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Araştırma Makalesi / Research Article

Design Study To Develop The Proof Skills Of Mathematics Pre-Service Teachers

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Abstract – Mathematical proof, in addition to its duties such as verifying and explaining a claim, helps to understand mathematics and constitute new information, thereby increasing its importance in mathematics education. Proving skills need to be developed to learn mathematics and find effective solutions in real-life problem situations. Given the importance of proof in mathematics education, it is critical to develop appropriate teaching practices to increase pre-service teachers' knowledge of proof, proving, and understanding levels, particularly in mathematics teacher training programs, to provide effective and permanent mathematics learning for future students. The effect of designed proof tasks on the development of pre-service mathematics teachers' proving skills was investigated in this study. This research, the design-based research, was carried out with the participation of three volunteer mathematics teacher candidates studying at a state university in Ankara. The data obtained from the proof tasks and individual interviews during four weeks period. Proof tasks designed according to the findings obtained from the data analyzed with the qualitative analysis method contributed to the development of the participants' proving skills.

Key words: mathematical proof, proving skills, proof task, pre-service mathematics teacher.

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1 This study was produced from the first author's PhD dissertation.

Introduction

Mathematical proof is an important mathematical argument, a connected sequence of assertions against a mathematical claim, that has set of accepted statements as true and does not require justification, and employs known and valid reasoning forms, as well as forms of expression that are appropriate in communication (Stylianides, 2007). The development of proof skills is not only important in mathematics, but also among the primary objectives of the mathematics teaching curriculum (NCTM, 2000). The most important reason for this is that reasoning and proof help students relate their previous knowledge to new ones, make inferences and make sense of their new knowledge (Brodahl et al., 2020). Thus, learning and teaching proof is critical in mathematics education (Hanna & de Villiers, 2008, 2012; Yan, Mason & Hanna, 2017) in terms of providing learning and making sense of mathematics, as well as assisting in the discovery of effective solutions in real-life problem situations (Mariotti, 2006).

In the process of mathematical proof, there are processes such as verifying a proposition, explaining why it is true, systematizing the results obtained (de Villiers, 1990; Selden & Selden, 2003), discovering new results and hypotheses through inferences, and expressing the results using mathematical language and notation. (Harel and Sowder, 2007; Ko and Knuth, 2009). This complex nature of the proof process (Moore, 1994; Dreyfus, 1999) causes students, pre-service teachers and even teachers to experience difficulties in this process. (Knapp, 2005). In the conducted studies, it was revealed that difficulties such as not knowing where to start proving and how to continue (Healy and Hoyles, 2007), not using the existing preliminary information strategically (Weber, 2001), lack of knowledge of definition (Dane, 2008; Polat and Akgün, 2016; Cihan, 2019), inability to use mathematical language and notation (Moore, 1994; Knapp, 2005), inability to determine the appropriate proof method (Baki and Kutluca, 2009), inability to understand the nature of proof (Çontay and Duatepe-Paksu, 2019), lack of mathematical concept readiness and lack of proof image (Pala, Aksoy and Narlı, 2021) occur. It is thought that the teaching methods, techniques, and tools used are effective in understanding the proof conceptually (Weber, 2004), gaining the ability to prove (Harel, Selden, & Selden, 2006) and overcoming difficulties experienced with proof (Hanna and de Villiers, 2008; Skott, Larsen and Østergaard, 2020). The difficulties that arise in the studies show that there are problems, deficiencies, or mistakes in teaching proof (Cihan, 2019; Zeybek Şimşek, 2020).

Related Literature

Although the difficulties experienced in the proving process are revealed in the studies, there are very few studies to find solutions to these difficulties and improve the current situation (Selden, Selden and Benkhalti, 2017; Yan, Mason and Hanna; 2017). Since the concept of proof is an important part of mathematics education, it is emphasized that appropriate teaching practices should be designed, effective application guidelines should be developed, and such researches should be increased to enhance pre-service teachers' knowledge of proof, ability to make proof, and concept comprehension levels, especially in mathematics teachers training departments (Yan, Mason and Hanna; 2017; Zeybek Şimşek, 2020). In this direction, it is important for pre-service teachers to deal with proof (Harel, Selden, & Selden, 2006), and to examine and develop the proving processes for the development of proof skills (Sarı Uzun, 2020). In the studies conducted in this field, proof schemes (Balacheff, 1988; Harel & Sowder, 1998; Weber, 2004), ways of thinking in the proof process (Raman, 2003), categories of explanations presented as proof (Miyazaki, 2000) have been constructed, and methods, strategies, and techniques (Dean, 1996; Schabel, 2001; Selden and Selden, 1995) for teaching proof have been presented. Many studies have been conducted in the recent past to use the methods, techniques, and strategies obtained as a result of these studies (Arslantaş İlter, 2020; Selden, Selden, and Benkhalti, 2017; Yan, Mason, and Hanna, 2017;), but no study has been found that provides a guide for the teaching of proof and the development of the proving process.

Proof tasks were developed and implemented in this study in response to the difficulties experienced by pre-service mathematics teachers during the proving process, such as not knowing where to begin the proof, not being able to continue the proof, using mathematical language and symbols, a lack of knowledge, and not being able to distinguish between hypotheses and judgments. These proof tasks are designed as a mathematical task (Greenberg, 1993; Weber, 2005) that covers the stages of applying inference rules and completing the proof until the desired result is obtained by providing some preliminary information (e.g., inferences, axioms, definitions).

Research Problem

This research is a part of a PhD dissertation that aims to improve pre-service mathematics teachers' proving skills. In the scope of this study, the contribution of designed proof tasks to the development of proof skills in pre-service mathematics teachers was investigated. For these purposes, the research problem and its subproblems are as follows:

- How did the designed proof tasks affect pre-service mathematics teachers' proving skills?
    - How did the 1st designed proof task affect pre-service mathematics teachers' proving skills?
    - How did the 2nd designed proof task affect pre-service mathematics teachers' proving skills?
    - How did the 3rd designed proof task affect pre-service mathematics teachers' proving skills?
- What are the opinions of pre-service mathematics teachers on design?

Method

This section covers the research method, stages, study group, data collection tools and data analysis.

Research Design

In this research, proof tasks including theorems involving basic mathematics subjects and designed as a guide in the proof process were applied to pre-service mathematics teacher in this research. The study focuses to contribute to the development of prospective teachers' proof skills with enhancing the design of these tasks by applying and evaluating it. In this sense, the model of the research was determined as design-based research (DBR).

As a constructivist approach that emphasizes learning by systematically designing teaching strategies and tools, design-based research (DBR) contributes to the creation, development, acceptance, and continuity of knowledge in learning environments (Brown, 1992; Collins, 1992). In this research, the ADDIE model (Branch, 2009), which includes the stages of analysis, design, development, implementation, and evaluation, was used as the DBR model.

The use of a design-based research approach (Doorman, Bakker, Drijvers & Wıjaya, 2016), which provides a systematic approach to the proof process and allows for the generalization of findings on specific contexts, was thought to be beneficial in emphasizing learning, creating, and developing knowledge.

Mathematical tasks are defined as tasks that provide a student with the opportunity to learn new mathematical content such as concepts and procedures or to develop mathematical processes such as analytical skills, creativity, and cognitive skills (Stylianides, 2016). According to the literature, mathematical tasks improve the quality of mathematics education and are an effective learning and teaching tool for learning mathematical concepts (NCTM, 2000; Stylianides and Ball, 2008; Stylianides, 2016). Therefore, in this study, it was considered that proof tasks are an appropriate tool for developing the proof skills of pre-service mathematics teachers.

Stages of Research

The analysis part of the DBR model, in which applications are made and evaluated to determine the current situation for pre-service mathematics teachers, the concept of proof and being able to make proof, and the difficulties and deficiencies experienced in the context of proof as a result of the analysis of the findings, are not presented within the scope of this research. However, it is explained in detail in the PhD dissertation.

The course content and proof tasks were prepared during the design and development phase as a result of the researcher's scan of the field studies (Almeida, 2000; Atwood, 2001; Sarı, 2011; Selden, Selden and Benkhalti, 2017; Selden, Selden and McKee, 2008; Stylianides and Stylianides, 2009), the examination of the relevant books (Hammack, 2013; Houston, 2009; Vellenman, 2006), and the expert opinions, in order to eliminate the deficiencies identified as a result of the applications, to assist in overcoming the difficulties encountered, and to prepare for the proof process.

