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Research Article

Tripathi connection in Finsler geometry

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Dedicated to Professor Nabil L. Youssef on the occasion of his 75th Birthday

Abstract

Adopting the pullback formalism, a new linear connection in Finsler geometry has been introduced and investigated. Such connection unifies all formerly known Finsler connections and some other connections not introduced so far. Also, our connection is a Finslerian version of the Tripathi connection introduced in Riemannian geometry. The existence and uniqueness of such connection is proved intrinsically. An explicit intrinsic expression relating this connection to Cartan connection is obtained. Some generalized Finsler connections are constructed from Tripathi Finsler connection, by applying the P^1-process and C-process introduced by Matsumoto. Finally, under certain conditions, many special Finsler connections are given.

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1. Introduction

In Riemannian geometry, the Levi-Civita connection is the unique torsion-free connection that preserves the Riemannian metric. Hayden introduced a metric connection with torsion [6]. Folland in [3], with the help of a 1-form, explored a symmetric connection that is non-metric. Yano in [21] investigated a certain type of Hayden connection which is known as semi-symmetric metric connection. Then, semi-symmetric non-metric connection had been studied in cf. [1, 15]. A further extension of semi-symmetric connections is the notion of quarter-symmetric connection [23] which includes Ricci quarter-symmetric connection. Recently, a generalized quarter-symmetric connection has been introduced in [18]. Fortunately, in [19], Tripathi defined a new connection which included all these connections and more as particular cases.

Finsler geometry is a natural generalization of Riemannain geometry [14]. The connection theory in the context of Finsler geometry has been enormously developed in both

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the Klein-Grifone approach and the pullback approach, see for example [5,9–11,17,27,29]. There are four celebrated Finsler connections, namely, Cartan, Berwald, Chern-Rund and Hashiguchi connections. Some types of Riemannian connections have been extended to the Finslerian context [13,14,20,26]. In [16], Tripathi connection has been introduced locally in the Finsler framework. In this paper we generalize [16] in Finsler global formalism.

Global formalism deals with the entire manifold where geometric properties do not change from coordinate system to another. It is known that global formalism is more general than local formalism in the sense that one can obtain local results from the global ones but the converse is not true in general. One reason for this phenomenon is that some terms existing globally disappear when they are treated locally, such as the Lie bracket of the natural basis vector fields which is used in the expressions of the torsion and curvature tensors. Another reason is that certain problems are local by its very nature and hence can by no means be globalized.

Based on the above discussion, this paper is devoted to a further development of the theory of connections in Finsler geometry. Our geometric treatment avoids the use of coordinate indices. We provide a Finslerain extension of Tripathi connection (Theorem 3.2) which not only includes all the aforementioned connections but also much more connections. Successfully, we derive the relation between some geometric objects associated with Finsler Tripathi connection and the corresponding ones associated with Cartan connection. We follow this by an investigation of its spray, nonlinear connection, torsion and curvature tensors along with Bianchi identities. Then, in §4, using the P^1-process and C-process defined in [10], we give a generalized version of the four celebrated Finsler connections. Finally, in §5 as a consequence (26) particular regular Finsler connections are given.

2. Connections, Sprays and Finsler metrics

Here we recall the necessary material for better understanding the present paper. For an n dimensional smooth manifold M, consider the tangent bundle $\pi : TM \longrightarrow M$ and its differential $d\pi : TTM \longrightarrow TM$. The vertical bundle $V(TM)$ of TM is just $\ker(d\pi)$. Let us denote the pullback bundle of the tangent bundle by $\pi^{-1}(TM)$. Let $\mathfrak{F}(TM)$ denote the algebra of C^∞ functions on TM and $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and denoted by barred letters $\overline{X}$.

Now, we recall the short exact sequence of vector bundle morphisms [4,17]

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where $\mathcal{T}M$ is the slit tangent bundle, γ is the natural injection and $\rho := (\pi_{TM}, \pi)$.

The tangent structure of TM or the vertical endomorphism is the endomorphism $J : T\mathcal{T}M \mapsto T\mathcal{T}M$ defined by $J = \gamma \circ \rho$. Note that $J^2 = 0$, $[J, J] = 0$ and $\ker J = \operatorname{Im} J = V(\mathcal{T}M)$. The Liouville vector field is the vector field given by $\mathcal{C} := \gamma\,\overline{\eta}$, where $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

Linear connections on the pullback bundle $\pi^{-1}(TM)$ [17,29].

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. The map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}.$$

is called the connection map of D. The connection D is regular if at each $u \in \mathcal{T}M$, we have the splitting

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M),$$

where $H_u(\mathcal{T}M) := \{X \in T_u(\mathcal{T}M) \,|\, K(X) = 0\}$ is the horizontal space at u .

When M is equipped with a regular connection D, the maps γ, $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are vector bundle isomorphisms. In this case, $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ is called the horizontal map of D.

Definition 2.1. The torsion tensor $\mathbb{T}$ of a regular connection D on $\pi^{-1}(TM)$ with horizontal map β has the following two counterparts:

(a) (h)h-torsion tensor $Q(\overline{X},\overline{Y}) = \mathbb{T}(\beta\overline{X},\beta\overline{Y})$,
(b) (h)hv-torsion tensor $T(\overline{X},\overline{Y}) = \mathbb{T}(\gamma\overline{X},\beta\overline{Y})$.

The curvature tensor $\mathbb{K}$ of D has the following three counterparts:

(c) h-curvature tensor $R(\overline{X},\overline{Y})\overline{Z} = \mathbb{K}(\beta\overline{X},\beta\overline{Y})\overline{Z}$,
(d) hv-curvature tensor $P(\overline{X},\overline{Y})\overline{Z} = \mathbb{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z}$,
(e) v-curvature tensor $S(\overline{X},\overline{Y})\overline{Z} = \mathbb{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z}$.

Consequently, the contracted curvature tensors of a connection D (denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$) are given, respectively, by

$$\widehat{R}(\overline{X},\overline{Y}) = R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) = P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) = S(\overline{X},\overline{Y})\overline{\eta}$$

and are called the $v(h)$-torsion, $v(hv)$-torsion and $v(v)$-torsion, respectively.

Geometry of sprays and Finsler metrics [4,5,17,27,29].

A vector field G on TM is said to be spray on M if $\rho\circ G = \overline{\eta}$ and $[\mathcal{C},G] = G$. Each spray induces canonically a nonlinear connection $\Gamma := [J,G]$, which is homogeneous (i.e., $[\mathcal{C},\Gamma] = 0$). The existence of Γ is equivalent to the existence of an n-dimensional distribution $H : u \in \mathcal{T}M \longrightarrow H_u \in T_u(\mathcal{T}M)$ supplementary to the vertical distribution; it is called the horizontal distribution. The corresponding horizontal and vertical projectors are given, respectively, by

$$h := \frac{1}{2}(Id_{TM} + \Gamma), \quad v := \frac{1}{2}(Id_{TM} - \Gamma). \tag{2.1}$$

Definition 2.2. A smooth Finsler structure on M is a map $L : TM \longrightarrow [0,\infty)$ such that:

(a) L is C^∞ on $\mathcal{T}M$, C^0 on TM,
(b) L is positively homogeneous of degree 1 in the directional argument y, that is $\mathcal{L}_{\mathcal{C}}L = L$, where $\mathcal{L}$ is the Lie derivative,
(c) The Hilbert 2-form $\Omega := \frac{1}{2}\, dd_J L^2$ has a maximal rank.

The Finsler metric g induced by L on $\pi^{-1}(TM)$ is defined as follows

$$g(\rho X, \rho Y) := \Omega(JX, Y), \ \forall X, Y \in \mathfrak{X}(TM).$$

Unlike Riemannian geometry which has one canonical linear connection on M, Finsler geometry admits at least four linear connections on $\pi^{-1}(TM)$: Cartan, Chern-Rund, Hashiguchi and Berwald connections [29]. It should be noted that these four connections are regular with $T(\overline{X},\overline{\eta}) = 0$.

Every Finsler structure determines uniquely a spray G, called the geodesic or canonical spray [7]. For the geodesic spray G there exists a unique homogenous nonlinear connection $\Gamma = [J,G]$, called the Barthel or canonical connection associated with the Finsler structure L.

Definition 2.3. Let (M,L) be a Finsler manifold and D be a regular connection on $\pi^{-1}(TM)$ with horizontal map β. Then, the vector field defined by $G = \beta\,\overline{\eta}$ is called the spray associated with D. In addition, the vector valued 1-form $\Gamma_D := 2\,\beta\circ\rho - I$ is a nonlinear connection, called associated with D.

