



On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers

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(2nd International Conference on Applied Engineering and Natural Sciences ICAENS 2022, March 10-13, 2022)

(DOI: 10.31590/ejosat.1082643)

ATIF/REFERENCE: Büyük, H., Taskara, N., On The Solutions of Three-Dimensional Difference Equation Systems Via Pell Numbers, (2022). Xxxx Xxxx Xxxx. *European Journal of Science and Technology*, (34), 433-440.

Abstract

In this study, we investigate the form of the solutions of the following rational difference equation system

$$x_n = \frac{z_{n-1}z_{n-3}}{x_{n-2} + 2z_{n-3}}, y_n = \frac{x_{n-1}x_{n-3}}{-y_{n-2} + 6x_{n-3}}, z_n = \frac{y_{n-1}y_{n-3}}{z_{n-2} + 14y_{n-3}}, n \in N_0$$

where initial values $x_{-3}, x_{-2}, x_{-1}, y_{-3}, y_{-2}, y_{-1}, z_{-3}, z_{-2}, z_{-1}$ are nonzero real numbers, such that their solutions are associated with Pell numbers. We also give a relationships between Pell numbers and solutions of systems

Keywords: System of difference equations, Pell numbers, Representation of solutions, Binet formula, Solutions.

Üç Boyutlu Fark Denklem Sistemlerinin Pell sayıları yardımıyla Çözümleri

Öz

Bu çalışmada

$$x_n = \frac{z_{n-1}z_{n-3}}{x_{n-2} + 2z_{n-3}}, y_n = \frac{x_{n-1}x_{n-3}}{-y_{n-2} + 6x_{n-3}}, z_n = \frac{y_{n-1}y_{n-3}}{z_{n-2} + 14y_{n-3}}, n \in N_0$$

rasyonel fark denklem sisteminin çözüm formlarını Pell sayılarıyla ilişkili olacak şekilde araştırıldı. Burada $x_{-3}, x_{-2}, x_{-1}, y_{-3}, y_{-2}, y_{-1}, z_{-3}, z_{-2}, z_{-1}$ başlangıç değerleri sıfırdan farklı reel sayılardır. Aynı zamanda sistemin çözümleri ile Pell sayıları arasındaki ilişkiler de verildi.

Anahtar Kelimeler: Fark denklem sistemleri, Pell sayıları, Çözümlerin temsili, Binet formülü, Çözümler.

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1. Introduction

Nonlinear difference equations and equation systems have long interested researchers in the field of mathematics as well as in other sciences. There are many recent investigations and interest in the field of nonlinear difference equations from several authors. (see, for example, [1]-[27] and the related references therein). For example,

Cinar [1] studied the difference equation and found its solutions

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, n \in \mathbb{N}_0 \tag{1.1}$$

In [2], Tollu et.al. studied the solutions and asymptotic of the difference equations

$$x_{n+1} = \alpha x_{n-k} + \frac{\delta x_{n-k} - x_{n-k-l}}{\beta x_{n-k-l} + x_n x_{n-1}} \tag{1.2}$$

In [3], Elsayed obtained the solutions of the difference equation

$$x_{n+1} = \alpha x_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}} \tag{1.3}$$

In addition, [9]-[12] studies have had different approaches to the solution of difference equations. There are also studies on the solutions of difference equation systems. For example;

Sahinkaya et al. [4], in their study, starting conditions are real numbers, they defined the difference equation system

$$\begin{aligned} x_{n+1} &= \frac{x_n y_n + a}{x_n + y_n}, \\ y_{n+1} &= \frac{y_n z_n + a}{y_n + z_n}, \\ z_{n+1} &= \frac{z_n x_n + a}{z_n + x_n} \end{aligned} \tag{1.4}$$

They also obtained the general solutions in closed form.