Only a part of the implementation phase is included in this research. In this part, the explanation of the proving process with basic proof methods and the implementation of proof tasks were carried out in this part over a four-week period, with the researchers and participants meeting every week after the completion of the preparation for proof part, to achieve the following goals:

- To know and to apply the direct proof method.
- To know and to apply the method of proof by contradiction.
- To apply proof methods to fundamental subjects.

During the evaluation phase, the data obtained from the proof task practices in this study were evaluated, and interviews were conducted with three volunteers at the end of each proof task, using a semi-structured individual interview form developed by reviewing the relevant literature, considering the parts in the designs, and experts' opinions. In these interviews, they were asked to explain the proof process, which includes stages such as understanding the theorem, determining the appropriate proof method, determining the desired result, and the proof task inference process. Beside from that, participants were asked about the difficulties they encountered during the proof process, type of information they required, and their opinions on proof tasks. Following the evaluation of the findings, changes were made to the proof task design and implemented again with a different theorem.

Participants

In this research, volunteered three pre-service mathematics teachers who are at peace with symbolic logic, who can express themselves, who are willing to learning and improvement, were selected as the participants of the study, by taking the opinions of the instructors who gave proof-related courses at a state university in Ankara. During the research, pseudonyms as S1, S2 and S3 were defined for the participants in accordance with ethical rules.

Data collection

Data were collected through proof tasks and semi-structured individual interviews in the context of the courses mentioned above.

Proof tasks

Theorems determined by examining the related books (Houston, 2009; Hammack, 2013; Vellenman, 2006) and taking expert opinions are included in the proof tasks prepared to examine and improve the proving processes of prospective mathematics teachers. Three faculty members who are experts in the field evaluated the theorems for suitability and mathematical language, and their final form was determined based on their feedback. The use of different theorems in each design is critical for the development and objective evaluation of the proving process participants. The following are the theorems that were used for each design:

1. "Let A, B, C be sets and $A \setminus B \subseteq C$. Then $A \setminus C \subseteq B$."
2. "Let $f: X \to Y$ be a function and $A \subset X$. If f is one-to-one, then $f(X \setminus A) \subset Y \setminus f(A)$."
3. "*Let* $\lim_{x\to a} f(x) = A$ *and* $\lim_{x\to a} g(x) = B$. *Then* $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A + B$ *dir.*"

Semi-Structured Individual Interviews

The purpose of semi-structured interviews is to thoroughly examine pre-service mathematics teachers' proof processes. Interviews were conducted three times, at the end of the first, second, and third proof tasks, using the semi-structured individual interview form, and audio recorded with the participants' permission. The interview form, which was prepared considering the literature review and the design components in mind, took its final form after three faculty members who were experts in the field weighed in on its suitability for the purpose, and the following questions were included in the form:

1. *Can you explain the proof task process step by step?*
2. *What kind of external sources did you use when you had difficulties while working on proof tasks? And how has the guidance you received from these outside sources helped you?*
3. *What kind of information would you need that you thought you would have done better if "....." was included in the proof tasks?*

In addition to these questions, during the last interview, the participants were asked some questions such as "*How would you describe your progress in proving in general through this study?*" and "*Did the tasks change your point of view towards proof? Please explain.*" to get the opinions of the participants.

Data Analysis

The research data obtained through proof tasks and recordings of interviews conducted with participants. Content analysis was used as a data analysis method. The data were analyzed by the arrangement of the data, coding of the data and the creation of themes, and the data were interpreted by associating them with each other.

The codes obtained during the analysis process were determined by the researcher's and consultant's shared opinions, and themes and categories were created.

Analysis of Proof Tasks

In the proof tasks, it has been requested that the participants should identify the propositions in the given theorem and express them as conditional propositions in the form of $p \Rightarrow q$, determine the hypothesis and conclusion, write the first sentence together with the reason as the introductory sentence of the proof process, make inferences from the first sentence, write the last sentence and the reason, and complete the proof. These steps determined for the process are presented in the proof task design. The following are the first proof task responses:

In the part of determining the components of the theorem, the expression of the theorem as $p \Rightarrow q$ is "A, B, C is a set and $A \setminus B \subseteq C$ is $A \setminus C \subseteq B$", hypothesis (p) : "$A \setminus B \subseteq C$", conclusion (q) : "$A \setminus C \subseteq B$".

Let the first sentence, which is the introductory sentence to the proof process, be the hypothesis, that is, the assumption: "$A \setminus B \subseteq C$".

It is expected that the inferences that can be made in the proof process will be made by using the properties of the sets according to the selected proof method. For example, let "$a \in A \setminus B$" for direct proof. Therefore, it can be started as "$a \in A$" and "$a \notin B$". By using the inclusion property in sets, the desired result is obtained. Let's show that there is no "$A \setminus C \subseteq B$" according to the method of proof by contradiction. Let's take an element a, with "$a \in A \setminus C$" and "$a \notin B$". Hence "$a \in A, a \notin C$" and "$a \notin B$". From here,"$a \in A \setminus B$" and "$a \notin C$" are obtained. So to be "$a \in A \setminus B \subseteq C$" it becomes "$a \in C$" and contradicts "$a \notin C$". By making inferences in this way, a contradiction is obtained, and the desired result is achieved.

The last sentence of the proof process, namely the desired result, is "Thus $A \setminus C \subseteq B$".

Changes were made to the design of the second proof task in response to the evaluation of the first proof task and the data obtained from the individual interviews. Due to the difficulties encountered after writing the first sentence to start the proof, steps were added to the inference section for the step of writing the second sentence and for making inferences step by step while making inferences. As in the first proof task, the evaluation was done as true, partly true, and false for each step.

The second proof task responses are given below.

Answers for determining the components of the theorem; restate the theorem: "If f is one-to-one, then $f(X \setminus A) \subset Y \setminus f(A)$ for $\forall A \subseteq X$." Hypothesis (p): "f is a one-to-one function." Conclusion (q): "$f(X \setminus A) \subset Y \setminus f(A)$".

The introductory sentence of the proof process (assumption): "Let f be a one-to-one function."

The second sentence of the proof process (according to the proof by contradiction method): "Let us assume that there is no $f(X \setminus A) \subset Y \setminus f(A)$."

The implications that can be made in the proving process are: "There is $\exists y \in X$ such that $y \in f(X \setminus A)$ and $y \notin Y \setminus f(A)$ (from the definition of inclusion). There is $\exists x \in X \setminus A$ such that $y = f(x)$ (from the image set definition). $y \in f(A)$ (from the difference set definition). There is $\exists a \in A$ such that $y = f(a)$ (from the image set definition). From here, if $f(x) = y = f(a), x = a$ and $x \in A$ contradiction is obtained."

The last sentence of the proof process: "Then $f(X \setminus A) \subset Y \setminus f(A)$".

In line with the evaluation of the second task and the data obtained from the interviews, it was determined that the participants had difficulties while making inference due to the lack of information about the proposition/theorem given. Accordingly, in addition to the second design, it was considered appropriate to provide the information that may be needed while making the proof, and an information box was added in the third design. Evaluation was made in the same way as the first and second designs.

The third proof task responses are given below.

"Let $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$. Then

$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B$"

Determining the components of the theorem; restate the theorem: "If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ then $\lim_{x \to a} [f(x) + g(x)] = A + B$"

Hypothesis (p):$\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$"

Conclusion (q): "$\lim_{x \to a} [f(x) + g(x)] = A + B$"

The introductory sentence to the proof process (assumption):

" Let $\lim_{x \to a} f(x) = A$ *and* $\lim_{x \to a} g(x) = B$."

Second sentence of the proof process (according to the direct proof method): Give the number ε>0 (from the limit definition) to show that $\lim_{x \to a} [f(x) + g(x)] = A + B$.

Inferences that can be made in the proving process: $if \lim_{x \to a} f(x) = A\ then\ \exists \delta\_1 > 0\ so\ if\ 0 < |x - a| < \delta\_1\ then\ |f(x) - A| < \varepsilon / L\ and\ if\ \lim_{x \to a} g(x) = B\ then\ \exists \delta\_2 > 0\ so\ if\ 0 < |x - a| < \delta\_2 then\ |f(x) - B| < \varepsilon / L$ (from the limit definition).

