D the normal connection. Clearly $h(X,Y) = h(Y,X)$ and A is related to h as $\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle$, where $\langle\, ,\rangle$ denotes the Riemannian metrics of M and $\widetilde{M}$ [1].

Let $\{e_1, e_2, ..., e_n, e_{n+1}, ..., e_m\}$ be an local orthonormal frame field on M where $\{e_1, e_2, ..., e_n\}$ are tangent vector and $\{e_{n+1}, ..., e_m\}$are normal vector. The connection form w_i^j are defined by

$$\widetilde{\nabla}_{e_i} = \sum w_i^j e_j \ ; \ w_i^j = -w_j^i, \quad 1 \le i, j \le m \quad (3)$$

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} w_j^k (e_i) e_k \ , \quad (4)$$

$$D_{e_i} e_\alpha = \sum_{\beta=n+1}^{m} w_\alpha^\beta (e_i) e_\beta \ . \quad (5)$$

The covariant derivations of h is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z) \ , \qquad (6)$$

where *X, Y, Z* tangent vector fields over *M* and $\overline{\nabla}$ is the van der Waerden Bortolotti connection. Then we have

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z) = (\overline{\nabla}_Z h)(Y,X) \qquad (7)$$

which is called *codazzi equations*.

If $\overline{\nabla} h = 0$ then *M* is said to have parallel second fundamental form (i.e.*1-parallel*) [2].

It is well known that $\overline{\nabla} h$ is a normal bundle valued tensor of type (0,3).We define the second covariant derivative of *h* by

$$\begin{aligned}(\overline{\nabla}_W \overline{\nabla}_X h)(Y,Z) = D_W ((\overline{\nabla}_X h)(Y,Z)) - (\overline{\nabla}_X h)(\nabla_W Y,Z) - \\ - (\overline{\nabla}_X h)(Y,\nabla_W Z) - (\overline{\nabla}_{\nabla_W X} h)(Y,Z)\end{aligned} \qquad (8)$$

For the orthonormal frame $\{e_1, e_2, ..., e_n\}$ of T_pM the mean curvature vector *H* of *f* is defined by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \, . \qquad (9)$$

2.CURVES OF *AW(k)* TYPE

Let $\gamma = \gamma(s) : I \subset IE \to IE^m$ be a unit speed curve in IE^m . The curve γ is called *Frenet curve of osculating order d* if its higher order derivatives $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s),\ldots,\gamma^{(d)}(s)$ are linearly independent and $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s),\ldots,\gamma^{(d+1)}(s)$ are linearly dependent for all $s \in I$. For each Frenet curve of order d one can associate an orthonormal *d*-frame $\nu_1, \nu_2, \ldots, \nu_d$ along γ (such that $T = \gamma'(s) = \nu_1$) called *the Frenet frame* and *d-1* functions $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$: *I→IR* called *the Frenet curvatures*, such that the Frenet formulas are defined in the usual way;

$$T'(s) = \nu_1' = \kappa_1(s)\nu_2(s) \qquad (10)$$

$$\nu_2'(s) = -\kappa_1(s)T(s) + \kappa_2(s)\nu_3(s) \qquad (11)$$

$$\nu_i'(s) = -\kappa_{i-1}(s)\nu_{i-1}(s) + \kappa_i(s)\nu_{i+1}(s) \qquad (12)$$

$$\nu_{i+1}'(s) = -\kappa_i(s)\nu_i(s). \qquad (13)$$

A regular curve $\gamma = \gamma(s) : I \subset IE \to IE^m$ is called a *W-curve of rank d*, if γ is a Frenet curve of osculating order d and the Frenet curvatures κ_i, $1 \le i \le d-1$ are non zero constant and $\kappa_d = 0$. In particular, a W-curve $\gamma(s)$ of rank 2 is called a *geodesic circle*. A W-curves of rank 3 is a *right circular helix*.