Halim et al. [5], in their study; investigated the solutions of on a three-dimensional system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{z_{n-1}}{a + by_n z_{n-1}}, \\ y_{n+1} &= \frac{x_{n-1}}{a + bz_n x_{n-1}}, \\ z_{n+1} &= \frac{y_{n-1}}{a + bx_n y_{n-1}} \end{aligned} \tag{1.5}$$

In addition, in the studies [13]-[20], there have been approaches from different perspectives on the solutions of difference equation systems. There are also studies that related the solutions of difference equations or systems of difference equations with some integer sequences (see, for example, [20]-[26] and the related references therein). For example,

Yazlık et al. in [6], considered the following difference equation systems

$$x_{n+1} = \frac{x_{n+1} \pm 1}{y_n x_n - 1}, y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}} \tag{1.6}$$

They also show that their solutions are associated to Padovan numbers.

Tollu et al., in [7], studied the systems of difference equations

$$x_{n+1} = \frac{1 + p_n}{q_n}, y_{n+1} = \frac{1 + r_n}{s_n}, n \in \mathbb{N}_0 \tag{1.7}$$

and solved fourteen systems out of sixteen possible systems. In particularly, the representation formulae of solutions of twelve systems were stated via Fibonacci numbers.

Okumus et al., in [8], considering four rational difference equations

$$x_{n+1} = \frac{\mp 1}{x_n(x_{n-1} - 1) \pm 1} \tag{1.8}$$

$$x_{n+1} = \frac{\mp 1}{x_n(x_{n-1} + 1) \pm 1},$$

they examined their solutions with Tribonacci numbers.

Now we give information about Pell numbers that establish a large part of our study.

Pell sequences are given such that

$$P_{n+1} = 2P_n + P_{n-1} ; n \geq 1 \tag{1.9}$$

with initial conditions $P_0=0, P_1=1$.

It can be easily obtained that the characteristic equation of (1.9) has the form

$$\lambda^2 - 2\lambda - 1 = 0$$

having the roots

$$\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2} \tag{1.10}$$

Binet formula for Pell sequences

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.11}$$

In this study, we consider the solutions of the following three-dimensional difference equation systems

$$x_n = \frac{z_{n-1} z_{n-3}}{x_{n-2} + 2z_{n-3}}, \tag{1.12}$$

$$y_n = \frac{x_{n-1} x_{n-3}}{-y_{n-2} + 6x_{n-3}},$$

$$z_n = \frac{y_{n-1} y_{n-3}}{z_{n-2} + 14y_{n-3}}, n \in \mathbb{N}_0$$

such that their solutions are associated with Pell numbers. We also establish a relationship between Pell numbers and solutions of systems.

Our aim is to show that system (1.12) is solvable by finding its closed-form formulas through an analytical approach and solutions are relationship between Pell numbers, in our present paper.

2. Material and Method

In our study, we investigated the solution and the relationship between solutions and Pell numbers, by using the apply the change variables, the Binet formula and recurrence relations of Pell numbers.

3. Results and Discussion

Assume that $\{x_n, y_n, z_n\}_{n \in \mathbb{N}_0}$ is a well-defined solution to system (1.12). Then we have

$$\frac{z_{n-1}}{x_n} = \frac{x_{n-2} + 2z_{n-3}}{z_{n-3}}, \quad (3.1)$$

$$\frac{x_{n-1}}{y_n} = \frac{-y_{n-2} + 6x_{n-3}}{x_{n-3}},$$

$$\frac{y_{n-1}}{z_n} = \frac{z_{n-2} + 14y_{n-3}}{y_{n-3}}$$

If we apply the change variables for $n \geq -2$, we have

$$u_n = \frac{z_{n-1}}{x_n}, v_n = \frac{x_{n-1}}{y_n}, w_n = \frac{y_{n-1}}{z_n} \quad (3.2)$$

Then system (1.12) can be written as

$$\begin{aligned} u_n &= \frac{1}{u_{n-2}} + 2, \\ v_n &= \frac{-1}{v_{n-2}} + 6, \\ w_n &= \frac{1}{w_{n-2}} + 14 \end{aligned} \quad (3.3)$$

If we arrange the equations (3.3), we have

$$\begin{aligned} u_n &= \frac{2u_{n-2} + 1}{u_{n-2}}, \\ v_n &= \frac{6v_{n-2} - 1}{v_{n-2}}, \\ w_n &= \frac{14w_{n-2} + 1}{w_{n-2}} \end{aligned} \quad (3.4)$$