$\delta\_i = min\{\delta\_1, \delta\_2\}, if\ 0 < |x - a| < \delta \quad then\ |(f(x) + g(x)) - (A + B)| = |(f(x) - A) + (g(x) - B)| \leq |f(x) - A| + |g(x) - B| < \varepsilon/L + \varepsilon/L = \varepsilon$ (triangle inequality and the definition of limit)"

The last sentence of the proof process (desired conclusion):

"Therefore $\lim_{x \to a}[f(x) + g(x)] = A + B$."

In order to ensure the internal validity of the research and to evaluate the stages in the proof process more objectively, different theorems were used in each design.

Analysis of Semi-Structured Individual Interviews

First and foremost, the interview recordings were converted and edited into written text. The audio recordings were listened to several times by the researchers while they were transcribed in order to ensure the reliability of the research. The researchers identified the steps in the proof tasks as themes and collected data around these themes. As a result, more detailed information about the participants' thoughts was obtained at each stage of the evidence process.

In the final interviews, participants were asked to evaluate the working process and explain how it helped them, and the data gathered was also analyzed.

Findings

The findings and comments obtained as a result of the analysis of the data collected from three pre-service teachers who participated in the four-week classroom studies that included the evidence tasks in the research and where individual interviews were conducted are presented in this section.

Findings of the First Design

In the first proof-of-design event, the participants were asked "$Let\ A, B, C\ be\ sets, and\ A \setminus B \subseteq C.\ Then\ the\ theorem\ "A \setminus C \subseteq B"\ is\ given.$" The findings obtained from the answers of the participants are given below.

Determining the Components of the Theorem

Participants were asked to identify and express the hypothesis and clause components of a given theorem in the form of $p \Rightarrow q$ conditional propositions. The following is the response of participant S1.

***p*:** A,B,C küme ve A\B⊆C	***q*:** A\C⊆B'dir.
Express the theorem in the form $p \Rightarrow q$. A,B,C küme ve A\B⊆C ⇒ A\C⊆B'dir.	

Figure 1: S1's response to determining the components of the first proof task theorem

While the participant correctly determined the hypothesis, s/he made a false inference by accepting the sets mentioned in the theorem statement as an inference, despite working in the set theory universe. Furthermore, s/he stated in this section that the proving process described in the lecture came to mind and that when s/he saw the theorem, s/he determined how to determine the propositions p and q in his/her mind as follows.

S1: *"As soon as I saw the theorem since it is a familiarity after you explained it, I first thought about how to separate "p" from "q". Since the theorem consists of two different sentences, when I saw the 'Let $A \setminus B \subseteq C$' part, I directly sensed that it was "p" and the other was "q"."*

The participant named S2 correctly determined the conclusion and hypothesis components of the theorem, as seen in Figure 2.

p: A\B⊆C 'dir.	q: A\C ⊆ B'dir.
Express the theorem in the form $p \Rightarrow q$. A,B,C Küme ve A\B⊆C ⇒ A\C⊆B 'dir.	

Figure 2: S2's response to determining the components of the first evidence task theorem

As shown in Figure 2, participant S2 correctly determined propositions p and q and correctly restated the theorem. According to the participant, s/he first attempted to understand the theorem and use the conjunction "*if*'.

S2: "First of all, I tried to understand the theorem. It came to me as a special expression. There are no mathematical conjunctions in this theorem, so it is not a mathematical statement. I even considered it when I asked in the first question for it to be written as " $if\ p, then\ q.$ " With the conjunction "if…,then…" I stated the theorem."

Although the participant named S3 determined the hypothesis correctly, she made a mistake by accepting the sets in the theorem statement as assumptions.

p: A,B,C küme ve A\B⊆C olsun.	***q***: A\C⊆B dir.
Express the theorem in the form $p \Rightarrow q$. A,B,C küme ve A\B⊆C ise A\C⊆B dir.	

Figure 3: S3's response to determining the components of the first proof task theorem

In this part, the participant explained that s/he tried to understand the theorem by drawing a figure before starting the proof as follows:

S3*: "I read the theorem first, then immediately thought of drawing a figure to visualize it." I gave it a few tries to see what would happen if the shapes were subsets of each other and what would happen if they intersected."*

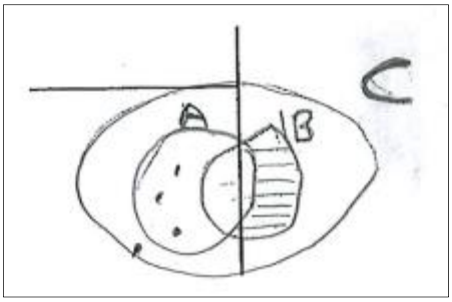


Figure 4: The drawing S3 drew for the theorem.

To understand the theorem, participants were observed using strategies such as looking at keywords such as "*if* ", "*then*", "*in that case*" and drawing a figure.

Introduction Sentence of the Proof Process

Participants were asked to write the first sentence of the proof process according to the proof method they chose.

When the participants' answers were examined, it was revealed that they chose the theorem's assumption as the first sentence of the proving process. As shown in Figure 5, the participant S1 chose the method of proof by contradiction and provided a correct answer based on the method.

Write the first sentence to prove the theorem.	**Reason:**
∀A,B,C küme ve A\B⊆C olsun ve A\C⊄B olsun.	Olmayana Ergi kanıt yöntemini seçtiğim için.

Figure 5: First sentence response of S1's first proof task

The participant stated that s/he first turned to direct evidence but decided to continue with the method of proof by contradiction and expressed his/her first sentence with the reason as “I thought I should accept that p is true and arrive at q, but after separating p and q, I thought I should use the method of proof by contradiction and wrote my first sentence accordingly.”.

The first sentence response of the participant named S2 is as in Figure 6.

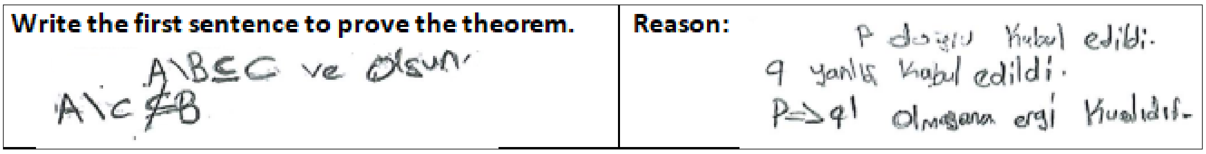


Figure 6: First sentence response of S2's first proof task

When the participant first looked at the theorem, s/he thought that s/he would use method of proof by contradiction as the solution method but stated that s/he tried the direct proof method but could not progress and explained that s/he decided to continue with the method of proof by contradiction as “When I first looked at the theorem, I had guessed that it would be solved with the method of proof by contradiction. But still, since I didn't try with direct proof first, I decided to do it with the method of proof by contradiction. After writing it as $p \Rightarrow q$, I decided to show that $q' \Rightarrow p'$ is true to show that this is true.” and it was seen that his/her answer was appropriate.

The answer of the participant named S3 is given below.

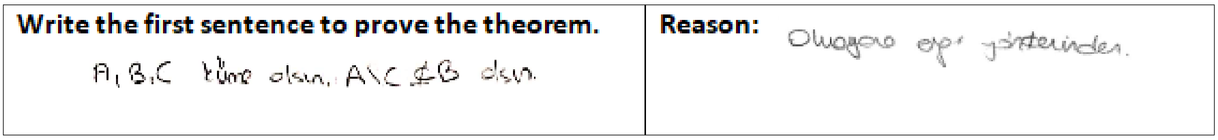


Figure 7: First sentence response of S3's first proof task

S3 also stated that, like S2, s/he attempted the direct proof method first; however, s/he was unable to prove, and thus chose the method of proof by contradiction. Figure 7 shows his/her response to the first sentence. The participant's explanation for this section is as follows:

S3: *“When I read the theorem, I wanted to try the direct proof method first. I saw that I was unable to settle a matter. I thought I'd try it the method of proof by* *contradiction. So, I took the proposition $p \Rightarrow q$ as $q' \Rightarrow p'$ and wrote my first sentence."*

The introductory sentence to the proof process, known as the assumption, is determined by the proof method used. Knowing the chosen proof method is critical for taking the first step in the proof. Looking at the answers, it was discovered that the chosen proof method was written as the reason for the introductory sentence rather than "assumption".

Inferences Made in the Proof Process

In this part, the participants are expected to make inferences from the introductory sentence they wrote and explain their reasons. The answer of the participant named S1 is given below.