Let *M* be a smooth *n*-dimensional submanifold in (*n+d*)-dimensional Euclidean space IE^{n+d}. For $x \in M$ and a unit vector $X \in T_xM$, the vector *X* and the normal space N_xM determine a (*d+1*)-dimensional *affine subspace IE(x,X)* of IE^{n+d}. The intersection of *M* and *IE*(*x*,*X*) gives rise to a curve $\gamma(s)$ (in a neighborhood of *x*) called the *normal section* of *M* at *x* in the direction of *X* , where *s* denotes the arc length of γ [1].

Definition 1. If each normal section γ of *M* is a Frenet curve of osculating order d then *M* is said to have *d-planar normal sections* (*d-PNS*). For every normal sections γ of *M* if γ is a *W*-curve of rank d in *M* then *M* is called *weak helical submanifold of order d*.

Definition 2. If each d-planar normal section is γ a geodesic of *M* then *M* is said to have *geodesic d-planar normal sections* (*Gd-PNS*). For every geodesic normal sections γ of *M* if γ is a *W*-curve of rank d in *M* then *M* is called *weak geodesic helical submanifold of order d*.

From now on we consider the Frenet curve of osculating order 3 of IE^m.

Proposition 3. Let γ be a Frenet curve of IE^m of osculating order 3 then we have

$$\gamma''(s)=\kappa_1\nu_2,\quad \gamma'(s)=\nu_1(s)$$

$$\gamma'''(s)=-\kappa_1^2\nu_1+\kappa_1'\nu_2+\kappa_1\kappa_2\nu_3 \tag{14}$$

$$\gamma^{iv}(s)=-3\kappa_1\kappa_1'\nu_1+(-\kappa_1^3+\kappa_1''-\kappa_1\kappa_2^2)\nu_2+(2\kappa_1'\kappa_2+\kappa_1\kappa_2')\nu_3. \tag{15}$$

Notation: Let us write

$$N_1(s)=\kappa_1\nu_2$$

$$N_2(s)=\kappa_1'\nu_2+\kappa_1\kappa_2\nu_3$$

$$N_3(s)=(-\kappa_1^3+\kappa_1''-\kappa_1\kappa_2^2)\nu_2+(2\kappa_1'\kappa_2+\kappa_1\kappa_2')\nu_3.$$

Corollary 4. $\gamma'(s)$, $\gamma''(s)$, $\gamma'''(s)$ and $\gamma^{iv}(s)$ are linearly dependent if and only if $N_1(s)$, $N_2(s)$ and $N_3(s)$ are linearly dependent.

Theorem 5. Let γ be a Frenet curve of IE^m of osculating order 3 then

$$N_3(s)=\left\langle N_3(s),N_1^*(s)\right\rangle N_1^*(s)+\left\langle N_3(s),N_2^*(s)\right\rangle N_2^*(s)$$

where

$$N_1^*(s)=\frac{N_1(s)}{\|N_1(s)\|},\qquad N_2^*(s)=\frac{N_2(s)-\left\langle N_2(s),N_1^*(s)\right\rangle N_1^*(s)}{\left\|N_2(s)-\left\langle N_2(s),N_1^*(s)\right\rangle N_1^*(s)\right\|}$$

[3].

Definition 6. Frenet curves (of osculating order 3) are

i) of type *weak AW(2)* if they satisfy

$$N_3(s)=\left\langle N_3(s),N_2^*(s)\right\rangle N_2^*(s),$$

ii) of type *weak AW(3)* if they satisfy

$$N_3(s)=\left\langle N_3(s),N_1^*(s)\right\rangle N_1^*(s)$$ [3].

Corollary 7. Let γ be a Frenet curve of type weak *AW(2)*. If γ is a plane curve then $\kappa_1''(s)-\kappa_1^3(s)=0$, and the solution of this differential equation is

$$\kappa_1=\pm\frac{\sqrt{2}}{s+c},\quad c=\text{Const.}$$ [3].