Lemma 2.4. *Let (M,L) be a Finsler manifold and D be a regular connection on $\pi^{-1}(TM)$ whose connection map is K and whose horizontal map is β. A necessary and sufficient condition for the $(1,1)$-type tensor defined by $\Gamma = \beta\circ\rho - \gamma\circ K$ to be a nonlinear connection* *on M is that the $(h)hv$-torsion T of D satisfies $T(\overline{X},\overline{\eta})=0$. Thereby, Γ coincides with the nonlinear connection associated with D. That is, $\Gamma=\Gamma_D=2\beta\circ\rho-I$. Consequently, $h_\Gamma=h_D=\beta\circ\rho$ and $v_\Gamma=v_D=\gamma\circ K$.*

3. Finsler Tripathi connection

Let us start with the following definition before going to the main result of the paper (Theorem 3.2).

Definition 3.1. A regular connection $\overline{D}$ on $\pi^{-1}(TM)$ is said to be quarter-symmetric if there exist a scalar 1-form u and a vector 1-form φ on $\pi^{-1}(TM)$ such that the (h)h-torsion $\overline{Q}$ of $\overline{D}$ satisfies*:*

$$\overline{Q}(\overline{X},\overline{Y})=u(\overline{Y})\varphi(\overline{X})-u(\overline{X})\varphi(\overline{Y})\quad\forall\,\overline{X},\,\overline{Y}\in\mathfrak{X}(\pi(M)). \tag{3.1}$$

The 1-forms u and φ are called the quarter-symmetric forms of $\overline{D}$.

In particular, if $\varphi=id_{\pi^{-1}(TM)}$, then $\overline{D}$ is called semi-symmetric. Moreover, if $\varphi=0$ or $u=0$, then $\overline{D}$ is called symmetric. Further, if the pullback bundle is equipped with a Finsler structure and $\varphi=Ric_o$, where Ric_o is the horizontal Ricci $(1,1)$-type tensor of the Cartan connection, then $\overline{D}$ is said to be Ricci quarter-symmetric.

Theorem 3.2. *Let (M,L) be a Finsler manifold. For given functions $f_1,f_2\in\mathfrak{F}(TM)$, scalar 1-forms A,B,u and a vector 1-form φ on $\pi^{-1}(TM)$, there exists a unique regular connection $\overline{D}(f_1,f_2,A,B,u,\varphi)$, or simply $\overline{D}$, on $\pi^{-1}(TM)$ such that*

(I) *The horizontal covariant derivative of g with respect to $\overline{D}$ has the form:*

$$(\overline{D}_{\bar{\beta}\overline{X}}\,g)(\overline{Y},\overline{Z})=2f_1A(\overline{X})\,g(\overline{Y},\overline{Z})+f_2\{B(\overline{Y})\,g(\overline{Z},\overline{X})+B(\overline{Z})\,g(\overline{X},\overline{Y})\},$$

(II) *The metric g is $\overline{D}$-vertically parallel, that is $\overline{D}_{\gamma\overline{X}}\,g=0$,*

(III) *$\overline{D}$ is quarter-symmetric with quarter-symmetric forms u and φ,*

(IV) *The $(h)hv$-torsion $\overline{T}$ of $\overline{D}$ satisfies $g(\overline{T}(\overline{X},\overline{Y}),\overline{Z})=g(\overline{T}(\overline{X},\overline{Z}),\overline{Y})$.*

This connection will be named Finsler Tripathi connection and denoted by $GC\Gamma$.

Proof. Suppose that (M,L) admits some regular connection $\overline{D}$ satisfying (I) - (IV). We prove the the uniqueness of $\overline{D}$.

Making use of (II), (IV) and Lemma 2.4, the associated nonlinear connection $\Gamma_{\overline{D}}$ is given by

$$\Gamma_{\overline{D}}=\bar{\beta}\circ\rho-\gamma\circ\bar{K}=\bar{h}-\bar{v}.$$

Here, $\bar{h}$, $\bar{v}$, $\bar{\beta}$, $\bar{K}$ are the horizontal projector, vertical projector, horizontal map and connection map associated with $\overline{D}$. From (II), (IV) and applying the Christoffel trick, we obtain for all $\overline{X},\overline{Y},\overline{Z}\in\mathfrak{X}(\pi(M))$

$$2g(\overline{D}_{\gamma\overline{X}}\overline{Y},\overline{Z})\;=\;\gamma\overline{X}\cdot g(\overline{Y},\overline{Z})+g(\overline{Y},\rho[\bar{\beta}\,\overline{Z},\gamma\overline{X}])+g(\overline{Z},\rho[\gamma\overline{X},\bar{\beta}\,\overline{Y}]). \tag{3.2}$$

Since the difference between two nonlinear connections is a semi-basic vector 1-form on TM [4, 29], we get

$$\bar{\beta}\,\overline{X}=\beta\overline{X}+\gamma\overline{X}_t,\;\text{ for some }\overline{X}_t\in\mathfrak{X}(\pi(M)), \tag{3.3}$$

where β is the horizontal map of the Cartan connection ∇. As the vertical endomorphism J satisfies $\rho\circ J=0$, we have

$$\rho[\gamma\overline{X},\bar{\beta}\,\overline{Y}]=\rho[\gamma\overline{X},\beta\overline{Y}]. \tag{3.4}$$

Considering [27, Theorem 4(a)] together with (3.2) and (3.4), it follows that the vertical counterpart of $\overline{D}$ and ∇ coincides, that is

$$\overline{D}_{\gamma\overline{X}}\overline{Y}=\nabla_{\gamma\overline{X}}\overline{Y}. \tag{3.5}$$

Now, by (I) and (III), we conclude that

$$\begin{aligned} 2\,g(\overline{D}_{\bar\beta\,\overline{X}}\overline{Y},\overline{Z}) &= \bar\beta\,\overline{X}\cdot g(\overline{Y},\overline{Z})+\bar\beta\,\overline{Y}\cdot g(\overline{Z},\overline{X})-\bar\beta\,\overline{Z}\cdot g(\overline{X},\overline{Y})-2\,f_2\,B(\overline{Z})\,g(\overline{X},\overline{Y})\\ &-u(\overline{Z})\{g(\overline{X},\varphi(\overline{Y}))+g(\overline{Y},\varphi(\overline{X}))\}+g(\overline{Z},\rho[\bar\beta\,\overline{X},\bar\beta\,\overline{Y}])\\ &+u(\overline{Y})\,\{g(\overline{X},\varphi(\overline{Z}))+g(\overline{Z},\varphi(\overline{X}))\}+g(\overline{Y},\rho[\bar\beta\,\overline{Z},\bar\beta\,\overline{X}])\\ &+u(\overline{X})\{g(\overline{Y},\varphi(\overline{Z}))-g(\overline{Z},\varphi(\overline{Y}))\}-g(\overline{X},\rho[\bar\beta\,\overline{Y},\bar\beta\,\overline{Z}])\\ &-2\,f_1\{A(\overline{X})\,g(\overline{Y},\overline{Z})+A(\overline{Y})\,g(\overline{Z},\overline{X})-A(\overline{Z})\,g(\overline{X},\overline{Y})\}. \end{aligned} \tag{3.6}$$

Formula (3.3) gives rise to

$$\rho[\bar\beta\,\overline{X},\bar\beta\,\overline{Y}]=\rho[\beta\overline{X},\beta\overline{Y}]+\rho[\beta\overline{X},\gamma\overline{Y}_t]+\rho[\gamma\overline{X}_t,\beta\overline{Y}].$$

Then, by [27, Theorem 4(b) and Theorem 6], (3.6) becomes

$$\begin{aligned} g(\overline{D}_{\bar\beta\,\overline{X}}\overline{Y},\overline{Z}) &= g(\nabla_{\beta\overline{X}}\overline{Y},\overline{Z})+g(D^\circ_{\gamma\overline{X}_t}\overline{Y},\overline{Z})-f_2\,B(\overline{Z})\,g(\overline{X},\overline{Y})\\ &+\mathbf{T}(\overline{X}_t,\overline{Y},\overline{Z})+\mathbf{T}(\overline{Y}_t,\overline{Z},\overline{X})-\mathbf{T}(\overline{Z}_t,\overline{X},\overline{Y})\\ &-u(\overline{Z})\,g(\varphi_1(\overline{X}),\overline{Y})+u(\overline{Y})\,g(\varphi_1(\overline{X}),\overline{Z})-u(\overline{X})\,g(\varphi_2(\overline{Y}),\overline{Z})\\ &-f_1\{A(\overline{X})\,g(\overline{Y},\overline{Z})+A(\overline{Y})\,g(\overline{Z},\overline{X})-A(\overline{Z})\,g(\overline{X},\overline{Y})\}, \end{aligned} \tag{3.7}$$