Let

$$\begin{aligned} u_n \text{ for } u_m^{(i)} &= u_{2m+i}, \\ u_{n-2} \text{ for } u_{m-1}^{(i)} &= u_{2(m-1)+i} \end{aligned} \quad (3.5)$$

$$\begin{aligned} v_n \text{ for } v_m^{(i)} &= v_{2m+i}, \\ v_{n-2} \text{ for } v_{m-1}^{(i)} &= v_{2(m-1)+i} \end{aligned} \quad (3.6)$$

$$\begin{aligned} w_n \text{ for } w_m^{(i)} &= w_{2m+i}, \\ w_{n-2} \text{ for } w_{m-1}^{(i)} &= w_{2(m-1)+i} \end{aligned} \quad (3.7)$$

where $n \in \mathbb{N}_0, i \in \{0,1\}$

By using equations (3.5), (3.6), (3.7), we can write the equation (3.4) as

$$\begin{aligned} u_m^{(i)} &= \frac{2u_{m-1}^{(i)} + 1}{u_{m-1}^{(i)}}, \\ v_m^{(i)} &= \frac{6v_{m-1}^{(i)} - 1}{v_{m-1}^{(i)}}, \\ w_m^{(i)} &= \frac{14w_{m-1}^{(i)} + 1}{w_{m-1}^{(i)}} \end{aligned} \quad (3.8)$$

Through an analytical approach, we put

$$u_{m-1}^{(i)} = \frac{r_m}{r_{m-1}} \quad (3.9)$$

$$v_{m-1}^{(i)} = \frac{s_m}{s_{m-1}} \quad (3.10)$$

$$w_{m-1}^{(i)} = \frac{t_m}{t_{m-1}} \quad (3.11)$$

Using the equations (3.9), (3.10) and (3.11), we can write the equation (3.8) as

$$r_{m+1} = 2r_m + r_{m-1} \quad (3.12)$$

$$s_{m+1} = 6s_m - s_{m-1} \quad (3.13)$$

$$t_{m+1} = 14t_m + t_{m-1} \quad (3.14)$$

Thus, we converted it to linear equations.

Firstly, let's find to root of linear equation (3.12).

If we write the characteristic equation

$$\lambda^2 - 2\lambda - 1 = 0,$$

roots of the equation are

$$\lambda_1 = 1 + \sqrt{2} = \alpha, \lambda_2 = 1 - \sqrt{2} = \beta$$

It turns out to that the roots of (3.12) are the same as the roots of the Pell number sequence.

Then general solution of the equation (3.12)

$$r_m = c_1(1 + \sqrt{2})^m + c_2(1 - \sqrt{2})^m \quad (3.15)$$

where r_0, r_{-1} are initial values.

Hence

$$r_0 = c_1 + c_2$$

$$r_{-1} = c_1(\sqrt{2} - 1) - c_2(\sqrt{2} + 1)$$

Using the initial values r_0, r_{-1} with some calculations and written at (3.15), we get

$$\begin{aligned} r_m &= \frac{r_{-1} + (\sqrt{2} + 1)r_0}{2\sqrt{2}} (1 + \sqrt{2})^m + \\ &\quad \frac{r_{-1} + (1 - \sqrt{2})r_0}{-2\sqrt{2}} (1 - \sqrt{2})^m \\ &= \frac{r_0[(1 + \sqrt{2})^{m+1} - (1 - \sqrt{2})^{m+1}]}{2\sqrt{2}} + \end{aligned}$$

$$\frac{r_{-1} \left[(1 + \sqrt{2})^m - (1 - \sqrt{2})^m \right]}{2\sqrt{2}}$$

Hence we have

$$r_m = \frac{r_0[(\alpha)^{m+1} - (\beta)^{m+1}]}{\alpha - \beta} + \frac{r_{-1}[(\alpha)^m - (\beta)^m]}{\alpha - \beta}$$