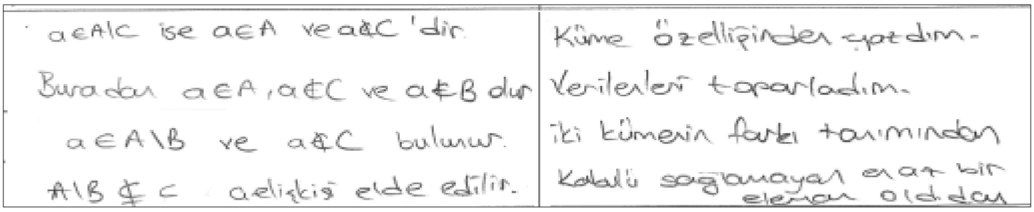


Figure 8: S1's inference response to the first proof task

As in Figure 8, it was seen that the participant made inferences using the properties of sets and difference definition. This part was explained as:

S1: "*I found that if I use the difference set property, if it is an element of set A, it will not be an element of set C. Then I gathered the givens and found at least one element that did not meet the requirement."*

The inferences of the participant named S2 during the proving process are given in Figure 9.

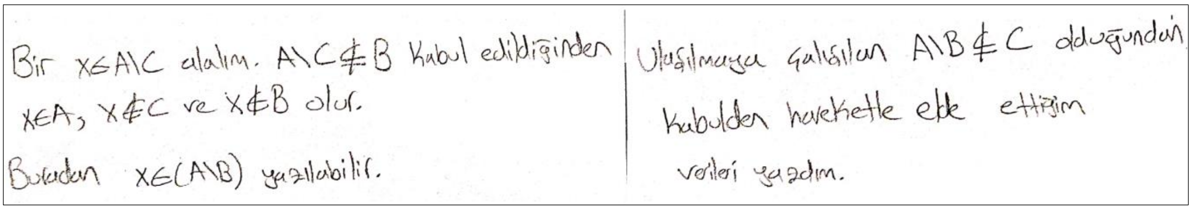


Figure 9: S2's inference response to the first proof task

The answer of the participant named S2 that s/he started the inference process by taking x element and made inferences given in Figure 9. S/he made a statement about this part as "*I thought if I didn't get the x element, I wouldn't be able to make any inferences. So, I started by getting an element.*"

S3's response, in which s/he obtained a contradiction by making inferences, is as follows:

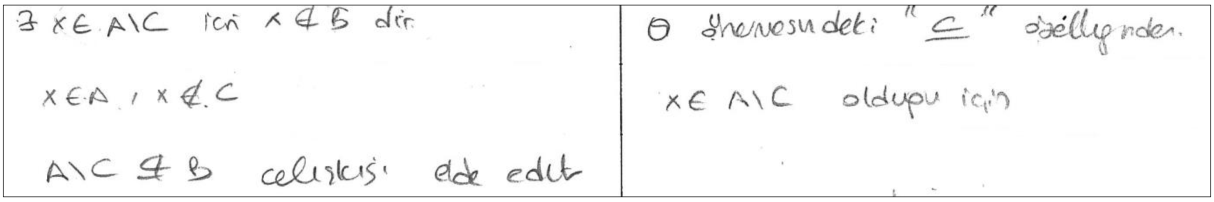


Figure 10: S3's inference response to the first proof task

The participant obtained a contradiction by writing his/her reasons and making inferences and described this process as "*I thought about what I could achieve using the properties of the set and wrote them down.*"

The participants made inferences based on the properties and meanings of the sets, just as they did in their answers and explanations.

Last sentence of the Proof Process

The last sentence of the theorem regarding the proving process was asked to each participant. Figure 11 shows, for example, the last sentence of participant S1 and the reason for the proof process. By writing the desired outcome, the participant provided the correct answer.

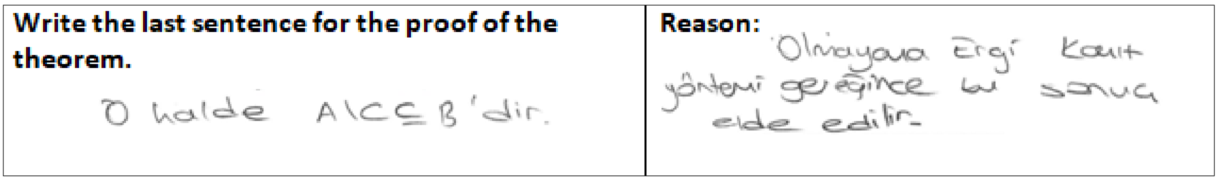


Figure 11: S1's last sentence response to the first evidence task

The participant stated how s/he wrote this result as "This result is obtained according to the method of proof by contradiction".

The participant named S2 stated that when he separated the theorem as p and q in his/her last sentence, s/he thought that the result he wanted to reach was q and gave the correct answer as in Figure 12.

Write the last sentence for the proof of the theorem.	Reason:
O halde A\B⊆C → A\C⊆B sağlanır	

Figure 12: S2's last sentence response to the first evidence task

The participant stated that in order to reach this result, he established the following relationship between the data in the theorem:

> S2: *"When I separated the theorem as p and q, I realized that q is the result I wanted to reach. I interrelated the data and came to this conclusion."*

As shown in Figure 13, the participant named S3, like the other participants, correctly answered his/her last sentence as the result she aimed to achieve after selecting the method.

Write the last sentence for the proof of the theorem.	Reason:
A\B ⊈ C elde edilir. O halde A\B⊆C ise A\C⊆B dir.	

Figure 13: S3's last sentence response to the first evidence task

During the interview, the participant explained how she determined this result by saying, *"At first, I determined the proof method and according to it, I determined what I wanted to achieve in the last sentence while writing the first sentence."*

When the participants' opinions on the proof process were examined, it was determined that the steps provided in the task were useful, but they struggled with the inference process right after the first sentence. Participants S2 and S3 stated that restating the theorem in design is an important step in understanding the theorem, and that the given titles help with the proof process.

Findings of the Second Design

According to the results of the first proof-of-design task application and the individual interviews, the participants stated that they had difficulty making inferences and writing their reasons after the first sentence and were hesitant how to proceed.

> *S1: "After I wrote the first sentence, I had a hard time making inferences right away. Once I started, I was able to continue, but of course it took a while."*

Changes were made to the second design in comparison to the first design for this purpose. Following the introductory sentence to the proof, it was requested to write the second sentence using what was given in the theorem to support the inference process and provide a connection. Following that, it was asked to make gradual inferences from the first two sentences and prove them. The following steps are designed to write the final sentence and the entire proof.

In the proof task prepared in accordance with the second design, "*Let* $f: X \to Y$ *be a function and* $A \subset X$. *If* f *is one-to-one, then* $f(X \setminus A) \subset Y \setminus f(A)$" is given. The findings from this task are given below.

Determining the Components of the Theorem

Participants were asked to identify and express the hypothesis and conclusion components of a given theorem in the form of $p \Rightarrow q$ conditional propositions. Figure 14 illustrates the response of participant S1.

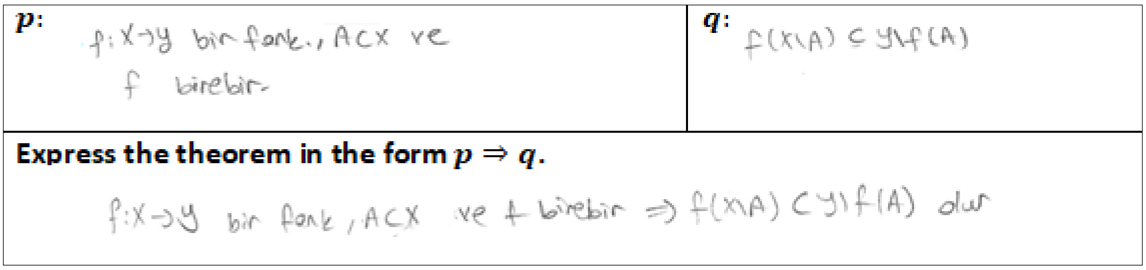


Figure 14: S1's response to determining the components of the second proof-of-design efficiency theorem

When the response of the participant was examined, it was seen that although she wrote the proposition q correctly, she wrote the statement in the first line extra in the proposition p, and she used the conjunction "and" incorrectly while restating the theorem as $p \Rightarrow q$.

S1: *"I had some trouble understanding the theorem. I had a hard time where to put the expression* $A \subset X$ *when making the distinction between* p *and* q*. Then I thought it should be* p*, but I felt like I had to prove it too."*

As stated by the participant named S1, the inclusion of the expression $A \subset X$ in the theorem caused difficulties in determining the components of the theorem.