The curvature vector field of γ (the mean curvature vector field of γ) is defined by

$$h(T,T)=H(s)=\gamma''(s)=\kappa_1(s)\nu_2(s). \tag{16}$$

One can use the Frenet equations (15) to compute

$$\gamma'^{v\perp}(s) = (-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2)v_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')v_3 \qquad (17)$$

Definition 8. Curves are of type *AW(1)* if they satisfy

$$\gamma'^{v\perp}(s) = 0, \qquad (18)$$

of type *AW(2)* if they satisfy

$$\gamma'^{v\perp}(s) \Lambda \gamma'''^{\perp}(s) = 0 \qquad (19)$$

and of type *AW(3)* if they satisfy

$$\gamma'^{v\perp}(s) \Lambda \gamma''(s) = 0. \qquad (20)$$

Proposition 9. Let γ be a Frenet curve of type *AW(1)* if and only if

$$-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2 = 0 \qquad (21)$$

$$2\kappa_1'\kappa_2 + \kappa_1\kappa_2' = 0. \qquad (22)$$

Proof. Substituting (17) into (18) we get the result.

Corollary 10. Let γ be a Frenet curve of type *AW(1)*.

i) If $\kappa_1 = 0$ then γ is a straight line.

ii) If $\kappa_1 \neq 0$, $\kappa_2 = 0$ then $\kappa_1'' - \kappa_1^3 = 0$. That is

$$\kappa_1 = \pm\frac{\sqrt{2}}{s+c}, \ c\text{=Const.}$$

[3].

iii) If $\kappa_1, \kappa_2 \neq 0$ then by (21) and (22) we obtain

$$\kappa_2 = \frac{c}{\kappa_1^2}, \quad \kappa_1'' - \kappa_1^3 - \frac{c^2}{\kappa_1^3} = 0. \qquad (23)$$

Putting $\kappa_1 = y$ into (23) we get

$$y'' - y^3 - \frac{c^2}{y^3} = 0. \qquad (24)$$

Thus solving the differential equation (24) one gets

$$\int^{y(x)} \frac{2\_a}{\sqrt{2\_a^6 - 4c^2 + 4\_C1\_a^2}} d\_a - x - \_C2 = 0,$$

$$\int^{y(x)} -\frac{2\_a}{\sqrt{2\_a^6 - 4c^2 + 4\_C1\_a^2}} d\_a - x - \_C2 = 0.$$

Using $\kappa_1 = y$, $\kappa_2 = \frac{c}{\kappa_1^2}$, we get the result.

Corollary 11. Every plane curve of *AW(1)* type is also of weak *AW(2)* type [3].

Proposition 12. Let γ be a Frenet curve of type *AW(2)* if and only if

$$-\kappa_1^3 + \kappa_1'' - \kappa_1\kappa_2^2 = \delta_1\kappa_1' \qquad (25)$$

$$2\kappa_1'\kappa_2+\kappa_1\kappa_2'=\delta_1\kappa_1\kappa_2. \qquad (26)$$

Proof. Substituting (14) and (17) into (19) we get the result.

Corollary 13. Let γ be a Frenet curve of type *AW(2)*.

i)If $\kappa_1=0$ then γ is a straight line.

ii) If $\kappa_1\neq 0$, $\kappa_2=0$ then by (25) we obtain

$$\kappa_1''-\kappa_1^3-\delta_1\kappa_1'=0 \qquad (27)$$

Putting $\kappa_1=y$ into (27) we get

$$y''-y^3-\delta_1 y'=0 \qquad (28)$$

Thus solving the differential equation (28) one gets

$$y=c_1e^{\frac{\delta_1+\sqrt{4+\delta_1^2}}{2}x}+c_2e^{\frac{\delta_1-\sqrt{4+\delta_1^2}}{2}x}.$$

Using $\kappa_1=y$ we get the result.

iii) If $\kappa_1,\kappa_2\neq 0$ then by (25) and (26) we obtain

$$\kappa_1'''\kappa_1+\kappa_1''(3\kappa_1'-3\delta_1\kappa_1)+\kappa_1'(-3\delta_1\kappa_1'-6\kappa_1^3+2\delta_1^2\kappa_1)+2\delta_1\kappa_1^4=0 \qquad (29)$$

Putting $\kappa_1=y$ and $\delta_1=c$ into (29) we get

$$y'''y+y''(3y'-3cy)+y'(-3cy'-6y^3+2c^2y)+2cy^4=0 .(30)$$

Thus solving the differential equation (30) one gets

$$y(x)=0,\ y(x)=\_b(\_a)\ \ where[\{-\_b(\_a)^6e^{(-2c\_a)}-\_b(\_a)^3e^{(-2c\_a)}c(\frac{d}{d\_a}\_b(\_a))$$

$$+\_b(\_a)^3e^{(-2c\_a)}(\frac{d^2}{d\_a^2}\_b(\_a))+\_C1=0\},$$

$$\{\_a=x,\ \_b(\_a)=y(x)\},\ \{x=\_a,\ y(x)=\_b(\_a)\}].$$

Using $\kappa_1=y$ we get the result.

