SURFACES SATISFYING $\bar{R}(X,Y).H=0$

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Abstract

In this study we consider the surfaces M^n in IE^5 satisfying the condition $\bar{R}(X,Y).H=0$ where H is the mean curvature vector of M.
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Özet

Bu çalışmada, H ortalama eğrilik vektörü olmak üzere, $\bar{R}(X,Y).H=0$ şartını sağlayan IE^5 deki M^n yüzeyleri gözönüne alındı.
Anahtar Kelimeler: Semi-paralel, Semi-simetrik uzay.

1- INTRODUCTION

Let $x: M^n \to E^m$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into m-dimensional Euclidean space IE^m. Denote by $\bar{R}$ the curvature tensor of the van der Waerden-Bortolotti connection $\bar{\nabla}$ of x and by h the second fundamental form of x. x is called *semi-parallel* if $\bar{R}.h=0$, i.e. $\bar{R}(X,Y).h=0$ for all tangent vectors X and Y to M, where $\bar{R}(X,Y)$ acts as a derivation on h. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R.R=0$, and a direct generalization of parallel immersions, i.e. isometric immersions for which $\bar{\nabla} h=0$. In [1], J. Deprez showed the fact that $x: M \to E^m$ is semi-parallel implies that M is semi-symmetric.

For references on semi-symmetric space, see [2]; for references on parallel immersions, see [3]. In [1], J. Deprez gave a local classification of semi-parallel hypersurfaces in Euclidean space. It is easily seen that all surfaces are semi-symmetric. In [4] J. Deprez gave a full classification of semi-parallel surfaces in IE^m.

In the present study we consider the surfaces M^n in IE^5 satisfying the condition

$$\bar{R}(X,Y).H=0 \qquad (1)$$

where H is the mean curvature vector of M. We have shown that surfaces in IE^5 satisfying the property (1) are minimal or totally umbilic or has trivial normal connections.

2-BASIC RESULTS

Let $x\colon M^n \to E^m$ be an isometric immersion of an n-dimensional (connected) Riemannian manifold M^n into m-dimensional Euclidean space IE^m. Let ν be a local unit normal section on *M*. In the sequel *X, Y, Z, U, V* denote vector fields which are tangent to M^n. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2}$$

and

$$\widetilde{\nabla}_X \nu = -A_\nu X + D_X \nu \tag{3}$$

respectively, where $\widetilde{\nabla}$ is the Levi Civita connection on IE^m, ∇ the Levi Civita connection on M^n and *D* the normal connection of *x*. The second fundamental tensor A_ν is related to the second fundamental form *h* by

$$< A_\nu X, Y >=< h(X,Y), \nu > \tag{4}$$

where < , > is a standart metric of IE^m.

If *M* is a surface, the Gaussian curvature of *M* at $x \in M$ becomes

$$K(x) =< R(X,Y)X, Y > \tag{5}$$

where *X* and *Y* form an orthonormal basis for $T_x M$. The mean curvature vector *H* of *x* is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{6}$$

where $e_1, e_2, ..., e_n$ is the orthonormal basis of $T_x M$. The mean curvature α of *x* becomes

$$\alpha = \sqrt{< H, H >}.$$

A totaly geodesic immersion *x* is an isometric immersion for which *h=0.* If *H=0* then *x* is called *minimal* and *x* is called *totally umbilical* if

$$h(X,Y) =< X, Y > H$$

where *X, Y* is an orthonormal basis of *M*. The immersion *x* is called *isotropic* (in the sense of O'Neill [5]) if for each *x* in *M* $\|h(X,X)\|$ is independent of the choice of a unit vector *X* in $T_x M$.

Let $X \Lambda Y$ denote the endomorphism $Z \to < Z, Y > X - < Z, X > Y$. Then the curvature tensor *R* of *M* is given by the equation of Gauss:

$$R(X,Y) = \sum_{i=1}^{p} A_i X \;\Lambda\; A_i Y \tag{7}$$

where $A_i = A_{\nu_i}$ and $\{\nu_1, ..., \nu_p\}$ is a local orthonormal basis for $T_x^{\perp} M$. The equation of Ricci becomes

$$< R^{\perp}(X,Y)\nu, \eta >=< [A_\nu, A_\eta] X, Y > \tag{8}$$

for ν and η normal vectors to *M*. An isometric immersion *x* is said to have trivial normal connection if $R^{\perp} = 0$. (8) shows that triviality of the normal connection of *x* is equivalent to the fact that all second fundamental tensors mutually commute and they are simultaneous diagonalizable.

Let *M* be an n-dimensional Riemannian manifold and *T* be a (0, k)-type tensor on *M*. The tensor *R.T* is defined by

$$(R.T)(X_1, X_2, X_3, X_4; X, Y) = (\widetilde{R}(X,Y).T)(X_1, X_2, X_3, X_4)$$
$$= -T(\widetilde{R}(X,Y)X_1, X_2, X_3, X_4) - T(X_1, \widetilde{R}(X,Y)X_2, X_3, X_4)$$
$$- T(X_1, X_2, \widetilde{R}(X,Y)X_3, X_4) - T(X_1, X_2, X_3, \widetilde{R}(X,Y)X_4) \quad (9)$$

where $X_1, X_2, X_3, X_4, X, Y \in \chi(M)$.