where $\mathbf{T}$ is the Cartan tensor defined by $\mathbf{T}(\overline{X},\overline{Y},\overline{Z}):=g(T(\overline{X},\overline{Y}),\overline{Z})$, $g(\varphi_1(\overline{X}),\overline{Y})$ and $g(\varphi_2(\overline{X}),\overline{Y})$ are the symmetric and antisymmetric parts of $g(\varphi(\overline{X}),\overline{Y})$, respectively.
Setting $\overline{X}=\overline{Y}=\overline{\eta}$ in (3.7), using the property that $\mathbf{T}$ is indicatory, as in [28, Lemma 4.9], and $\bar K\circ\bar\beta=K\circ\beta=0$, we get

$$\overline{\eta}_t = f_1\{2\,A(\overline{\eta})\,\overline{\eta}-L^2\,\overline{a}\}+f_2\,L^2\,\overline{b}+L\,\ell(\varphi_1(\overline{\eta}))\,\overline{u}-u(\overline{\eta})\,(\varphi_1-\varphi_2)(\overline{\eta}), \tag{3.8}$$

where $\ell:=L^{-1}\,i_{\overline{\eta}}g$, $g(\overline{a},\overline{X}):=A(\overline{X})$, $g(\overline{b},\overline{X}):=B(\overline{X})$, $g(\overline{u},\overline{X}):=u(\overline{X})$.
Set $\overline{Y}=\overline{\eta}$ again in (3.7) and consider (3.8), we obtain

$$\begin{aligned} \overline{X}_t &= f_1\{A(\overline{X})\,\overline{\eta}+A(\overline{\eta})\,\overline{X}-L\,\ell(\overline{X})\,\overline{a}+L^2\,T(\overline{a},\overline{X})\}\\ &+f_2\{L\,\ell(\overline{X})\,\overline{b}-L^2\,T(\overline{b},\overline{X})\}-u(\overline{\eta})\,\{\varphi_1(\overline{X})+T((\varphi_2-\varphi_1)(\overline{\eta}),\overline{X})\}\\ &+L\,\{\ell(\varphi_1(\overline{X}))\,\overline{u}-\ell(\varphi_1(\overline{\eta}))\,T(\overline{u},\overline{X})\}+u(\overline{X})\varphi_2(\overline{\eta}). \end{aligned} \tag{3.9}$$

Therefore, the Cartan tensor satisfies

$$\begin{aligned} \mathbf{T}(\overline{X}_t,\overline{Y},\overline{Z}) &= f_1\,\{A(\overline{\eta})\,\mathbf{T}(\overline{X},\overline{Y},\overline{Z})-L\,\ell(\overline{X})\,\mathbf{T}(\overline{a},\overline{Y},\overline{Z})+L^2\,\mathbf{T}(T(\overline{a},\overline{X}),\overline{Y},\overline{Z})\}\\ &+f_2\{L\,\ell(\overline{X})\,\mathbf{T}(\overline{b},\overline{Y},\overline{Z})-L^2\,\mathbf{T}(T(\overline{b},\overline{X}),\overline{Y},\overline{Z})\}+u(\overline{X})\mathbf{T}(\varphi_2(\overline{\eta}),\overline{Y},\overline{Z})\\ &+L\,\ell(\varphi_1(\overline{X}))\,\mathbf{T}(\overline{u},\overline{Y},\overline{Z})-L\,\ell(\varphi_1(\overline{\eta}))\,\mathbf{T}(T(\overline{u},\overline{X}),\overline{Y},\overline{Z})\\ &+u(\overline{\eta})\,\mathbf{T}(T((\varphi_1-\varphi_2)(\overline{\eta}),\overline{X})-\varphi_1(\overline{X}),\overline{Y},\overline{Z}). \end{aligned} \tag{3.10}$$

In addition, the Berwald connection D° can be written as follows

$$\begin{aligned} {D^\circ}_{\gamma\overline{X}_t}\overline{Y} &= f_1\,\{A(\overline{X})\,{D^\circ}_{\gamma\overline{\eta}}\overline{Y}+A(\overline{\eta})\,{D^\circ}_{\gamma\overline{X}}\overline{Y}-L\,\ell(\overline{X}){D^\circ}_{\gamma\overline{a}}\overline{Y}+L^2\,{D^\circ}_{\gamma T(\overline{a},\overline{X})}\overline{Y}\}\\ &+f_2\{L\,\ell(\overline{X})\,{D^\circ}_{\gamma\overline{b}}\overline{Y}-L^2\,{D^\circ}_{\gamma T(\overline{b},\overline{X})}\overline{Y}\}-L\,\ell(\varphi_1(\overline{\eta}))\,{D^\circ}_{\gamma T(\overline{u},\overline{X})}\overline{Y}\\ &+u(\overline{X})\,{D^\circ}_{\gamma\varphi_2(\overline{\eta})}\overline{Y}-u(\overline{\eta})\,{D^\circ}_{\gamma\varphi_1(\overline{X})-\gamma T((\varphi_1-\varphi_2)(\overline{\eta}),\overline{X})}\overline{Y}\\ &+L\,\ell(\varphi_1(\overline{X}))\,{D^\circ}_{\gamma\overline{u}}\overline{Y}. \end{aligned} \tag{3.11}$$

Using the above three relations (3.9) - (3.11), together with the formula $\nabla_{\gamma\overline{X}}\overline{Y} = D^{\circ}{}_{\gamma\overline{X}}\overline{Y} + T(\overline{X},\overline{Y})$ [27], we conclude from (3.7) that

$$\begin{aligned}\overline{D}_{\bar{\beta}\overline{X}}\overline{Y} \;=\; & \nabla_{\beta\overline{X}}\overline{Y} + f_1\{A(\overline{\eta})\,\nabla_{\gamma\overline{X}}\overline{Y} + A(\overline{X})\,\nabla_{\gamma\overline{\eta}}\overline{Y} - A(\overline{X})\,\overline{Y} - A(\overline{Y})\,\overline{X} + \overline{a}\,g(\overline{X},\overline{Y})\\ & -L\,\{\ell(\overline{X})\,\nabla_{\gamma\overline{a}}\overline{Y} + \ell(\overline{Y})\,T(\overline{a},\overline{X})\} + L^2\{\nabla_{\gamma T(\overline{a},\overline{X})}\overline{Y} + S(\overline{X},\overline{a})\overline{Y}\} + \mathbf{T}(\overline{a},\overline{X},\overline{Y})\overline{\eta}\}\\ & -f_2\,\{g(\overline{X},\overline{Y})\bar{b} - L\,\{\ell(\overline{X})\,\nabla_{\gamma\bar{b}}\overline{Y} + \ell(\overline{Y})\,T(\bar{b},\overline{X})\} + L^2\,\{S(\overline{X},\bar{b})\overline{Y} + \nabla_{\gamma T(\bar{b},\overline{X})}\overline{Y}\}\\ & +\mathbf{T}(\bar{b},\overline{X},\overline{Y})\overline{\eta}\} - g(\varphi_1(\overline{X}),\overline{Y})\,\overline{u} + u(\overline{X})\{\nabla_{\gamma\varphi_2(\overline{\eta})}\overline{Y} - \varphi_2(\overline{Y})\}\\ & +u(\overline{Y})\{\varphi_1(\overline{X}) + T(\varphi_2(\overline{\eta}),\overline{X})\} + L\,\ell(\varphi_1(\overline{X}))\,\nabla_{\gamma\overline{u}}\overline{Y} + L\,\ell(\varphi_1(\overline{Y}))\,T(\overline{u},\overline{X})\\ & +L\,\ell(\varphi_1(\overline{\eta}))\,\{S(\overline{u},\overline{X})\overline{Y} - \nabla_{\gamma T(\overline{u},\overline{X})}\overline{Y}\} - \mathbf{T}(\overline{u},\overline{X},\overline{Y})\,\varphi_1(\overline{\eta}) - \mathbf{T}(\varphi_2(\overline{\eta}),\overline{X},\overline{Y})\,\overline{u}\\ & +u(\overline{\eta})\{\varphi_1(T(\overline{X},\overline{Y})) - T(\varphi_1(\overline{Y}),\overline{X}) + S(\overline{X},(\varphi_1-\varphi_2)(\overline{\eta}))\overline{Y} - \nabla_{\gamma\varphi_1(\overline{X})}\overline{Y}\\ & +\nabla_{\gamma T((\varphi_1-\varphi_2)(\overline{\eta}),\overline{X})}\overline{Y}\},\end{aligned} \tag{3.12}$$

where S is the v-curvature tensor of Cartan connection.