In Figure 15, the answer of the participant named S2 regarding the determination of the components of the theorem is given.

p: $f: X \to Y$ birebir fonksiyon ve $f(X-A) \subset Y - f(A)$	**q:** $A \subseteq X$
Express the theorem in the form $p \Rightarrow q$. $f: X \to Y$ birebir fonksiyon ve $f(X-A) \subseteq Y - f(A) \Rightarrow A \subset X$ dir.	

Figure 15: S2's response to determining the components of the second proof-of-design efficiency theorem

When the participant's response is examined, it is clear that he made a mistake in restating the theorem and distinguishing between p and q. This section is explained further below.

S2: *"I tried to express the theorem with an if, but I was stuck in a dilemma. There are not two expressions, there are three here. I didn't know which of these to include in p and which in q."*

Similar to the answer of participant S1, participant S3 gave the answer in Figure 16 that while the statement "f is one-to-one" was sufficient in the proposition p, which is the components of the theorem, she also wrote the statement "Let $f: X \to Y$ be a function and $A \subset X$".

p: $f: X \to Y$ bir fonk ve $A \subset X$ ol. f birebir dir.	**q:** $f(X \setminus A) \subset Y \setminus f(A)$ dır.
Express the theorem in the form $p \Rightarrow q$.	

Figure 16: S3's response to determining the components of the second proof-of-design efficiency theorem

When the participant's response was examined, she made the proposition q correctly, but did not write the part of the theorem that should be expressed as $p \Rightarrow q$. In this part, the participant expressed what she did to understand the theorem as follows:

S3: *"I understood the theorem by drawing a figure. Visually it was better. I thought p as a one-to-one function, and that A is a subset of X, and I took the rest as q."*

Participants S1 and S2 mentioned difficulty distinguishing between p and q. It has been observed that the theorem is composed of three propositions, which makes it difficult to make this distinction based on two propositions. The participant named S3 stated that she distinguished between p and q through the use of visuals.

Introduction Sentence of the Proof Process

Participants were asked to write the first sentence of the proof process based on the method of proof they chosen.

It was discovered that the participants preferred the direct proof method in their responses, and they attempted to write the first sentence of the proving process as a result. Figure 17 shows the answer of participant S1.

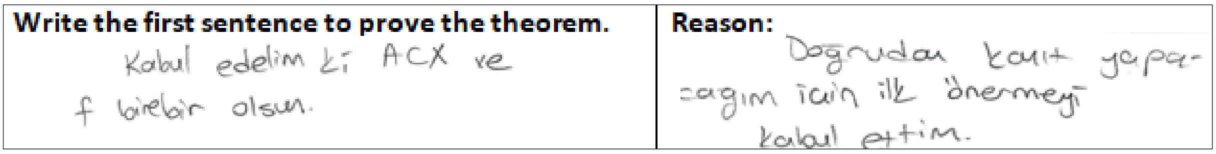


Figure 17: S1's first sentence response to the second proof-of-design task

In his/her first sentence, the participant used the conjunction "and" to combine the expressions. The answer in this section is simply to accept that "f is a one-to-one function." As a result, it has been rated as partially correct. S/he explained the first sentence she determined according to the direct proof method as *"I thought to do it according to the direct proof method, so I thought it was appropriate to accept p as true."*.

S2 also made a mistake in the first sentence of the proof process as a result of his/her error in determining the components of the theorem. Figure 19 shows the participant's response.

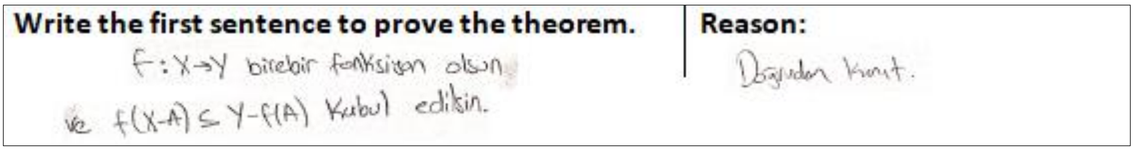


Figure 18: S2's first sentence response to the second proof-of-design task

As can be seen in the participant's response, the mistake s/he made in the distinction between p and q was reflected in the first sentence and s/he was aware of this situation and said, *"I accepted p as correct because I thought I had to prove it directly. But I am not sure that p is correct."*

Participants S1 and S2 stated that they used the direct proof method and that the first sentence should be accepted as true as a consequence. However, it was revealed that neither of the participant wrote a completely correct assumption.

It was discovered that, similar to S1, participant S3 added the expression "$A \subset X$ " to the assumption that s/he had determined as the first sentence of the proof process. Figure 19 depicts the participant's response.

Write the first sentence to prove the theorem.	**Reason:**
$f: X \to Y$ bir fonk ve $A \subset X$ o... f birebir olsun. $f(X \setminus A) \subset Y \setminus f(A)$ old. gösterelim.	Doğrudan kanıt yönteminden; Alt küme tanımından.

Figure 19: S3's first sentence response to the second proof-of-design task

When the participant's response was examined, it was revealed that there were statements that were partially correct but indicated the desired result to be hypothesized. According to the participant, s/he first determined what s/he wanted to show at the end and then expressed what he needed to start the proof as the first sentence.

S3: "*First I determined what I wanted to show. To show this, I thought I had to use f is one-to-one and A is a subset of X, or I wouldn't be able to start the proof.*"

When the answers of the participants were examined, it was discovered that there were no participants who could write the proof completely and correctly along with the reason, even if they assumed correctly.

Writing the Second Sentence

According to the data obtained after the first design was implemented, the part of writing the second sentence was added after the introduction sentence, that is, the first sentence, to assist the inference process. Figure 20 shows the response of participant S1 to this section.

Write the second sentence to prove the theorem.	**Reason:**
$y \in f(X \setminus A)$ alalım.	Göstermek istediğim ifade bir kapsama old. dan bir eleman aldım.

Figure 20: S1's second sentence response to the second proof-of-design task

It was seen that the participant formed two sentences correctly after the assumption s/he wrote in the first sentence for the introduction to the proof.

The participant named S2 formed the second sentence incorrectly because of the mistakes he made in the first parts. The participant's response is given in Figure 21.

Write the second sentence to prove the theorem.	Reason:
Bir $x \in X-A$ ele alalım, $x \in X-A \Rightarrow f(x) \in f(X-A)$ olur	Verileri kullanarak gerekli ilişkilendirmeyi yapmak

Figure 21: S2's second sentence response to the second proof-of-design task

In the second sentence response, the participant stated that s/he correlated the data for the outcome s/he desired.

The answer of the participant named S3 is given in Figure 22.

Write the second sentence to prove the theorem.	Reason:
En az bir x elemanı vardır ki $x \in X-A$ ve $y = f(x)$ dir.	f fonksiyon olduğundan görüntü kümesi tanımından

Figure 22: S3's second sentence response to the second proof-of-design task

When the participant's response was examined, it was discovered that s/he used the image set definition to write the second sentence; however, it was determined that the expression "$x \in X \setminus A$" was written correctly while the expression "$y = f(x)$" was incorrect. His/her answer was rated as partially correct, as the expression "$y \in f(X \setminus A)$" was expected to be written instead. The explanation of the participant regarding this part is given below.

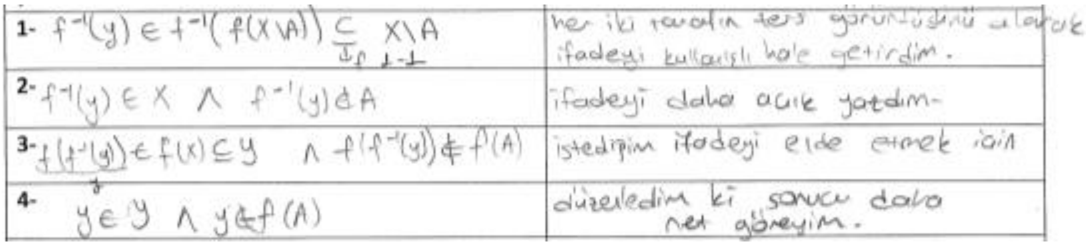
S3: *"After writing the first sentence, I wrote the second sentence by looking at the givens in the theorem."*

As seen in Figures 21 and 22, participants S2 and S3 stated that they wrote their second sentences according to the information given in the theorem. It was observed that the participants had difficulties and made mistakes while writing the second sentence, as in the first sentence, due to the mistakes they made in the part of determining the components of the theorem.