Proposition 14. Let γ be a Frenet curve of type *AW(3)* if and only if

$$-\kappa_1^3+\kappa_1''-\kappa_1\kappa_2^2=\delta_2\kappa_1 \qquad (31)$$

$$2\kappa_1'\kappa_2+\kappa_1\kappa_2'=0. \qquad (32)$$

Proof. Substituting (16) and (17) into (20) we get the result.

Corollary 15. Let γ be a Frenet curve of type *AW(3)*.

i)If $\kappa_1=0$ then γ is a straight line.

ii) If $\kappa_1\neq 0$, $\kappa_2=0$ then by (31) we obtain

$$\kappa_1''-\kappa_1^3-\delta_2\kappa_1=0 \qquad (33)$$

Putting $\kappa_1=y$ and $\delta_2=c$ into (33) we get

$$y''-y^3-cy=0 \qquad (34)$$

Thus solving the differential equation (34) one gets

$$\int^{y(x)}\frac{2}{\sqrt{2\_a^4+4\_a^2c+4\_C1}}d\_a-x-\_C2=0,$$

$$\int^{y(x)} -\frac{2}{\sqrt{2\_a^4 + 4\_a^2 c + 4\_C1}} d\_a - x - \_C2 = 0 .$$

Using $\kappa_1 = y$ we get the result.

iii) If $\kappa_1, \kappa_2 \neq 0$ then by (31) and (32) we obtain

$$\kappa_2 = \frac{c}{\kappa_1^2}, \quad \kappa_1'' - \kappa_1^3 - \frac{c^2}{\kappa_1^3} - \delta_2 \kappa_1 = 0 . \tag{35}$$

Putting $\kappa_1 = y$ and $\delta_2 = d$ into (35) we get

$$y'' - y^3 - \frac{c^2}{y^3} - dy = 0 . \tag{36}$$

Thus solving the differential equation (36) one gets

$$\int^{y(x)} -\frac{2\_a}{\sqrt{4\_C1\_a^2 + 2\_a^6 - 4c^2 + 4d\_a^4}} d\_a - x - \_C2 = 0$$

$$\int^{y(x)} \frac{2\_a}{\sqrt{4\_C1\_a^2 + 2\_a^6 - 4c^2 + 4d\_a^4}} d\_a - x - \_C2 = 0 .$$

Using $\kappa_1 = y$, $\kappa_2 = \frac{c}{\kappa_1^2}$, we get the result.

Corollary 16. Every Frenet curve of weak *AW(3)* type is also of *AW(3)* type [3].

3. SURFACES OF *AW(k)* TYPE

In this part we consider surfaces of *AW(k)* type.

Let us write

$$H(X)=h(X,X) \tag{37}$$

$$\nabla H(X) = (\overline{\nabla}_X h)(X,X) \tag{38}$$

$$J(X) = (\overline{\nabla}_X \overline{\nabla}_X h)(X,X) + 3h(A_{h(X,X)} X, X) \tag{39}$$

so that $H : T(M) \to N(M)$, $\nabla H : T(M) \to N(M)$ and $J : T(M) \to N(M)$ are fibre maps whose restriction to each fibre $T_X(M)$ is a homogeneous polynomial map, H is of degree 2 and ∇H is of degree 3 and J is of degree 4.