Consequently, from (3.5) and (3.12), taking into account (3.9), the full expression of $\overline{D}_X\overline{Y}$ in terms of Cartan connection is the following

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\; & \nabla_X\overline{Y} + f_1\,\{g(\rho X,\overline{Y})\,\overline{a} - A(\rho X)\,\overline{Y} - A(\overline{Y})\,\rho X - L\,\ell(\overline{Y})\,T(\overline{a},\rho X)\\ & +\mathbf{T}(\overline{a},\rho X,\overline{Y})\,\overline{\eta} + L^2\,S(\rho X,\overline{a})\overline{Y}\} - \mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}) - \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\,\overline{u}\\ & -f_2\,\{g(\rho X,\overline{Y})\,\bar{b} - L\,\ell(\overline{Y})\,T(\bar{b},\rho X) + \mathbf{T}(\bar{b},\rho X,\overline{Y})\,\overline{\eta} + L^2 S(\rho X,\bar{b})\overline{Y}\}\\ & -g(\varphi_1(\rho X),\overline{Y})\,\overline{u} - u(\rho X)\,\varphi_2(\overline{Y}) + u(\overline{Y})\,\{\varphi_1(\rho X) + T(\varphi_2(\overline{\eta}),\rho X)\}\\ & -u(\overline{\eta})\,\{T(\varphi_1(\overline{Y}),\rho X) + S((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\overline{Y} + \varphi_1(T(\rho X,\overline{Y}))\}\\ & +L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) + L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}.\end{aligned} \tag{3.13}$$

Hence, $\overline{D}_X\overline{Y}$ is uniquely determined by the right-hand side of (3.13).

In order to prove the existence of $\overline{D}$, just define $\overline{D}$ by the above formula. Then, it is easy to check that $\overline{D}$ is a regular Finsler connection that satisfies the conditions (I) - (IV). This completes the proof. $\square$

Remark 3.3. It is worth mentioning that the connection $GC\Gamma$ is the Finslerian version of the Riemannian Tripathi connection [19] and generalizes the local study provided in [16].

Corollary 3.4. *The $GC\Gamma$-connection $\overline{D}$ and the Cartan connection ∇ are related by*

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} + N(\rho X,\overline{Y}),$$

where

$$\begin{aligned}N(\rho X,\overline{Y}) \;=\; & f_1\,\{g(\rho X,\overline{Y})\,\overline{a} - A(\rho X)\overline{Y} - A(\overline{Y})\,\rho X - L\,\ell(\overline{Y})\,T(\overline{a},\rho X) + \boldsymbol{T}(\overline{a},\rho X,\overline{Y})\overline{\eta}\\ & +L^2\,S(\rho X,\overline{a})\overline{Y}\} - f_2\{g(\rho X,\overline{Y})\,\bar{b} - L\,\ell(\overline{Y})\,T(\bar{b},\rho X) + \boldsymbol{T}(\bar{b},\rho X,\overline{Y})\,\overline{\eta}\\ & +L^2\,S(\rho X,\overline{bY})\} - \{g(\varphi_1(\rho X),\overline{Y}) + \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\}\overline{u} - u(\rho X)\,\varphi_2(\overline{Y})\\ & +L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) - u(\overline{\eta})\{S((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\overline{Y} + T(\varphi_1(\overline{Y}),\rho X)\\ & -\varphi_1(T(\rho X,\overline{Y}))\} + u(\overline{Y})\{T(\varphi_2(\overline{\eta}),\overline{X}) + \varphi_1(\rho X)\}\\ & +L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y} - \boldsymbol{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

Proposition 3.5. *Consider a Finsler manifold (M,L) and let $\overline{D}$ be the Finsler Tripathi connection. Then,*

(a) *The canonical spray $\bar{G}$ associated with $\overline{D}$ is given by*

$$\bar{G} = G + f_1\,\{2A(\overline{\eta})\,\gamma\overline{\eta} - L^2\,\gamma\overline{a}\} + f_2\,L^2\,\gamma\bar{b} + L\,\ell(\varphi_1(\overline{\eta}))\,\gamma\overline{u} - u(\overline{\eta})\,\gamma(\varphi_1-\varphi_2)(\overline{\eta}),$$

where G is the geodesic spray of the Finsler structure.

(b) *The canonical nonlinear connection $\overline{\Gamma}$ associated with $\overline{D}$ is given by:*

$$\begin{aligned}\overline{\Gamma}(X) := \Gamma(X) \quad + \quad & 2\Big(f_1\{A(\rho X)\,\gamma\overline{\eta} + A(\overline{\eta})\,JX - L\,\ell(\rho X)\,\gamma\overline{a} + L^2\gamma T(\overline{a},\rho X)\} \\ & +f_2\,\{L\,\ell(\rho X)\,\gamma\overline{b} - L^2\gamma T(\overline{b},\rho X)\} + L\,\ell(\varphi_1(\rho X))\,\gamma\overline{u} - u(\overline{\eta})\gamma\varphi_1(\rho X) \\ & +u(\rho X)\gamma\varphi_2(\overline{\eta}) - L\ell(\varphi_1(\overline{\eta}))\gamma T(\overline{u},\rho X) + u(\overline{\eta})\,\gamma T((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\Big),\end{aligned}$$

where Γ is the Barthel connection of (M,L).

Proof. (a) It follows from (3.3), by replacing $\overline{X}$ by $\overline{\eta}$, and using (3.8).
(b) The expression of $\overline{\Gamma}$ is obtained by applying Lemma 2.4, taking into account Equations (3.3) and (3.9). □

Proposition 3.6. *Let $\overline{D}$ be the Finsler Tripathi connection. Then, the following hold:*

(a) *the $(h)hv$-torsion $\overline{T}$ of $\overline{D}$ coincides with the $(h)hv$-torsion T of Cartan connection.*
(b) *the $(h)h$-torsion $\overline{Q}$ of $\overline{D}$ has the form: $\overline{Q}(\overline{X},\overline{Y}) = u(\overline{Y})\varphi(\overline{X}) - u(\overline{X})\varphi(\overline{Y})$.*
(c) *the $(v)v$-torsion $\widehat{\overline{S}}$ of $\overline{D}$ vanishes identically.*
(d) *the $(v)hv$-torsion $\widehat{\overline{P}}$ of $\overline{D}$ has the form:*

$$\widehat{\overline{P}}(\overline{X},\overline{Y}) = \widehat{P}(\overline{X},\overline{Y}) - \nabla_{\gamma\overline{Y}}\overline{X}_t - N(\overline{X},\overline{Y}) + N(\rho[\beta\overline{X},\gamma\overline{Y}],\overline{\eta}).$$

(e) *the $(v)h$-torsion $\widehat{\overline{R}}$ of $\overline{D}$ has the form:*

$$\widehat{\overline{R}}(\overline{X},\overline{Y}) = \widehat{R}(\overline{X},\overline{Y}) + N(\rho[\bar{\beta}\,\overline{X},\bar{\beta}\,\overline{Y}],\overline{\eta}) + K([\beta\overline{X},\gamma\overline{Y}_t] + [\gamma\overline{X}_t,\beta\overline{Y}] + [\gamma\overline{X}_t,\gamma\overline{Y}_t]),$$

where $\widehat{P}$ and $\widehat{R}$ are the $(v)hv$ and $(v)h$ torsions of the Cartan connection, respectively.

Proof. (a) Follows from the definition of $\overline{T}$, together with Equations (3.4) and (3.5).

(b) Follows directly by condition (III) of Theorem 3.2.

(c) Using (a) above and [30, Proposition 2.5], we obtain

$$\begin{aligned}\overline{S}(\overline{X},\overline{Y})\overline{Z} \;&=\; (\overline{D}_{\gamma\overline{Y}}\overline{T})(\overline{X},\overline{Z}) - (\overline{D}_{\gamma\overline{X}}\overline{T})(\overline{Y},\overline{Z}) + \overline{T}(\overline{X},\overline{T}(\overline{Y},\overline{Z})) \\ &\quad -\overline{T}(\overline{Y},\overline{T}(\overline{X},\overline{Z})) + \overline{T}(\widehat{\overline{S}}(\overline{Y},\overline{X}),\overline{Z}).\end{aligned} \tag{3.14}$$

Setting $\overline{Z} = \overline{\eta}$ into (3.14), taking into account (a) together with the properties of T and the fact that $\bar{K}\circ\gamma = id_{\mathfrak{X}(\pi(M))}$, the result follows.