By using the Binet formula in equation (1.11)

$$r_m = r_0 P_{m+1} + r_{-1} P_m \tag{3.16}$$

When we substitute (3.16) using (3.9)

$$u_{-1} = \frac{r_0}{r_{-1}},$$

$$u_{m-1}^{(i)} = \frac{r_0 P_{m+1} + r_{-1} P_m}{r_0 P_m + r_{-1} P_{m-1}} = \frac{\frac{r_0}{r_{-1}} P_{m+1} + P_m}{\frac{r_0}{r_{-1}} P_m + P_{m-1}} \tag{3.17}$$

$$= \frac{u_{-1} P_{m+1} + P_m}{u_{-1} P_m + P_{m-1}}$$

Using (3.8) and (1.9) we have

$$u_m^{(i)} = \frac{u_{-1} P_{m+2} + P_{m+1}}{u_{-1} P_{m+1} + P_m} \tag{3.18}$$

Using (3.5), (3.17) and (3.18), we have

$$u_{2m}^{(i)} = \frac{u_{-1} P_{2m+2} + P_{2m+1}}{u_{-1} P_{2m+1} + P_{2m}} \tag{3.19}$$

$$u_{2m+1}^{(i)} = \frac{u_{-1} P_{2m+3} + P_{2m+2}}{u_{-1} P_{2m+2} + P_{2m+1}} \tag{3.20}$$

Theorem 3.1 Let $\{u_n, v_n, w_n\}_{n \geq -2}$ be a well-defined solution to the system (3.4). Then, for u_n ,

$$u_{6n} = \frac{z_{-1} P_{6n+1} + x_0 P_{6n}}{z_{-1} P_{6n} + x_0 P_{6n-1}}$$

$$u_{6n+1} = \frac{z_{-1} P_{6n+2} + x_0 P_{6n+1}}{z_{-1} P_{6n+1} + x_0 P_{6n}}$$

$$u_{6n+2} = \frac{z_{-1} P_{6n+3} + x_0 P_{6n+2}}{z_{-1} P_{6n+2} + x_0 P_{6i-3}}$$

$$u_{6n+3} = \frac{z_{-1} P_{6n+4} + x_0 P_{6n+3}}{z_{-1} P_{6n+3} + x_0 P_{6i-4}}$$

$$u_{6n+4} = \frac{z_{-1} P_{6n+5} + x_0 P_{6n+4}}{z_{-1} P_{6n+4} + x_0 P_{6i-5}}$$

$$u_{6n+5} = \frac{z_{-1} P_{6n+6} + x_0 P_{6n+5}}{z_{-1} P_{6n+5} + x_0 P_{6i-6}}$$

Proof. Putting in place $m = \{3n, 3n+1, 3n+2\}$, the equation $u_{-1} = \frac{x_0}{z_{-1} - 2x_0}$ and Pell sequence in the equations (3.19) and (3.20), the desired equality is found.

Also the equations $\{u_{6i}, u_{6i-1}, u_{6i-2}, u_{6i-3}, u_{6i-4}, u_{6i-5}\}$ are found by taking $m = \{3i, 3i-1, 3i-2\}$, $i \in \{1, 2, 3, \dots\}$ in (3.19) in a similar way. ■

Now, we find the roots of linear equation (3.13). If we write characteristic equation of (3.13), we have $\lambda^2 - 6\lambda + 1 = 0$

Roots of the equation are

$$\lambda_1 = 3 + 2\sqrt{2} = \alpha^2,$$

$$\lambda_2 = 3 - 2\sqrt{2} = \beta^2$$

It turns out to that the roots of (3.13) are related the roots of the Pell number sequence.

Then general solution of the equation (3.13) is

$$s_m = c_1(3 + 2\sqrt{2})^m + c_2(3 - 2\sqrt{2})^m \tag{3.21}$$

where s_0, s_1 are initial values.