Then

$$J_1 (\overline{\nabla}_{e_1} \overline{\nabla}_{e_1} h)(e_1, e_1) + 3h(A_{h(e_1,e_1)} e_1, e_1) \tag{40}$$

$$J_2 (\overline{\nabla}_{e_2} \overline{\nabla}_{e_2} h)(e_2, e_2) + 3h(A_{h(e_2,e_2)} e_2, e_2) . \tag{41}$$

Definition 17. [4] Submanifolds are of type *AW(1)* if they satisfy

$$J \equiv 0 \tag{42}$$

submanifolds are of type *AW(2)* if they satisfy

$$\|\nabla H\|^2 J \equiv \langle J, \nabla H \rangle \nabla H \tag{43}$$

and of type *AW(3)* if they satisfy

$$\|H\|^2 J \equiv \langle J, H \rangle H . \tag{44}$$

Proposition 18. [5] Let M be a connected normally flat surfaces in IE^4 . e_3 is parallel to the mean curvature vector H of M such that

$$A_{e_3} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad A_{e_4} = \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}. \tag{45}$$

We give the following results.

Lemma 19. From the Codazzi equations and using (4), (5) and (45)

$$(\lambda - \mu) w_1^2(e_2) = e_1(\mu) + \beta\, w_3^4(e_1) \tag{46}$$

$$2\beta\, w_1^2(e_2) = -e_1(\beta) + \mu\, w_3^4(e_1) \tag{47}$$

$$(\lambda - \mu) w_1^2(e_1) = e_2(\lambda) - \beta\, w_3^4(e_2) \tag{48}$$

$$2\beta\, w_1^2(e_1) = e_2(\beta) + \lambda\, w_3^4(e_2)\,. \tag{49}$$

Lemma 20. If $M \subset IE^4$ is normally flat surfaces then

$$\begin{aligned} J_1 = \{ & e_1^2(\lambda) - \lambda\,(w_3^4(e_1))^2 - 2e_1(\beta) w_3^4(e_1) - \beta\, e_1(w_3^4(e_1)) - 3w_1^2(e_1) e_2(\lambda) \\ & + 3\beta\, w_1^2(e_1) w_3^4(e_2) + 3\lambda\,(\lambda^2 + \beta^2) \} e_3 \\ + \{ & e_1^2(\beta) - \beta\,(w_3^4(e_1))^2 + 2e_1(\lambda) w_3^4(e_1) + \lambda e_1(w_3^4(e_1)) - 3w_1^2(e_1) e_2(\beta) \\ & - 3\lambda\, w_1^2(e_1) w_3^4(e_2) + 3\beta\,(\lambda^2 + \beta^2) \} e_4 \end{aligned} \tag{50}$$

and

$$\begin{aligned} J_2 = \{ & e_2^2(\mu) - \mu\,(w_3^4(e_2))^2 + 2e_2(\beta) w_3^4(e_2) + \beta\, e_2(w_3^4(e_2)) + 3w_1^2(e_2) e_1(\mu) \\ & + 3\beta\, w_1^2(e_2) w_3^4(e_1) + 3\mu\,(\mu^2 + \beta^2) \} e_3 \\ + \{ & -e_2^2(\beta) + \beta\,(w_3^4(e_2))^2 + 2e_2(\mu) w_3^4(e_2) + \mu\, e_2(w_3^4(e_2)) - 3w_1^2(e_2) e_1(\beta) \\ & + 3\mu\, w_1^2(e_2) w_3^4(e_1) - 3\beta\,(\mu^2 + \beta^2) \} e_4\,. \end{aligned} \tag{51}$$

Proof. Substituting (4), (5), (6), (8) and (45) into (40) and (41) we get the result.

Proposition 21. Let $M \subset IE^4$ be a normally flat surfaces. If M is $AW(1)$ type then $J_1 = 0$ and $J_2 = 0$. That is

$$\begin{aligned} & e_1^2(\lambda) - (\lambda + 2\mu)\,(w_3^4(e_1))^2 + 4\beta\, w_3^4(e_1) w_1^2(e_2) - \beta\, e_1(w_3^4(e_1)) \\ & - 3(\lambda - \mu)(w_1^2(e_1))^2 + 3\lambda(\lambda^2 + \beta^2) = 0\,, \end{aligned}$$