(d) According to Corollary 3.4 and Proposition 3.5 together with $\bar{K}\circ\bar{\beta} = 0$, we get

$$\begin{aligned}\widehat{\overline{P}}(\overline{X},\overline{Y}) \;&=\; -\overline{D}_{\bar{\beta}\,\overline{X}}\overline{D}_{\gamma\overline{Y}}\,\overline{\eta} + \overline{D}_{\gamma\overline{Y}}\overline{D}_{\bar{\beta}\,\overline{X}}\,\overline{\eta} + \overline{D}_{[\bar{\beta}\,\overline{X},\gamma\overline{Y}]}\,\overline{\eta} = -\overline{D}_{\bar{\beta}\,\overline{X}}\overline{Y} + \overline{D}_{[\bar{\beta}\,\overline{X},\gamma\overline{Y}]}\overline{\eta} \\ &=\; -\nabla_{\beta\,\overline{X}}\overline{Y} + \nabla_{[\beta\,\overline{X},\gamma\overline{Y}]}\overline{\eta} - \nabla_{\gamma\overline{X}_t}\overline{Y} + \nabla_{[\gamma\overline{X}_t,\gamma\overline{Y}]}\overline{\eta} - N(\overline{X},\overline{Y}) + N(\rho[\bar{\beta}\,\overline{X},\gamma\overline{Y}],\overline{\eta}) \\ &=\; \widehat{P}(\overline{X},\overline{Y}) - \nabla_{\gamma\overline{Y}}\overline{X}_t - N(\overline{X},\overline{Y}) + N(\rho[\beta\,\overline{X},\gamma\overline{Y}],\overline{\eta}).\end{aligned}$$

Hence, the result follows by taking into account (c) above.

(e) The proof is similar to that of (d) above. □

Proposition 3.7. *Let $\overline{S}$, $\overline{P}$ and $\overline{R}$ (S, P and R) be the v-, hv- and h-curvatures of the $GC\Gamma$-connection $\overline{D}$ (the Cartan connection ∇), then we have*

(a) $\overline{S}(\overline{X},\overline{Y})\overline{Z} = S(\overline{X},\overline{Y})\overline{Z}$,
(b) $\overline{P}(\overline{X},\overline{Y})\overline{Z} = P(\overline{X},\overline{Y})\overline{Z} + (\nabla_{\gamma\overline{Y}}N)(\overline{X},\overline{Z}) + N(T(\overline{Y},\overline{X}),\overline{Z})$

$$\begin{aligned}&+ f_1\{A(\overline{\eta})\,S(\overline{X},\overline{Y})\overline{Z} - L\,\ell(\overline{X})\,S(\overline{a},\overline{Y})\overline{Z} + L^2\,S(T(\overline{a},\overline{X}),\overline{Y})\overline{Z}\} \\ &+ f_2\{L\,\ell(\overline{X})\,S(\overline{b},\overline{Y})\overline{Z} + L^2\,S(T(\overline{b},\overline{X}),\overline{Y})\overline{Z}\} + L\,\ell(\varphi_1(\overline{X}))\,S(\overline{u},\overline{Y})\overline{Z} \\ &+ u(\overline{X})\,S(\varphi_2(\overline{\eta}),\overline{Y})\overline{Z} - L\,\ell(\varphi_1(\overline{\eta}))\,S(T(\overline{u},\overline{X}),\overline{Y})\overline{Z} \\ &+ u(\overline{\eta})\,\{S(T((\varphi_1-\varphi_2)(\overline{\eta}),\overline{X}),\overline{Y})\overline{Z} - S(\varphi_1(\overline{X}),\overline{Y})\overline{Z}\},\end{aligned}$$

(c) $\overline{R}(\overline{X},\overline{Y})\overline{Z} = R(\overline{X},\overline{Y})\overline{Z} + P(\overline{X},\overline{Y}_t)\overline{Z} - P(\overline{Y},\overline{X}_t)\overline{Z} + S(\overline{X}_t,\overline{Y}_t)\overline{Z}$
$+ \mathfrak{U}_{\overline{X},\overline{Y}}\{(\nabla_{\bar\beta\overline{Y}}N)(\overline{X},\overline{Z}) + N(\overline{Y},N(\overline{X},\overline{Z})) + N(T(\overline{Y}_t,\overline{X}),\overline{Z})\}$,

where $\mathfrak{U}_{X,Y}\{B(X,Y)\} := B(X,Y) - B(Y,X)$.

Proof. (a) Follows from (3.5). (b) and (c) follow from both (3.5) and (3.12). □

Proposition 3.8. *For the $GC\Gamma$-connection, the following identities hold*:

(a) $\overline{P}(\overline{X},\overline{Y})\overline{Z} - \overline{P}(\overline{Z},\overline{Y})\overline{X} = (\overline{D}_{\bar\beta\overline{Z}}T)(\overline{Y},\overline{X}) - (\overline{D}_{\bar\beta\overline{X}}T)(\overline{Y},\overline{Z}) - (\overline{D}_{\gamma\overline{Y}}Q)(\overline{Z},\overline{X})$
$+ T(\overline{Y},\overline{Q}(\overline{Z},\overline{X})) - T(\widehat{\overline{P}}(\overline{Z},\overline{Y}),\overline{X}) + T(\widehat{\overline{P}}(\overline{X},\overline{Y}),\overline{Z})$
$- \overline{Q}(\overline{Z},T(\overline{Y},\overline{X})) + \overline{Q}(\overline{X},T(\overline{Y},\overline{Z}))$,

(b) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\overline{R}(\overline{X},\overline{Y})\overline{Z} - T(\widehat{\overline{R}}(\overline{X},\overline{Y}),\overline{Z})\} = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{\overline{Q}(\overline{X},\overline{Q}(\overline{Y},\overline{Z})) - (\overline{D}_{\bar\beta\overline{X}}Q)(\overline{Y},\overline{Z})\}$,

(c) $(\overline{D}_{\bar\beta\overline{Z}}S)(\overline{X},\overline{Y},\overline{W}) - \overline{P}(\overline{Z},\widehat{S}(\overline{X},\overline{Y}))\overline{W} =$
$= \mathfrak{U}_{\overline{X},\overline{Y}}\{(\overline{D}_{\gamma\overline{Y}}\overline{P})(\overline{Z},\overline{X},\overline{W}) + \overline{P}(T(\overline{X},\overline{Z}),\overline{Y})\overline{W} + S(\widehat{\overline{P}}(\overline{Z},\overline{X}),\overline{Y})\overline{W}\}$,

(d) $(\overline{D}_{\gamma\overline{X}}\overline{R})(\overline{Y},\overline{Z},\overline{W}) = S(\widehat{\overline{R}}(\overline{Y},\overline{Z}),\overline{X})\overline{W} - \overline{P}(\overline{Q}(\overline{Y},\overline{Z}),\overline{X})\overline{W} +$
$+ \mathfrak{U}_{\overline{Z},\overline{Y}}\{(\overline{D}_{\bar\beta\overline{Z}}\overline{P})(\overline{Y},\overline{X},\overline{W}) + \overline{P}(\overline{Z},\widehat{\overline{P}}(\overline{Y},\overline{X}))\overline{W} + \overline{R}(T(\overline{X},\overline{Z}),\overline{Y})\overline{W}\}$,

(e) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{(\overline{D}_{\bar\beta\overline{X}}\overline{R})(\overline{Y},\overline{Z},\overline{W}) + \overline{P}(\overline{X},\widehat{\overline{R}}(\overline{Y},\overline{Z}))\overline{W} + \overline{R}(\overline{Q}(\overline{X},\overline{Y}),\overline{Z})\overline{W}\} = 0$.

Proof. It results from [30, Propositions 2.5, 2.6], taking into account Corollary 3.4 and Propositions 3.6, 3.7 above. □

4. A generalization of the four celebrated Finsler connections

This section is devoted to constructing new Finsler connections from our general connection $GC\Gamma$ by means of the P^1-process and C-process introduced by Matsumoto [10]. First, let us denote Cartan, Berwald, Hasiguashi and Chern-Rund connections by $C\Gamma$, $B\Gamma$, $H\Gamma$ and $R\Gamma$, respectively.

Definition 4.1. Let $\overline{D}$ be the $GC\Gamma$-connection. The process of adding the associated (v)hv-torsion $\overline{P}(\overline{Y},\overline{X})$ to the horizontal part $\overline{D}_{\bar\beta\overline{X}}\overline{Y}$ of $GC\Gamma$ is called the $\overline{P}^1$-process. Moreover, the process of subtracting the associated (h)hv-torsion $\overline{T}(\overline{X},\overline{Y})$ from the vertical part $\overline{D}_{\gamma\overline{X}}\overline{Y}$ of $GC\Gamma$ is called $\overline{C}$-process.