Hence

$$s_0 = c_1 + c_2$$

$$s_{-1} = c_1(3 + 2\sqrt{2})^{-1} + c_2(3 - 2\sqrt{2})^{-1}$$

And we have

$$c_1 = \frac{(3+2\sqrt{2})s_0 - s_{-1}}{4\sqrt{2}}, \quad c_2 = \frac{s_{-1} - (3-2\sqrt{2})s_0}{4\sqrt{2}}$$

Using the initial values s_0, s_{-1} with some calculations and written at (3.21), we get

$$s_m = \frac{(3 + 2\sqrt{2})s_0 - s_{-1}}{4\sqrt{2}} (3 + 2\sqrt{2})^m + \frac{s_{-1} - (3 - 2\sqrt{2})s_0}{4\sqrt{2}} (3 - 2\sqrt{2})^m$$

$$= \frac{s_0 \left[(3 + 2\sqrt{2})^{m+1} - (3 - 2\sqrt{2})^{m+1} \right]}{4\sqrt{2}} - \frac{s_{-1} \left[(3 + 2\sqrt{2})^m - (3 - 2\sqrt{2})^m \right]}{4\sqrt{2}}$$

Hence we have

$$s_m = \frac{s_0[(\alpha^2)^{m+1} - (\beta^2)^{m+1}]}{2(\alpha - \beta)} - \frac{s_{-1}[(\alpha^2)^m - (\beta^2)^m]}{2(\alpha - \beta)}$$

By using the Binet formula in equation (1.11),

$$s_m = \frac{s_0 P_{2m+2} - s_{-1} P_{2m}}{2} \tag{3.22}$$

When we substitute (3.22) using (3.10),

Then general solution of the equation (3.14) is

$$v_{m-1}^{(i)} = \frac{s_m}{s_{m-1}}, \quad v_{-1} = \frac{s_0}{s_{-1}}$$

$$v_{m-1}^{(i)} = \frac{s_0 P_{2m+2} - s_{-1} P_{2m}}{s_0 P_{2m} - s_{-1} P_{2m-2}} = \frac{\frac{s_0}{s_{-1}} P_{2m+2} - P_{2m}}{\frac{s_0}{s_{-1}} P_{2m} - P_{2m-2}} \quad (3.24)$$

$$= \frac{v_{-1} P_{2m+2} - P_{2m}}{v_{-1} P_{2m} - P_{2m-2}}$$

Using (3.8) and (1.9) in (3.24), we have

$$v_m^{(i)} = \frac{v_{-1} P_{2m+4} - P_{2m+2}}{v_{-1} P_{2m+2} - P_m} \quad (3.25)$$

Then by considering (3.6), (3.24) and (3.25), we have

$$v_{2m}^{(i)} = \frac{v_{-1} P_{4m+4} - P_{4m+2}}{v_{-1} P_{4m+2} - P_{4m}} \quad (3.26)$$

$$v_{2m-1}^{(i)} = \frac{v_{-1} P_{4m+2} - P_{4m}}{v_{-1} P_{4m-2} - P_{4m-4}} \quad (3.27)$$

Theorem 3.2. Let $\{u_n, v_n, w_n\}_{n \geq -2}$ be a well-defined solution to the system (3.4). Then, for v_n ,

$$v_{6n} = \frac{x_{-1} P_{12n+2} - y_0 P_{12n}}{x_{-1} P_{12n} - y_0 P_{12n-2}}$$

$$v_{6n+1} = \frac{x_{-1} P_{12n+4} - y_0 P_{12n+2}}{x_{-1} P_{12n+2} - y_0 P_{12n}}$$

$$v_{6n+2} = \frac{x_{-1} P_{12n+6} - y_0 P_{12n+4}}{x_{-1} P_{12n+4} - y_0 P_{12n+2}}$$

$$v_{6n+3} = \frac{x_{-1} P_{12n+8} - y_0 P_{12n+6}}{x_{-1} P_{12n+6} - y_0 P_{12n+4}}$$

$$v_{6n+4} = \frac{x_{-1} P_{12n+10} - y_0 P_{12n+8}}{x_{-1} P_{12n+8} - y_0 P_{12n+6}}$$

$$v_{6n+5} = \frac{x_{-1} P_{12n+12} - y_0 P_{12n+10}}{x_{-1} P_{12n+10} - y_0 P_{12n+8}}$$

Proof. Putting in place $m = \{3n, 3n+1, 3n+2\}$, the equation $v_{-1} = \frac{y_0}{6y_0 - x_{-1}}$ and Pell sequence in the equations (3.26) and (3.27), the desired equality is found.