Let $\bar{\nabla}$ be the connection of van der Waerden-Bortolotti of *x*, denote the curvature tensor of $\bar{\nabla}$ by $\bar{R}$ then

$$(\bar{R}(X,Y).h)(U,V) = R^{\perp}(X,Y)h(U,V) - h(R(X,Y)U,V) - h(U,R(X,Y)V) \quad (10)$$

Lemma 1. Let *M* be a surface in IE^5 then

$$(\bar{R}(e_1,e_2).h)(e_1,e_1) = (\lambda-\mu)(a_2b_2 + a_3b_3)\nu_1 + [-\lambda(\lambda-\mu)b_2 + 2\beta a_3 + 2Kb_2]\nu_2 + [-\lambda(\lambda-\mu)b_3 - 2\beta a_2 + 2Kb_3]\nu_3 \quad (11)$$

and

$$(\bar{R}(e_1,e_2).h)(e_2,e_2) = -(\lambda-\mu)(a_2b_2 + a_3b_3)\nu_1 + [-\mu(\lambda-\mu)b_2 - 2\beta a_3 - 2Kb_2]\nu_2 + [-\mu(\lambda-\mu)b_3 + 2\beta a_2 - 2Kb_3]\nu_3 \quad (12)$$

where *K* is the Gaussian curvature of $M \subset IE^5$ and $\beta = a_2b_3 - a_3b_2$.

Proof. (see [6]).

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Definition 2. Let *M* be a surface in IE^5 then we define $\bar{R}.H$ by

$$\bar{R}(e_1,e_2).H = \frac{1}{2}\{(\bar{R}.h)(e_1,e_1) + (\bar{R}.h)(e_2,e_2)\} \quad (13)$$

where e_1, e_2 is an orthonormal basis of the surface *M*.

Corollary 3.

$$\bar{R}(e_1,e_2)H = \frac{1}{2}\{[-\lambda(\lambda-\mu)b_2 - \mu(\lambda-\mu)b_2]\nu_2 + [-\lambda(\lambda-\mu)b_3 - \mu(\lambda-\mu)b_3]\nu_3\}$$
$$= \frac{1}{2}\{-b_2(\lambda-\mu)(\lambda+\mu)\nu_2 - b_3(\lambda-\mu)(\lambda+\mu)\nu_3\} \quad (14)$$

Proof. By Lemma 1 and (6) we get the result.

Proposition 4. [7] Let *M* be a surfaces in IE^5 and $\nu_1,...,\nu_p$ orthonormal vectors in *N(M)* such that ν_1 is in the direction of the mean curvature vector and such that

$A_{\nu_4} = ... = A_{\nu_p} = 0$. If we choose an orthonormal basis of TM of eigenvectors of $A_1 = A_{\nu_1}$. Identifying linear transformations and their matrices in this basis, we obtain

$$A_1 = A_{\nu_1} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \; A_2 = A_{\nu_2} = \begin{bmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{bmatrix}, \; A_3 = A_{\nu_3} = \begin{bmatrix} a_3 & b_3 \\ b_3 & -a_3 \end{bmatrix}. \qquad (15)$$

Theorem 5. Let M be a surface in IE^5 satisfying the property $\bar{R}.H = 0$ then M is one of the following surfaces:

1) a totally umbilic surface with $\lambda = \mu$, or

2) a surfaces with trivial normal connection and $H = 2\lambda$, or

3) a minimal surface.

Proof. If $\bar{R}.H = 0$ then by previous Corollary we get

$$-b_2(\lambda-\mu)(\lambda+\mu)\nu_2 - b_3(\lambda-\mu)(\lambda+\mu)\nu_3 = 0. \qquad (16)$$

Thus, we have

$$b_2(\lambda-\mu)(\lambda+\mu) = 0 \;\text{ and }\; b_3(\lambda-\mu)(\lambda+\mu) = 0. \qquad (17)$$

Therefore we have three possibilities

1) If $b_2 = b_3 = 0, a_2 = a_3 = 0$ then the equations (16) and (17) are automatically satisfied. Therefore M is totally umbilic.

2) If $b_2 = b_3 = 0, a_2 \neq 0, a_3 \neq 0$ then by (15) we get

$$A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \; A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & -a_2 \end{bmatrix}, \; A_3 = \begin{bmatrix} a_3 & 0 \\ 0 & -a_3 \end{bmatrix}$$

which implies that $R^{\perp} = 0$ i.e. M has trivial normal connection.

If $\lambda = \mu$ then the equations (16) and (17) are automatically satisfied and by (15) we get $H=2\lambda$.

3) If $\lambda = -\mu$ then the equations (16) and (17) are automatically satisfied and by (15) we get $H=0$ (i.e. M is minimal).

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