Theorem 4.2. *By means of the $\overline{P}^1$-process and $\overline{C}$-process, we have:*

(a) *The $\overline{P}^1$-process of $GC\Gamma$ yields a generalized Hashiguchi connection ($GH\Gamma$).*
(b) *The $\overline{C}$-process of $GC\Gamma$ yields a generalized Chern-Rund connection ($GR\Gamma$).*
(c) *The $\overline{P}^1$-process followed by the $\overline{C}$-process of $GC\Gamma$ (or the $\overline{C}$-process followed by the $\overline{P}^1$-process) yields a generalized Berwald connection ($GB\Gamma$).*

Now, we define a vanishing condition (or in short VC) by setting $f_1 = f_2 = u = 0$. Based on Theorem 4.2 and Corollary 3.4, if the VC is satisfied, we obtain the following diagram:

$$\begin{tikzcd}
GB\Gamma \arrow[d, "\text{VC}"'] & GH\Gamma \arrow[l, "\overline{C}\text{-proc.}"'] \arrow[d, "\text{VC}"'] & GC\Gamma \arrow[l, "\overline{P}^1\text{-proc.}"'] \arrow[r, "\overline{C}\text{-proc.}"] \arrow[d, "\text{VC}"'] & GR\Gamma \arrow[r, "\overline{P}^1\text{-proc.}"] \arrow[d, "\text{VC}"] & GB\Gamma \arrow[d, "\text{VC}"] \\
B\Gamma & H\Gamma \arrow[l, "C\text{-proc.}"'] & C\Gamma \arrow[l, "P^1\text{-proc.}"'] \arrow[r, "C\text{-proc.}"] & R\Gamma \arrow[r, "P^1\text{-proc.}"] & B\Gamma
\end{tikzcd}$$

The arrows of the second row arise from the usual P^1-process and C-process and they are well known [10]. The arrows of the first row arise from Theorem 4.2 and they are completely new. Moreover, the connections of the second row come from the generalized connections of the first row under the VC.

One notes from the above discussion that the Finsler Tripathi connection may be considered as a generalized Cartan connection, and this justifies the symbol $GC\Gamma$ which was attributed to this connection.

5. Special cases

In the present section, we give some important particular cases (26 cases) of our connection $\overline{D}$ which result from certain choices of $f_1, f_2, A, B, u, \varphi_1, \varphi_2$. Some of the following cases have been already studied in the context of Finsler geometry while many others have not.

Generalized quarter-symmetric recurrent metric Finsler connection:
$A = B,\ f_1 = 1-t,\ f_2 = -t,\ t \in \mathbb{R}$

(1) In this case, $\overline{D}$ reduces to a Finslerian version of the connection introduced in [18] and is written in the form:

$$\begin{aligned}\overline{D}_X\overline{Y} \ &= \ \nabla_X\overline{Y} - (1-t)\,\{A(\rho X)\overline{Y} + A(\overline{Y})\,\rho X\} + g(\rho X,\overline{Y})\,\overline{a} - L\,\ell(\overline{Y})\,T(\overline{a},\rho X)\\ &+L^2\,S(\rho X,\overline{a})\overline{Y} - \{g(\varphi_1(\rho X),\overline{Y}) + \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\}\overline{u} - u(\rho X)\,\varphi_2(\overline{Y})\\ &+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) - u(\overline{\eta})\{S((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\overline{Y} + T(\varphi_1(\overline{Y}),\rho X)\\ &-\varphi_1(T(\rho X,\overline{Y}))\} + u(\overline{Y})\{T(\varphi_2(\overline{\eta}),\overline{X}) + \varphi_1(\rho X)\} + \mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta}\\ &+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y} - \mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

Quarter-symmetric metric Finsler connection: $f_1 = f_2 = 0$

(2) $\overline{D}$ becomes a Finslerian version of the Riemannian connection given in [23, formula (3.3)], that is,

$$\begin{aligned}\overline{D}_X\overline{Y} \ &= \ \nabla_X\overline{Y} - \{g(\varphi_1(\rho X),\overline{Y}) + \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\}\overline{u} - u(\rho X)\,\varphi_2(\overline{Y})\\ &+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) - u(\overline{\eta})\{S((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\overline{Y} + T(\varphi_1(\overline{Y}),\rho X)\\ &-\varphi_1(T(\rho X,\overline{Y}))\} + u(\overline{Y})\{T(\varphi_2(\overline{\eta}),\overline{X}) + \varphi_1(\rho X)\}\\ &+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y} - \mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(3) In addition, when $\varphi = Ric_o$, then we obtain a Finslerian version of the Ricci quarter-symmetric metric connection appeared in [12, formula (2.2)].

(4) When $\varphi_2 = 0$, we get a Finslerian version of the quarter-symmetric metric connection presented in [12, formula (1.6)]. Thus, $\overline{D}$ has the form:

$$\begin{aligned}\overline{D}_X\overline{Y} \ &= \ \nabla_X\overline{Y} - g(\varphi_1(\rho X),\overline{Y})\,\overline{u} + L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) - u(\overline{\eta})\{S(\varphi_1(\overline{\eta}),\rho X)\overline{Y}\\ &+T(\varphi_1(\overline{Y}),\rho X) - \varphi_1(T(\rho X,\overline{Y}))\} + u(\overline{Y})\,\varphi_1(\rho X)\\ &+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y} - \mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(5) When $\varphi_1 = 0$, $\overline{D}$ reduces to a Finslerian version of the connection given in [23, formula (3.6)]. That is,

$$\begin{aligned}\overline{D}_X\overline{Y} = \nabla_X\overline{Y} - \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\,\overline{u} - u(\rho X)\varphi_2(\overline{Y}) + u(\overline{Y})\,T(\varphi_2(\overline{\eta}),\overline{X})\\ + u(\overline{\eta})\,S(\varphi_2(\overline{\eta}),\rho X)\overline{Y}.\end{aligned}$$

Quarter-symmetric non-metric Finsler connection: $f_1 \neq 0,\ f_2 = 0$

(6) If $f_1 = \frac{1}{2}$, then we obtain an intrinsic formula of the quarter-symmetric h-recurrent Finsler connection presented in [16]. Thereby, $\overline{D}$ becomes

$$\begin{aligned}\overline{D}_X\overline{Y} \ &= \ \nabla_X\overline{Y} + \frac{1}{2}\{g(\rho X,\overline{Y})\,\overline{a} - A(\rho X)\overline{Y} - A(\overline{Y})\,\rho X - L\,\ell(\overline{Y})\,T(\overline{a},\rho X)\\ &+L^2\,S(\rho X,\overline{a})\overline{Y}\} - \{g(\varphi_1(\rho X),\overline{Y}) + \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\}\overline{u}\\ &-u(\rho X)\,\varphi_2(\overline{Y}) + L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X) - u(\overline{\eta})\{S((\varphi_1-\varphi_2)(\overline{\eta}),\rho X)\overline{Y}\\ &+T(\varphi_1(\overline{Y}),\rho X) - \varphi_1(T(\rho X,\overline{Y}))\} + u(\overline{Y})\{T(\varphi_2(\overline{\eta}),\overline{X}) + \varphi_1(\rho X)\}\end{aligned}$$

$$+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}-\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta})+\frac{1}{2}\mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta}.$$

(7) When $\varphi_2=0$, $\overline{D}$ reduces to a Finslerian version of the connection given in [19, §4.2 (4)]. Then, $\overline{D}$ can be written as follows

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}+f_1\,\{g(\rho X,\overline{Y})\,\overline{a}-A(\rho X)\overline{Y}-A(\overline{Y})\,\rho X-L\,\ell(\overline{Y})\,T(\overline{a},\rho X)\\ &+\mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta}+L^2\,S(\rho X,\overline{a})\overline{Y}\}+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X)+u(\overline{Y})\varphi_1(\rho X)\\ &-u(\overline{\eta})\{S(\varphi_1(\overline{\eta}),\rho X)\overline{Y}+T(\varphi_1(\overline{Y}),\rho X)-\varphi_1(T(\rho X,\overline{Y}))\}\\ &+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}-g(\varphi_1(\rho X),\overline{Y})\,\overline{u}-\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(8) If $f_1=1$, $A=u$ and $\varphi_2=0$, then we get a Finslerian version of the connection $\overline{D}$ presented in [19, §4.2 (5)]. Therefore, $\overline{D}$ is given by:

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}+g(\rho X-\varphi_1(\rho X),\overline{Y})\,\overline{u}-u(\rho X)\overline{Y}-u(\overline{Y})\,\rho X-L\,\ell(\overline{Y})\,T(\overline{u},\rho X)\\ &+\mathbf{T}(\overline{u},\rho X,\overline{Y})\overline{\eta}+L^2\,S(\rho X,\overline{u})\overline{Y}+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X)\\ &-u(\overline{\eta})\{S(\varphi_1(\overline{\eta}),\rho X)\overline{Y}+T(\varphi_1(\overline{Y}),\rho X)-\varphi_1(T(\rho X,\overline{Y}))\}\\ &+u(\overline{Y})\,\varphi_1(\rho X)+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}-\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(9) If $\varphi_1=0$, then we obtain a Finslerian version of the quarter-symmetric recurrent connection $\overline{D}$ given in [19, §4.2 (6)]. Thus, $\overline{D}$ has the form

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}+f_1\{g(\rho X,\overline{Y})\overline{a}-A(\rho X)\overline{Y}-A(\overline{Y})\,\rho X-L\ell(\overline{Y})T(\overline{a},\rho X)\\ &+\mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta}+L^2\,S(\rho X,\overline{a})\overline{Y}\}-u(\rho X)\varphi_2(\overline{Y})+u(\overline{Y})T(\varphi_2(\overline{\eta}),\overline{X})\\ &-\mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\overline{u}+u(\overline{\eta})\,S(\varphi_2(\overline{\eta}),\rho X)\overline{Y}.\end{aligned}$$