Inferences Made in the Proving Process

In the inference part, it has been rearranged to ensure that inferences are step-by-step after the initial design. Participants are expected to continue the proof process by making step-by-step inferences from the first and second sentences they wrote. When the answers were examined, it was discovered that the participants had difficulty making inferences, and that the majority of the participants were unable to make inferences fully and correctly.

In Figure 23, the inferences made by the participant named S1 during the proof process are given.



1-	$f^{-1}(y) \in f^{-1}(f(X \setminus A)) \subseteq X \setminus A$ (I_f 1-1)	her iki tarafın ters görüntüsünü alarak ifadeyi kullanışlı hale getirdim.
2-	$f^{-1}(y) \in X \wedge f^{-1}(y) \notin A$	İfadeyi daha açık yazdım.
3-	$f(f^{-1}(y)) \in f(X) \subseteq Y \wedge f(f^{-1}(y)) \notin f(A)$	istediğim ifadeyi elde etmek için
4-	$y \in Y \wedge y \notin f(A)$	düzenledim ki sonucu daha net göreyim.

Figure 23: S1's inference response to the second proof-of-design task

When the participant's inference response was examined, it was seen that he made the inference neatly. The participant made the following statements regarding this part in the individual interview:

> S1: *"In the first design, you had to write your first sentence, then make inferences before writing your last sentence." You expect us to make the inferences step by step, with justifications. Making a step-by-step inference by writing the reason was better and more planned for me. It was neater for us to express what we were thinking at what stage of the proof."*

With this explanation, s/he stated that making inferences step-by-step and writing them down along with the reasons allowed them to express their thoughts while also ensuring that the proof process was organized and planned.

The participant named S2 also suggested that s/he had difficulty in making inferences due to the lack of knowledge about the subject of the theorem by saying: "*I have deficiencies in the subject of functions. That's why I had a hard time making inferences.*" The mistakes s/he made at the beginning of the proof process caused him/her to have difficulty and make mistakes while making inference.

The inferences made by the participant named S3 during the proof process are given in Figure 24.

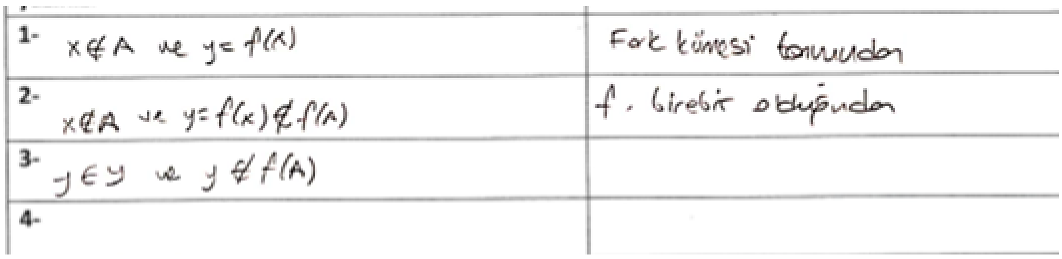


Figure 24: S3's inference response to the second proof-of-design task

The participant, on the other hand, stated that s/he was having difficulty writing his/her reasons while making inferences and that s/he assumed s/he was doing something inaccurately in this regard, as follows. S/he also stated that making inferences step by step ensures a consistent evidence process.

> *S3: "It was a little difficult to write down the reasons while making inferences. Because not all of them have a reason. I was wondering if I was making a mistake. But the step-by-step inference here, compared to the previous one, allowed us to make the proof more orderly. Otherwise, I would be confused."*

It can be stated that the participants had difficulty explaining their reasoning while making inferences, which was due to their lack of knowledge and previous mistakes.

Last sentence of the Proof Process

When the answers for the last sentence, namely writing the desired result, were analyzed, it was seen that S1 made a mistake, S2 did not give an answer, and S3 got it right.

The participant named S1 wrote inference instead of the desired result as the last sentence. While the expected answer was "$y \in Y \setminus f(A)$", writing "$y \notin Y \setminus f(A)$" caused his/her answer to be evaluated as wrong.

During the individual interview with participant S1, s/he expressed his thoughts on what he wanted to achieve while writing the last sentence and the first sentence as follows: "*Actually, I determined the last sentence in the first place. Because while I was writing the first sentence, I actually thought about what I wanted to achieve and wrote accordingly*." However, it was realized in his/her response that the reason for the last sentence was the inference of the previous theorem (the statement in the previous inference).

It was discovered that the participant S2 did not write an answer for the final sentence and made mistakes from the start of the proof process, resulting in the inability to reach a conclusion. "I did not write anything because I wasn't sure about the last sentence," s/he explained.

The participant named S3 explained that she used the definition of difference in sets by writing the expression "$y \in Y \setminus f\ (A)$" as the last sentence and that it was the desired result, saying "*I determined the last sentence according to what I wanted to see as a result.*"

Findings of the Third Design

In line with the data obtained after the implementation of the second design, it was observed that the participants had difficulties in making inferences due to the lack of information about the proposition/theorem given. For example, participant S2 stated that the inability to make inferences was due to a lack of knowledge about the function. As a result, in addition to the second design, it was thought appropriate to provide information that might be needed during the proving process, so an information box was added to the third design.

In the proof task prepared in accordance with the third design, theorem "$let \lim_{x\to a} f(x) = A\ and \lim_{x\to a} g(x) = B. So \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = A+B$", and the definitions and properties of mathematical concepts in this theorem are given as information.

Determining the Components of the Theorem

When the participants' answers to determining the components of the theorem were examined, it was discovered that they correctly distinguished between the hypothesis and the inference of the theorem, but the participant named S3 did not write the theorem as $p \Rightarrow q$. The answer to determining the components of the participatory theorem named S1 is shown in Figure 25.

p: $\lim_{x\to a} f(x)=A$ ve $\lim_{x\to a} g(x)=B$	**q:** $\lim_{x\to a}[f(x)+g(x)]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x) = A+B$
Express the theorem in the form $p \Rightarrow q$. $\lim_{x\to a} f(x)=A$ ve $\lim_{x\to a} g(x)=B \Rightarrow \lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x)+\lim_{x\to a} g(x) = A+B$	

Figure 25: Response of S1's third proof task to determine the components of the theorem

When the participant's response was examined, s/he restated the theorem and determined the hypothesis and assumption of the theorem correctly. The explanation for this part is as follows:

> S1: *"First I read my theorem. The distinction between the propositions p and q was clear. And immediately I wrote the theorem using it, and I wrote the p and q as well."*

Participant S2 simply wrote "$p \Rightarrow q$" in restating the theorem. However, the answer that makes the distinction between hypothesis and assumption correctly is given in Figure 26.

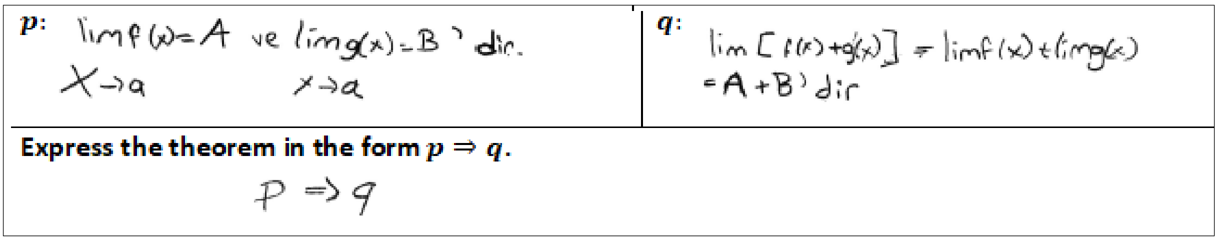


Figure 26: S2's response to the third proof task by determining the components of the theorem

The participant explained this part as follows, "*While I was reading the theorem, I separated p and q and wrote it.*"

The participant named S3 made the distinction between hypothesis and assumption correctly, similar to S2, but did not write anything in the part of restating the theorem. The participant's response is given below.