$$\begin{aligned} & 2e_1(\lambda) w_3^4(e_1) + (\lambda + \mu) e_1(w_3^4(e_1)) + (\lambda - 3\mu) w_3^4(e_1) w_1^2(e_2) \\ & - \beta\{ 2(w_3^4(e_1))^2 - 4(w_1^2(e_2))^2 + 2e_1(w_1^2(e_2)) + 6(w_1^2(e_1))^2 - 3(\lambda^2 + \beta^2) \} = 0, \end{aligned}$$

$$\begin{aligned} & e_2^2(\mu) - (\mu + 2\lambda)\,(w_3^4(e_2))^2 + 4\beta\, w_3^4(e_2) w_1^2(e_1) + \beta\, e_2(w_3^4(e_2)) \\ & + 3(\lambda - \mu)(w_1^2(e_2))^2 + 3\mu(\mu^2 + \beta^2) = 0, \end{aligned}$$

$$\begin{aligned} & 2e_2(\mu) w_3^4(e_2) + (\lambda + \mu) e_2(w_3^4(e_2)) + (3\lambda - \mu) w_3^4(e_2) w_1^2(e_1) \\ & - \beta\{ -2(w_3^4(e_2))^2 + 4(w_1^2(e_1))^2 + 2e_2(w_1^2(e_1)) - 6(w_1^2(e_2))^2 + 3(\mu^2 + \beta^2) \} = 0. \end{aligned}$$

Proof. Substituting (46), (47), (48), (49) into (50) and (51) and from *AW(1)* type definition we get the result.

Proposition 22. Let M be a normally flat and has got constant principal curvature submanifold. Then

$$J_1 = \{-\lambda\,(w_3^4(e_1))^2 - \beta\, e_1(w_3^4(e_1)) + 3\beta\, w_1^2(e_1)w_3^4(e_2) + 3\lambda(\lambda^2+\beta^2)\}e_3 \quad (52)$$
$$+\{-\beta(w_3^4(e_1))^2 + \lambda\, e_1(w_3^4(e_1)) - 3\lambda\, w_1^2(e_1)w_3^4(e_2) + 3\beta(\lambda^2+\beta^2)\}e_4,$$
$$J_2 = \{-\lambda\,(w_3^4(e_2))^2 + \beta\, e_2(w_3^4(e_2)) + 3\beta\, w_1^2(e_2)w_3^4(e_1) + 3\lambda(\lambda^2+\beta^2)\}e_3 \quad (53)$$
$$+\{\beta(w_3^4(e_2))^2 + \lambda\, e_2(w_3^4(e_2)) + 3\lambda\, w_1^2(e_2)w_3^4(e_1) - 3\beta(\lambda^2+\beta^2)\}e_4.$$

Lemma 23. Let M be a normally flat and has got constant principal curvature submanifold of *AW(1)* type

i) If $\lambda = \beta = 0$ then M is a plane,

ii) If $\lambda = -\beta$ then M has got vanishing Gaussian curvature ($K = 0$), mean curvature $H = \lambda$ or $(w_3^4(e_2))^2 = 3(\lambda^2+\beta^2)$,

iii) If $\lambda = \beta$ then M has got vanishing Gaussian curvature ($K = 0$), mean curvature $H = \lambda$ or $e_2(w_3^4(e_2)) = -3w_1^2(e_2)w_3^4(e_1)$.

Theorem 24. [3] Let γ be a Frenet curve of order 3 and of type *AW(k)* then the cylinder over γ is of type *AW(k),* where k=1,2,3.

Example 25. Let $\gamma(s) = (\int_0^s \cos(P_k(t))dt, \int_0^s \sin(P_k(t))dt)$ be a polinomial spiral with $\kappa_\gamma(s) = P_k'(t) = \pm\frac{\sqrt{2}}{s+c}$, c=*Const.* The Riemannian product of $\gamma(s)$ with the helicoid *x(w, t) = (wcost, wsint, at)* is of *AW(1)* type.