(10) When $f_1=1$, $A=u$ and $\varphi_1=0$, $\overline{D}$ reduces to a Finslerian version of the special quarter-symmetric recurrent connection presented in [19, §4.2 (7)] . That is,

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}+g(\rho X,\overline{Y})\overline{u}-u(\rho X)\overline{Y}-u(\overline{Y})\rho X-L\ell(\overline{Y})T(\overline{u},\rho X)\\ &+\mathbf{T}(\overline{u},\rho X,\overline{Y})\overline{\eta}+L^2\,S(\rho X,\overline{u})\overline{Y}-u(\rho X)\varphi_2(\overline{Y})+u(\overline{Y})T(\varphi_2(\overline{\eta}),\overline{X})\\ &-\mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\overline{u}+u(\overline{\eta})\,S(\varphi_2(\overline{\eta}),\rho X)\overline{Y}.\end{aligned}$$

Quarter-symmetric non-metric Finsler connection: $f_1=0,\ f_2\neq 0$

(11) If $\varphi_2=0$, we get a Finslerian version of the connection given in [19, §4.2 (8)]. Then, the connection $\overline{D}$ has the form:

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}-f_2\{g(\rho X,\overline{Y})\,\overline{b}-L\,\ell(\overline{Y})\,T(\overline{b},\rho X)+\mathbf{T}(\overline{b},\rho X,\overline{Y})\,\overline{\eta}\\ &+L^2\,S(\rho X,\overline{a})\overline{Y}\}-g(\varphi_1(\rho X),\overline{Y})\overline{u}+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X)\\ &-u(\overline{\eta})\{S(\varphi_1(\overline{\eta}),\rho X)\overline{Y}+T(\varphi_1(\overline{Y}),\rho X)-\varphi_1(T(\rho X,\overline{Y}))\}\\ &+u(\overline{Y}\,\varphi_1(\rho X)+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}-\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(12) When $B=u$ and $\varphi_2=0$, $\overline{D}$ becomes a Finslerian version of the connection appeared in [19, §4.2 (9)]. That is,

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}-f_2\{g(\rho X,\overline{Y})\,\overline{u}-L\,\ell(\overline{Y})\,T(\overline{u},\rho X)+\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\overline{\eta}\\ &+L^2\,S(\rho X,\overline{a})\overline{Y}\}-g(\varphi_1(\rho X),\overline{Y})\overline{u}+L\,\ell(\varphi_1(\overline{Y}))T(\overline{u},\rho X)\\ &-u(\overline{\eta})\{S(\varphi_1(\overline{\eta}),\rho X)\overline{Y}+T(\varphi_1(\overline{Y}),\rho X)-\varphi_1(T(\rho X,\overline{Y}))\}\\ &+u(\overline{Y})\,\varphi_1(\rho X)+L\,\ell(\varphi_1(\overline{\eta}))\,S(\overline{u},\rho X)\overline{Y}-\mathbf{T}(\overline{u},\rho X,\overline{Y})\,\varphi_1(\overline{\eta}).\end{aligned}$$

(13) If $\varphi_1=0$, we obtain a Finslerian version of the connection given in [19, §4.2 (10)]. The connection $\overline{D}$ can be written as

$$\begin{aligned}\overline{D}_X\overline{Y} \;=\;& \nabla_X\overline{Y}-f_2\{g(\rho X,\overline{Y})\,\overline{b}-L\,\ell(\overline{Y})\,T(\overline{b},\rho X)+\mathbf{T}(\overline{b},\rho X,\overline{Y})\,\overline{\eta}\\ &+L^2\,S(\rho X,\overline{a})\overline{Y}\}-\mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\,\overline{u}-u(\rho X)\,\varphi_2(\overline{Y})\\ &+u(\overline{\eta})\,S(\varphi_2(\overline{\eta}),\rho X)\overline{Y}+u(\overline{Y})\,T(\varphi_2(\overline{\eta}),\overline{X}).\end{aligned}$$

(14) If $B = u$ and $\varphi_1 = 0$, we get a Finslerian version of the connection presented in [19, §4.2 (11)]. The connection $\overline{D}$ is given by:

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} - f_2\{g(\rho X,\overline{Y})\,\overline{u} - L\,\ell(\overline{Y})\,T(\overline{u},\rho X) + \mathbf{T}(\overline{u},\rho X,\overline{Y})\,\overline{\eta} \\ & + L^2\,S(\rho X,\overline{a})\overline{Y}\} - \mathbf{T}(\varphi_2(\overline{\eta}),\rho X,\overline{Y})\,\overline{u} - u(\rho X)\,\varphi_2(\overline{Y}) \\ & + u(\overline{\eta})\,S(\varphi_2(\overline{\eta}),\rho X)\overline{Y} + u(\overline{Y})\,T(\varphi_2(\overline{\eta}),\overline{X}).\end{aligned}$$

Semi-symmetric metric Finsler connection: $f_1 = f_2 = 0,\ \varphi = id_{\pi^{-1}(TM)}$

- (15) We obtain the Finslerian version of the connection defined in [21]. That is,

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} - g(\rho X,\overline{Y})\overline{u} + u(\overline{Y})\rho X + L\ell(\overline{Y})T(\overline{u},\rho X) \\ & + L^2T(T(\rho X,\overline{Y}),\overline{u}) - L^2T(T(\overline{u},\overline{Y}),\rho X) - T(\overline{u},\rho X,\overline{Y})\overline{\eta}.\end{aligned}$$

(16) If $u = \ell$, we get an intrinsic version of the connection given in [16, §5.3].

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} - L^{-1}g(\rho X,\overline{Y})\overline{\eta} + \ell(\overline{Y})\rho X.$$

Note that special classes of semi-symmetric metric Finsler connections have been introduced and investigated in [24] and [25].

Semi-symmetric non-metric Finsler connection: $f_1 \neq 0,\ f_2 = 0, \varphi = id_{\pi^{-1}(TM)}$

(17) $\overline{D}$ reduces to a Finslerian version of the semi-symmetric recurrent connection given in [16, 19]. Thus, $\overline{D}$ is written in the form:

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} + f_1\{g(\rho X,\overline{Y})\overline{a} - A(\rho X)\overline{Y} - A(\overline{Y})\,\rho X - L\ell(\overline{Y})T(\overline{a},\rho X) \\ & + \mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta} + L^2\{T(T(\overline{a},\overline{Y}),\rho X) - T(T(\rho X,\overline{Y}),\overline{a})\}\} \\ & - g(\rho X,\overline{Y})\overline{u} + u(\overline{Y})\rho X + L\ell(\overline{Y})T(\overline{u},\rho X) \\ & - L^2T(T(\overline{u},\overline{Y}),\rho X) - \mathbf{T}(\overline{u},\rho X,\overline{Y})\overline{\eta} + L^2T(T(\rho X,\overline{Y}),\overline{u}).\end{aligned}$$

(18) If $f_1 = \frac{1}{2}$, we obtain a Finslerian version of the semi-symmetric recurrent connection $\overline{D}$ studied in [2, 8, 16]. In this case, $\overline{D}$ is given by:

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} + \frac{1}{2}\{g(\rho X,\overline{Y})\overline{a} - A(\rho X)\overline{Y} - A(\overline{Y})\,\rho X - L\ell(\overline{Y})T(\overline{a},\rho X) \\ & + \mathbf{T}(\overline{a},\rho X,\overline{Y})\overline{\eta} + L^2\{T(T(\overline{a},\overline{Y}),\rho X) - T(T(\rho X,\overline{Y}),\overline{a})\}\} \\ & - g(\rho X,\overline{Y})\overline{u} + u(\overline{Y})\rho X + L\ell(\overline{Y})T(\overline{u},\rho X) \\ & + L^2T(T(\rho X,\overline{Y}),\overline{u}) - L^2T(T(\overline{u},\overline{Y}),\rho X) - T(\overline{u},\rho X,\overline{Y})\overline{\eta}.\end{aligned}$$

(19) If $f_1 = \frac{1}{2}$ and $A = u = \ell$, we obtain a special semi-symmetric h-recurrent Finsler connection [16] given by:

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} - \frac{1}{2}\{L^{-1}g(\rho X,\overline{Y})\overline{\eta} + \ell(\rho X)\overline{Y} - \ell(\overline{Y})\rho X\}.$$

Semi-symmetric non-metric Finsler connection: $f_1 = 0,\ f_2 \neq 0,\ \varphi = id_{\pi^{-1}(TM)}$

(20) We obtain a Finslerian version of the semi-symmetric non-metric connection given in [19, §4.4 (14)], that is,