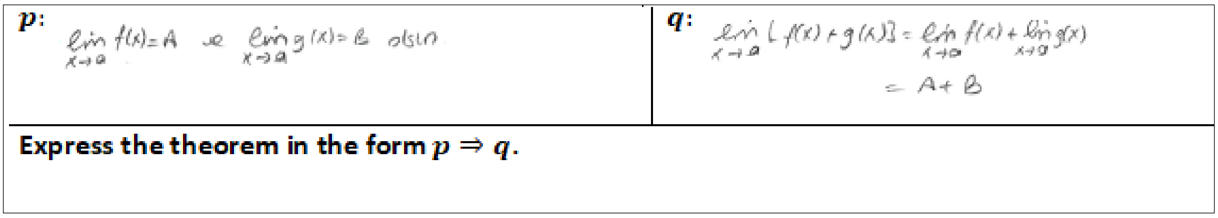


Figure 27: S3's response to the third proof task by determining the components of the theorem

The participant explained that the distinction between p and q was made while studying the theory, stating, *"I determined p and q as soon as I read the theorem at the start of the task."*

As stated by the participants, the hypothesis and assumption of the theorem were determined while reading the theorem, and thus the theorem could be rewritten. Participants

S2 and S3 wrote only the inferences p and q, but while restating the theorem, S2 wrote "$p \Rightarrow q$" while S3 did not write anything.

Introductory Sentence of the Proof Process

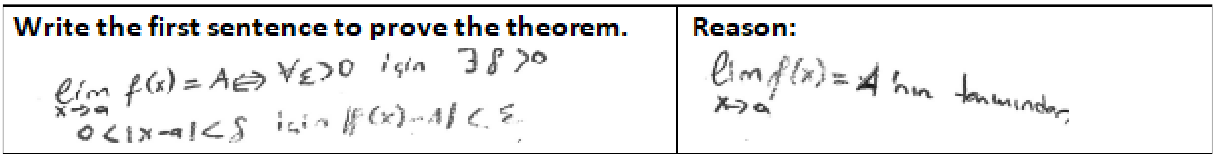
When the answers of the participants in the introductory sentence of the proving process, that is, the first sentence, were examined, it was determined that the participants named S1 and S2 wrote their inferences by choosing the direct proof method, but the participant named S3 made inferences instead of writing assumptions.

Participants S1 and S2 correctly wrote their assumptions in the introductory sentence of the proof process as "Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$".

Participant S1 explained that he preferred the method of direct proof and wrote his first sentence accordingly: "*When I read the theorem, the direct proof method came to my mind first. The assumptions p and q were already very clear. I thought using direct method would be easier. I would have difficulties while using the method of proof by contradiction. I started my first sentence by accepting the proposition p as true*".

Participant S2, similar to S1, made the following explanation regarding the first sentence response: "*It was more appropriate to use the direct proof method for this theorem. For direct proof, I have to accept p as true as the first sentence*".

The response of the participant named S3 as an introductory sentence to the proof process is given in Figure 28.



Write the first sentence to prove the theorem.	Reason:
$\lim_{x\to a} f(x) = A \Leftrightarrow \forall \varepsilon > 0$ için $\exists \delta > 0$ $0 < \lvert x-a \rvert < \delta$ için $\lvert f(x) - A \rvert < \varepsilon$	$\lim_{x\to a} f(x) = A$ 'nın tanımından

Figure 28: S3's first sentence response to the third proof-of-design task

When the participant's response was examined, it was revealed that, while he was expected to write a hypothesis, he instead made inferences and did not write assumptions. As a result, it was deemed incorrect. In the individual interview, the participant stated that he realized the error as follows.

S3: *"After writing the propositions p and q, the method by which I could show q using p was the direct proof method. Accordingly, I should have started by accepting p as true. But I started to make assumptions."*

Writing the Second Sentence

In this part, the participants were asked to write a second sentence after the introductory sentence of the proof process to help make inferences.

The second sentence response of the participant named S1 is given below.

Write the second sentence to prove the theorem.	**Reason:**
$\varepsilon>0$ alalım. $\lim_{x\to a} f(x)=A$ olduğundan $\delta_1>0$ vardır öyle ki $\lvert x-a\rvert<\delta_1 \Rightarrow \lvert f(x)-A\rvert<\varepsilon/2$ olur.	Varsayımdan

Figure 29: S1's second sentence response to the third proof-of-design task

As in Figure 29, the participant named S1 correctly wrote his second sentence using the definition and inference, and his explanation for this was "I looked at the information and when I saw the definition of the limit; I thought I could use it in the second sentence".

The second sentence written by the participant named S2 using the definition of the limit is given in Figure 30.

Write the second sentence to prove the theorem.	**Reason:**
Limit tanımından; $\lim_{x\to a} f(x)=A \Leftrightarrow \forall\varepsilon>0$ için $\exists\delta>0$ $0<\lvert x-a\rvert<\delta$ için $\lvert f(x)-A\rvert<\varepsilon$	Limit tanımı

Figure 30: Second sentence response to S2's third proof-of-design task

In his individual interview, the participant stated that s/he wrote the second sentence using the definition of the limit: *"I looked at the information given. I wrote the second sentence after using the definition of the limit and following the first sentence."*

The answer of the participant named S3 that s/he continues to make inferences in his/her second sentence is given in Figure 31.

Write the second sentence to prove the theorem.	**Reason:**
$\lim_{x \to a} g(x) = B \Leftrightarrow \forall \varepsilon > 0$ için $\exists \delta > 0$: $0 < \|x-a\| < \delta$ için $\|g(x) - B\| < \varepsilon$	$\lim_{x \to a} g(x) = B$ olduğundan.

Figure 31: S3's second sentence response to the third proof-of-design task

When the participant's response was examined, it was found that he continued to make inferences, but used the symbol "δ" both for the expression $f(x)$ and for the expression $g(x)$. In the individual interview, he answered as follows regarding this part:

> S3: *"When I saw the definition of the limit in the information, I thought I should use it."*.

According to the data obtained for the second sentence, the participants stated that they wrote their second sentences using the definition of the limit in the information box and making use of the assumption (first sentence).

Inferences Made in the Proving Process

In this section, participants are expected to continue the proof process by making step-by-step inferences from the first and second sentences they wrote. The following are the inferences made by the participant S1 during the proving process.

1-	Buradan $\lim g(x) = B$ olduğundan $\|x-a\| < \delta_1 \Rightarrow \|f(x) - A\| < \frac{\varepsilon}{2}$ olur.	limit tanımı gereğince
2-	O halde $\varepsilon > 0$ için $\|x-a\| < \delta_1$ için $\|f(x) - A + g(x) - B\| < \|f(x) - A\| + \|g(x) - B\| < \varepsilon$	Taraf tarafa toplama üçgen eşitsizliği
3-	Buradan $\varepsilon > 0$ için $\delta_1 > 0$ vardır öyle ki $\|x-a\| < \delta_1$ için $\|f(x) + g(x) - (A+B)\| < \varepsilon$	limitten
4-		

Figure 32: S1's inference response to the third proof-of-design task

As shown in Figure 32, the participant correctly assumed his/her reasons and stated that s/he benefited from the information in the information box as follows.

S1: *"It was very nice to have the information. So, I wrote my inferences using that information without any trouble."*

The answer that the participant named S2 made appropriate inferences by using the information given in the information box is given in Figure 33.

1-	$0<\|x-a\|<\delta$ için $\|f(x)-A\|<\varepsilon$	
2-	$0<\|x-a\|<\delta$ için $\|g(x)-B\|<\varepsilon$	
3-	$0<\|x-a\|<2\delta$ için $\|f(x)+g(x)-A-B\|<2\varepsilon$	
4-	$0<\|x-a\|<2\delta$ için $\|f(x)-A\|+\|g(x)-B\|<2\varepsilon$	üçgen eşitsizliği

Figure 33: S2's inference response to the third proof-of-design task

The participant made accurate inferences step by step, as shown in Figure 33, and stated this as follows.

S2: *"Having definitions of the expressions in the theorem and additional information was helpful while making inference."*

In Figure 34, the inferences made by the participant named S3 during the proving process are given.

1-	$A+B \Rightarrow \forall\varepsilon>0$ için $\exists\delta>0 : 0<\|x-a\|<\delta$ için	
2-	$\|f(x)-A\|+\|g(x)-B\|<2\varepsilon$	
3-	$\|f(x)-A+g(x)-B\|<2\varepsilon$	
4-	$\|f(x)-A+g(x)-B\| \leq \|f(x)-A\|+\|g(x)-B\|<2\varepsilon$ $\leq \lim_{x\to a} f(x)=A+\lim_{x\to a} g(x)=B$	

Figure 34: S3's inference response to the third proof-of-design task

When the participant's inferences were examined, it was discovered that s/he made errors, so it was considered incorrect. However, in the individual interview, s/he stated that s/he benefited from the information as follows:

S3: "The fact that the information was given in advance made me feel comfortable when making inferences. I just used what I needed to use. For example, the triangle inequality and the definition of limit helped me a lot."