Example 26. We define helical cylinder $\mathbf{H}^2$ embedded in IE^4 by

$$x(u,v)=\{(u, a\cos v, a\sin v, bv) : a, b \in IR\}$$

and we show that $\mathbf{H}^2$ is of type *AW(3).*

For

$$p=(u, a\cos v, a\sin v, bv)$$

$T_p(\mathbf{H}^2)$ is spanned by

$$x_u=(1, 0, 0, 0)$$
$$x_v=(0, -a\sin v, a\cos v, b)$$

and $N_p(\mathbf{H}^2)$ is spanned by

$$n_1=(0, \cos v, \sin v, 0)$$
$$n_2=(0, \frac{b}{a}\sin v, -\frac{b}{a}\cos v, 1).$$

We have the orthonormal frame X, Y, v_1, v_2 where

$$X = \frac{x_u}{\|x_u\|} = (1,0,0,0)$$

$$Y = \frac{x_v}{\|x_v\|} = \frac{1}{\sqrt{a^2+b^2}}(0,-a\sin v, a\cos v, b)$$

$$v_1 = \frac{n_1}{\|n_1\|} = (0, \cos v, \sin v, 0)$$

$$v_2 = \frac{n_2}{\|n_2\|} = \frac{a}{\sqrt{a^2+b^2}}(0, \frac{b}{a}\sin v, -\frac{b}{a}\cos v, 1).$$

Differentiating these we have

$$\tilde{\nabla}_X X = \tilde{\nabla}_X Y = \tilde{\nabla}_Y X = 0, \quad \tilde{\nabla}_Y Y = \frac{-a}{a^2+b^2} v_1$$

$$\tilde{\nabla}_X v_1 = \tilde{\nabla}_X v_2 = 0, \quad \tilde{\nabla}_Y v_1 = \frac{a}{a^2+b^2} Y - \frac{b}{a^2+b^2} v_2, \quad \tilde{\nabla}_Y v_2 = \frac{b}{a^2+b^2} v_1.$$

Combining these with (1) and (2) we get

$$\nabla_X X = \nabla_X Y = \nabla_Y X = \nabla_Y Y = 0 \quad (54)$$

$$h(X,X) = h(X,Y) = h(Y,X) = 0, \quad h(Y,Y) = \frac{-a}{a^2+b^2} v_1 \quad (55)$$

$$A_{v_1} X = A_{v_2} X = A_{v_2} Y = 0, \quad A_{v_1} Y = \frac{-a}{a^2+b^2} Y \quad (56)$$

$$D_X v_1 = D_X v_2 = 0, \quad D_Y v_1 = \frac{-b}{a^2+b^2} v_2, \quad D_Y v_2 = \frac{b}{a^2+b^2} v_1. \quad (57)$$

Substituting (6), (8), (54), (55), (56) and (57) into (40) and (41) we have

$$J(X) = J_1 = 0, \quad J(Y) = J_2 = \frac{a(b^2-3a^2)}{(a^2+b^2)^3} v_1. \quad (58)$$

Substituting (37) and (58) into (44) we get the result.

Example 27. We define surfaces embedded in IE^4 by

$$x(u,v) = (u, v, u\cos v, u\sin v)$$

and we show that surfaces is of type *AW(3).*
After some calculations we get

$$J(X) = J_1 = 0, \quad J(Y) = J_2 = \frac{-\sqrt{2}u}{(1+u^2)^3} v_1. \quad (59)$$

Substituting (37) and (59) into (44) we get the result.

Example 28. We define surfaces embedded in IE^4 by

$$x(u,v) = \{(u\cos v, u\sin v, \cos bv, \sin bv) : b \in IR\}$$

and we show that surfaces is of type *AW(3).*
After some calculations we get

$$J(X) = J_1 = 0, \quad J(Y) = J_2 = \frac{b^2(u^2b^2+8u^2-3b^4)}{(u^2+b^2)^3} v_1. \quad (60)$$

Substituting (37) and (60) into (44) we get the result.

Example 29. We define a Mobius band $\mathbf{M}^2$ embedded in IE^4 by

$$x(u,v) = (\cos u, \sin u, v\cos\frac{u}{2}, v\sin\frac{u}{2})$$

and we show that $\mathbf{M}^2$ is of type *AW(3)*.
After some calculations we get

$$J(X) = J_1 = \frac{-144}{(4+v^2)^3} v_1, \quad J(Y)=J_2=0. \tag{61}$$

Substituting (37) and (61) into (44) we get the result.

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