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} - f_2\{g(\rho X,\overline{Y})\overline{b} - L\ell(\overline{Y})T(\overline{b},\rho X) + \mathbf{T}(\overline{b},\rho X,\overline{Y})\overline{\eta} \\ & + L^2\{T(T(\overline{b},\overline{Y}),\rho X) - T(T(\rho X,\overline{Y}),\overline{b})\}\} \\ & - g(\rho X,\overline{Y})\overline{u} + u(\overline{Y})\rho X + L\ell(\overline{Y})T(\overline{u},\rho X) \\ & - L^2T(T(\overline{u},\overline{Y}),\rho X) - \mathbf{T}(\overline{u},\rho X,\overline{Y})\overline{\eta} + L^2T(T(\rho X,\overline{Y}),\overline{u}).\end{aligned}$$

(21) If $f_2 = -1$, we obtain a Finslerian version of the connection discussed in [15], that is,

$$\begin{aligned}\overline{D}_X\overline{Y} \;\;=\;\; & \nabla_X\overline{Y} + g(\rho X,\overline{Y})\overline{b} - L\ell(\overline{Y})T(\overline{b},\rho X) + \mathbf{T}(\overline{b},\rho X,\overline{Y})\overline{\eta} \\ & + L^2\{T(T(\overline{b},\overline{Y}),\rho X) - T(T(\rho X,\overline{Y}),\overline{b})\}\end{aligned}$$

$$-g(\rho X, \overline{Y})\overline{u} + u(\overline{Y})\rho X + L\ell(\overline{Y})T(\overline{u}, \rho X)$$
$$-L^2T(T(\overline{u}, \overline{Y}), \rho X) - \mathbf{T}(\overline{u}, \rho X, \overline{Y})\overline{\eta} + L^2T(T(\rho X, \overline{Y}), \overline{u}).$$

(22) If $f_2 = -1$ and $B = u$, then we get the Finslerian version of the connection studied in [1], that is,

$$\overline{D}_X\overline{Y} \quad = \quad \nabla_X\overline{Y} + u(\overline{Y})\rho X.$$

Symmetric non-metric Finsler connection: $u = 0$

(23) We obtain a Finslerian version of the connection appeared in [19, §4.5 (15)]. It is given by:

$$\begin{aligned}\overline{D}_X\overline{Y} \quad = \quad & \nabla_X\overline{Y} + f_1\{g(\rho X, \overline{Y})\,\overline{a} - A(\rho X)\overline{Y} - A(\overline{Y})\,\rho X - L\,\ell(\overline{Y})\,T(\overline{a}, \rho X) \\ & + \mathbf{T}(\overline{a}, \rho X, \overline{Y})\overline{\eta} + L^2\,S(\rho X, \overline{a})\overline{Y}\} - f_2\{g(\rho X, \overline{Y})\,\overline{b} - L\,\ell(\overline{Y})\,T(\overline{b}, \rho X) \\ & + \mathbf{T}(\overline{b}, \rho X, \overline{Y})\,\overline{\eta} + L^2\,S(\rho X, \overline{b})\overline{Y}\}.\end{aligned}$$

(24) If $f_1 = \frac{1}{2}$ and $f_2 = 0$, then $\overline{D}$ reduced to a Finslerian version of the symmetric recurrent connection (Weyl connection) investigated in [3, 13, 26]. That is,

$$\begin{aligned}\overline{D}_X\overline{Y} \quad = \quad & \nabla_X\overline{Y} + \frac{1}{2}\{g(\rho X, \overline{Y})\,\overline{a} - A(\rho X)\,\overline{Y} - A(\overline{Y})\,\rho X - L\,\ell(\overline{Y})\,T(\overline{a}, \rho X) \\ & + \mathbf{T}(\overline{a}, \rho X, \overline{Y})\,\overline{\eta} + L^2\,S(\rho X, \overline{a})\overline{Y}\}.\end{aligned}$$

(25) If $f_1 = f_2 = -1$ and $A = B$, then $\overline{D}$ is a Finslerian version of the connection considered in [22], that is,

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} + A(\rho X)\overline{Y} + A(\overline{Y})\,\rho X.$$

(26) If $f_1 = \frac{1}{2}$, $A = \ell$ and $f_2 = 0$, then we obtain a special symmetric h-recurrent Finsler connection studied in [13, 26], that is,

$$\overline{D}_X\overline{Y} = \nabla_X\overline{Y} + \frac{1}{2}\{L^{-1}g(\rho X, \overline{Y})\overline{\eta} - \ell(\rho X)\overline{Y} - \ell(\overline{Y})\rho X\}.$$

We end this work by the following remark: applying the P^1-process and C-process to each of the above mentioned special cases, one can get more new Finsler connections.

6. Concluding remarks

We conclude the present paper by some comments and remarks.

• In this paper we provide a general class of Finsler connections, large enough to include the classical four Finsler connections (Berwald, Cartan, Chern-Rund and Hashiguchi) and so many other generalizations.

• In Riemannian geometry, the Levi-Civita connection is the unique linear connection that is metric and symmetric. This connection can be generalized by replacing:
– the metricity condition (MR-condition) by requiring the covariant derivative of the metric tensor to be a prescribed $(0, 3)$-type tensor (symmetric in the first 2 arguments).
– the symmetric condition (TR-condition), by requiring the torsion tensor to be a given vector valued 2-form.
There are many choices for the $(0, 3)$-type tensor as well as for the vector valued 2-form. Such a choice was used by Tripathi in his formulae (2.5) and (2.6) of [19, Theorem 2.1]. Tripathi uses for the MR-condition:

$$\nabla g(X, Y, Z) = 2f_1u_1(Z)g(X, Y) + f_2[u_2(X)g(Y, Z) + u_2(Y)g(X, Z)], \tag{6.1}$$

which is a $(0, 3)$-type tensor symmetric in X and Y, and for the TR-condition:

$$T(X, Y) = u(X)\varphi(Y) - u(Y)\varphi(X), \tag{6.2}$$

which is a vector valued 2-form. Of course, these are not the only possible choices for the two MR- and TR-conditions, but any choice should be motivated.

• Similar ideas can be extended to the Finslerian context. A very general Finsler connection (including the classical four Finsler connections) can be defined by providing information about:
– the metricity condition (MF-condition): the horizontal and vertical covariant derivatives of the metric tensor are prescribed $(0,3)$-type tensor (symmetric in the first 2 arguments).
– the symmetric condition (TF-condition): horizontal and vertical counterparts of the torsion tensor field are given by vector valued 2-form.
In the main theorem of this paper (Theorem 3.2), for the MF-conditions, the horizontal covariant derivative is provided by condition (6.1) and the vertical covariant derivative vanishes (here of course one can generalize by providing a given tensor). For the TF-conditions, the h(h)-torsion uses condition (6.2), while the other torsion counterpart satisfies some symmetry.

• Justification of our choices for MF- and TF-conditions.
We have chosen the MF- and TF-conditions of Theorem 3.2 in such a way that the following two criteria are fulfilled.
(1) When the objects $f_1, f_2, A, B, u, \varphi$ vanish, we retrieve the Cartan connection. This corresponds to the fact that when these objects vanish in the Riemannian context, the Levi-Civita connection is retrieved. For this reason, Condition IV of Theorem 3.2 is mandatory and Condition II ($\overline{D}_{\gamma\overline{X}}g = 0$) is the optimal condition for the intrinsic proof to be feasible; if we replace the RHS by a nonzero object the proof becomes so complicated or even impossible.
(2) Generalizing [19] form Riemannian to Finslerian context and generalizing [16] form local to global formalism retaining all what was gained in [19] and [16] and, moreover, possessing the ability to produce other new Finsler connections that do not exist in the Finsler literature.

• The particular cases treated in the §5 fall in the following three categories:
1. Connections that have been generalized from Riemannian to Finsler geometry such as (1)-(5), (7)-(15), (20)-(23), (25).
2. Finsler Connections that have been generalized from local to global (or intrinsic) formalism such as (6), (16)-(19).
3. Finsler connections in the global formalism reobtained such as (24), (26).

• From the Finsler Tripathi connection $GC\Gamma$ (3.13), four new fundamental Finsler connections have been obtained: generalized Cartan (which is $GC\Gamma$ itself), generalized Berwald, generalized Hachiguchi, generalized Chern-Rund. From the later connections, the classical Finsler connections (Cartan, Berwald, Hachiguchi, Chern-Rund) are retrieved by placing $f_1 = f_2 = u = 0$ (§4). Many other new Finsler connections that do not exist neither locally nor globally in the literature can be obtained by various choices of the objects $f_1, f_2, A, B, u, \varphi$ (different from those considered in §5), which merit to be investigated. This indicate the great potentiality of the Finsler Tripathi connection.

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