Also, $\{v_{6i}, v_{6i-1}, v_{6i-2}, v_{6i-3}, v_{6i-4}, v_{6i-5}\}$ equations are found by giving $m = \{3i, 3i-1, 3i-2\}$, $i \in \{1, 2, 3, \dots\}$ values in (2.19) in a similar way. ■

Now, we are finding the roots of linear equation (3.14).

If we write characteristic equation of (3.14), we have

$$\lambda^2 - 14\lambda - 1 = 0$$

Roots of the equation are

$$\lambda_1 = 7 + 5\sqrt{2} = \alpha^3, \quad \lambda_2 = 7 - 5\sqrt{2} = \beta^3$$

It turns out to that the roots of (3.14) are related the roots of the Pell number sequence.

$$t_m = c_1(7 + 5\sqrt{2})^m + c_2(7 - 5\sqrt{2})^m \quad (3.28)$$

where t_0, t_1 are initial values.

$$t_0 = c_1 + c_2$$

$$t_{-1} = c_1(7 + 5\sqrt{2})^{-1} + c_2(7 - 5\sqrt{2})^{-1}$$

And we have

$$c_1 = \frac{(7+5\sqrt{2})t_0+t_{-1}}{10\sqrt{2}}, \quad c_2 = \frac{t_{-1}+(7-5\sqrt{2})t_0}{-10\sqrt{2}}$$

Using the initial values s_0, s_{-1} with some calculations and written at (3.28), we get

$$t_m = \frac{(7 + 5\sqrt{2})t_0 + t_{-1}}{10\sqrt{2}} (7 + 5\sqrt{2})^m$$

$$+ \frac{t_{-1} + (7 - 5\sqrt{2})t_0}{-10\sqrt{2}} (7 - 5\sqrt{2})^m$$

$$= \frac{t_0 [(7 + 5\sqrt{2})^{m+1} - (7 - 5\sqrt{2})^{m+1}]}{10\sqrt{2}}$$

$$+ \frac{t_{-1} [(7 + 5\sqrt{2})^m - (7 - 5\sqrt{2})^m]}{10\sqrt{2}}$$

Hence we have

$$t_m = \frac{t_0 [(\alpha^3)^{m+1} - (\beta^3)^{m+1}]}{5(\alpha - \beta)} + \frac{t_{-1} [(\alpha^3)^m - (\beta^3)^m]}{5(\alpha - \beta)}$$

By using the Binet formula in equation (1.11)

$$t_m = \frac{t_0 P_{3m+3} + t_{-1} P_{3m}}{5} \quad (3.29)$$

When we substitute (3.29) using (3.11)

$$w_{m-1}^{(i)} = \frac{t_m}{t_{m-1}}, \quad w_{-1} = \frac{t_0}{t_{-1}}$$

$$w_{m-1}^{(i)} = \frac{t_0 P_{3m+3} + t_{-1} P_{3m}}{t_0 P_{3m} + t_{-1} P_{3m-3}} = \frac{\frac{t_0}{t_{-1}} P_{3m+3} + P_{3m}}{\frac{t_0}{t_{-1}} P_{3m} + P_{3m-3}} \quad (3.30)$$

$$= \frac{w_{-1} P_{3m+3} + P_{3m}}{w_{-1} P_{3m} + P_{3m-3}}$$

Using (3.8) and (1.9) in (3.30), we have

$$w_m^{(i)} = \frac{w_{-1} P_{3m+6} + P_{3m+3}}{w_{-1} P_{3m+3} + P_{3m}} \quad (3.31)$$

Then by considering (3.7), (3.30) and (3.31), we have

$$w_{2m}^{(i)} = \frac{w_{-1} P_{6m+6} + P_{6m+3}}{w_{-1} P_{6m+3} + P_{6m}} \quad (3.33)$$

$$w_{2m-1}^{(i)} = \frac{w_{-1} P_{6m+3} + P_{6m}}{w_{-1} P_{6m} + P_{6m-3}} \quad (3.34)$$