The information box added to the design has been understood to be useful when making inferences based on both the answers given and the statements of the participants.

Last sentence of the Proof Process

When the last sentence responses of the participants were examined, the answers of the participants named S1 and S2 as "$Then\ [f(x)+g(x)] = f(x) + g(x) = A+B$" were evaluated as correct since it is the desired result. The participant named S3 did not respond to this part.

In the individual interviews, no one expressed an opinion on the last sentence, but it was mentioned that the information in the information box is useful in all parts.

Participants Views on Proving Process

In the findings obtained from the interviews regarding the proving process, the participants stated that these tasks were beneficial, gave them confidence while proving, made the proof process more organized, and contributed to their proving skills, with the addition of the information box in the final design.

S1: *"I've noticed that I think faster with proof events. Previously, I always had doubts when I was proving, I always got stuck when I wrote two sentences, I couldn't move forward. I think that I was more conscious about what I will do in these tasks. With the information part, my anxiety disappeared. When I saw that I was able to prove, I became more confident."*

The participant named S1 stated that s/he was nervous while proving beforehand, but s/he thought faster with these tasks, and when s/he saw that s/he was able to prove, s/he gained self-confidence.

S2: *"I realized that I used to be very messy when I was proving. In order to get the proof right, we had to establish its systematics. Evidence is actually a discipline, I realized that in these tasks. So, I found that when I progressed regularly, I was able to prove."*

S2, the participant, stated that s/he realized s/he worked scattered before the evidence tasks, that s/he was able to prove and get into an order with these tasks.

S3: *"Before this, when I was going to prove, I didn't know where to start and how to move forward. It always seemed like I was going to do it wrong. But with these tasks, I established a regularity. Each part facilitates the proof process. The addition of the information part also gave a confidence. It reinforced the feeling that I was doing the proof right. Actually, it wasn't that hard to prove. I realized that this way I could do the proof with an task. When you get it in order, the rest comes."*

S3 also stated that s/he did not know where to begin while proving beforehand and was concerned about making mistakes. S/he stated that the proof process became more regular as a result of these tasks, and the information part also provided confidence.

Discussions

The component of determining the components of the theorem in the proof tasks was added as a solution to the difficulty of starting the proof. It is important to determine and understand the components of the given theorem in order to start the proving process (Dean, 1996, Weber, 2012). In this section, it is thought that re-expressing the theorem given as a conditional statement as q if p and determining the hypothesis (p) and the assumption (q) will help to understand the theorem. Based on the findings of this part in the proof tasks used in the research, it was determined that this part aided them in starting the proof by contributing to the understanding of the theorem. This result, like the study of Benkhalti, Selden, and Selden (2017), can be said to be a solution to the problem of starting the proving process.

It was revealed that the participants followed strategies such as looking at the key words in the theorem (if, then, in that case, etc.) and drawing a figure in order to understand the theorem. Attempting to understand the theorem by looking at the keywords in it to begin the proof process is an example of Weber's (2004) approach, which he refers to as Syntactic Proof Generation. Weber's (2004) another approach is Generating Semantic Proofs, which he describes as persuading himself to the proof process by drawing pictures suitable for the statement of the theorem for a better understanding of mathematical expressions. The participant, who tries to understand the theorem by drawing a figure, is given an example for this Generating Semantic Proofs approach.

Writing the introductory sentence of the proof process, namely the assumption for the proof, is an important step after determining the components of the theorem (Selden, Selden, & Benkhalti, 2017). As a result, writing the first sentence was added to the tasks as an introduction to the proof process. In the findings for the first sentence, it was discovered that the theorem was more easily determined once it was understood, and the assumption was mostly written correctly according to the preferred proof method.

It was discovered that the participants struggled to continue after the first design, so the writing of the second sentence was added. This section can be considered the transition phase to making assumptions, and it contributes to making assumption by connecting with the first sentence.

In the inference part, in the first stages, the participants were expected to make inferences using their prior knowledge for the proof of the theorem. However, it was observed that in some cases, the participants could not choose the appropriate one to make inferences from their existing prior knowledge and use it strategically, and sometimes they could not continue due to lack of knowledge. This situation is similar to the results of the studies of Dane (2008), Polat and Akgün (2016), Sarı Uzun and Bülbül (2013), and Weber (2001) .

In line with the evaluations and opinions received, it was thought that adding an information box to the evidence task design would positively affect the proving process, which was difficult due to lack of information. According to the findings obtained in the final design, the information box aided in making inferences during the proofing process. Selden and Selden (2009) also stated that conducting proof studies with notes containing definitions and theorems related to proof is effective in their studies. Similarly, Karaoğlu (2010) concluded in the thesis study that, the key points and ideas that will help the pre-service teachers in the proving process were implicitly given, and the pre-service teachers completed the proof process by making use of them when needed.

In order to complete the proof after the inferences, it is important to write the last sentence of the proof process, that is, to state that the proof is over. According to the findings of this section, the last sentence is the desired result, which helps participants in filling the gap between the first sentence and contributes to the proof process by directing them to the point they need to reach while making inferences. Selden, Selden, and Benkhalti (2017) also stated that determining the beginning and ending sentences in the proof frames used to prove the theorem contributes to this process.

Regarding the design of the evidence task, participants stated that it gave them confidence in proving, that it provided a more regular proof process, that it was beneficial, and that their proving skills improved. This result is similar to the result of Benkhalti, Selden, and Selden's (2017) study that proof frameworks positively affect self-confidence in the proving process.

Conclusions and Suggestions

As a result, it was seen that the results obtained in this study supported the results of the studies conducted in the field. In addition, a proof task design was developed in this study to guide the proving process and to develop proving skills.

Based on the findings of the study, it is believed that the application of such studies to a larger number of students and their dissemination in mathematics courses will contribute to overcoming the difficulties encountered in the proof and proving process, and that the bias toward proof will change in a positive way.

In this study, proof tasks were applied only to pre-service mathematics teachers at a state university and their effect on the development of their proving skills was examined. This study can also be conducted with pre-service mathematics teachers studying at different universities, and its effect on the development of their proof skills can be examined.

On the other hand, due to the wide subject area of mathematics, similar proof tasks can be designed and applied with different subjects.

In future studies, it may be recommended to apply proving tasks to students at different grade levels, from primary school to university, in accordance with their level.

Matematik Öğretmen Adaylarınn Kanıtlama Becerilerinin Geliştirilmesine Yönelik Bir Tasarım Çalışması

Özet:

Matematiksel kanıtın bir iddiayı doğrulamak ve açıklamak gibi görevlerinin yanı sıra matematiği anlamaya yardımcı olması ve yeni bilgilerin oluşumunu sağlaması matematik eğitiminde önemini artırmaktadır. Matematiği öğrenmek ve gerçek yaşamda karşılaşılan problem durumlarında etkili çözüm yolları bulabilmek için de kanıtlama becerilerinin geliştirilmesi gerekir. Kanıtın matematik eğitimindeki önemi göz önüne alındığında, özellikle matematik öğretmeni yetiştiren bölümlerde, öğretmen adaylarının kanıt bilgisini, kanıt yapabilme ve anlama düzeylerini artırmak için uygun öğretim uygulamalarının tasarlanması gelecekteki öğrencilere etkili ve kalıcı matematik öğretimi sağlamaları açısından önemlidir. Bu araştırmada tasarlanan kanıt etkinliklerinin matematik öğretmen adaylarının kanıtlama becerilerinin gelişimine etkisi incelenmiştir. Tasarım tabanlı araştırma yönteminin kullanıldığı bu araştırma, Ankara'da bulunan bir devlet üniversitesinde öğrenim gören gönüllü üç matematik öğretmen adayının katılımıyla gerçekleştirilmiştir. Araştırmada, uygulanan kanıt etkinlikleri ve katılımcılarla yapılan bireysel görüşmelerden elde edilen veriler kullanılmıştır. Nitel analiz yöntemiyle analiz edilen verilerden elde edilen bulgulara göre tasarlanan kanıt etkinlikleri katılımcıların kanıtlama becerilerinin gelişimine katkı sağlamıştır.

Anahtar kelimeler: matematiksel kanıt, kanıtlama becerileri, kanıt etkinliği, matematik öğretmen adayları.

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