Theorem 3.3. Let $\{u_n, v_n, w_n\}_{n \geq -2}$ be a well-defined solution to the system (3.4). Then, for w_n ,

$$w_{6i} = \frac{y_{-1}P_{18i+3} + z_0P_{18i}}{y_{-1}P_{18i} + z_0P_{18i-3}}$$

$$w_{6i-1} = \frac{y_{-1}P_{18i} + z_0P_{18i-3}}{y_{-1}P_{18i-3} + z_0P_{18i-6}}$$

$$w_{6i-2} = \frac{y_{-1}P_{18i-3} + z_0P_{18i-6}}{y_{-1}P_{18i-6} + z_0P_{18i-9}}$$

$$w_{6i-3} = \frac{y_{-1}P_{18i-6} + z_0P_{18i-9}}{y_{-1}P_{18i-9} + z_0P_{18i-12}}$$

$$w_{6i-4} = \frac{y_{-1}P_{18i-9} + z_0P_{18i-12}}{y_{-1}P_{18i-12} + z_0P_{18i-15}}$$

$$w_{6i-5} = \frac{y_{-1}P_{18i-12} + z_0P_{18i-15}}{y_{-1}P_{18i-15} + z_0P_{18i-18}}$$

Proof. Putting in place $m=\{3n, 3n+1, 3n+2\}$, the equation $w_{-1} = \frac{z_0}{y_{-1}-14z_0}$ and Pell sequence in the equations (3.33) and (3.34), the desired equality is found.

Also the equations $\{w_{6i}, w_{6i-1}, w_{6i-2}, w_{6i-3}, w_{6i-4}, w_{6i-5}\}$ are found by taking $m=\{3i, 3i-1, 3i-2\}$, $i \in \{1, 2, 3, \dots\}$ in (3.19) in a similar way. ■

Corollary 3.1. Let $\{x_n, y_n, z_n\}$ be a well-defined solution to the system (1.12).

Now we take

$$x_n = \frac{z_{n-1}}{u_n}, \quad n \in \mathbb{N}_0 \tag{3.35}$$

$$y_n = \frac{x_{n-1}}{v_n}, \quad n \in \mathbb{N}_0 \tag{3.36}$$

$$z_n = \frac{y_{n-1}}{w_n}, \quad n \in \mathbb{N}_0 \tag{3.37}$$

Using equalities (3.36) and (3.37) in formula (3.35), after some calculations we have

$$x_{6n} = \frac{x_{6n-6}}{u_{6n}w_{6n-1}v_{6n-2}u_{6n-3}w_{6n-4}v_{6n-5}} \tag{3.38}$$

Using equalities (3.35) and (3.37) in formula (3.36), we get

$$y_{6n} = \frac{y_{6n-6}}{v_{6n}u_{6n-1}w_{6n-2}v_{6n-3}u_{6n-4}w_{6n-5}} \tag{3.39}$$

Using equalities (3.36) and (3.37) in formula (3.35), we get

$$z_{6n} = \frac{z_{6n-6}}{w_{6n}v_{6n-1}u_{6n-2}w_{6n-3}v_{6n-4}u_{6n-5}} \tag{3.40}$$

Multiplying the equalities which are obtained (3.38), (3.39) and (3.40) from 1 to n, respectively, it follows that

$$\tag{3.41}$$

$$x_{6n} = \frac{x_0}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}},$$

$$y_{6n} = \frac{y_0}{\prod_{i=1}^n v_{6i}u_{6i-1}w_{6i-2}v_{6i-3}u_{6i-4}w_{6i-5}}, \tag{3.42}$$

$$z_{6n} = \frac{z_0}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}} \tag{3.43}$$

Using the equalities (3.41), (3.42) and (3.43) in (3.35), (3.36) and (3.37), we get

$$x_{6n-1} = y_{6n}v_{6n} \tag{3.44}$$

$$= \frac{y_0 \left(\frac{x_{-1}P_{12n+2} - y_0P_{12n}}{x_{-1}P_{12n} - y_0P_{12n-2}} \right)}{\prod_{i=1}^n v_{6i}u_{6i-1}w_{6i-2}v_{6i-3}u_{6i-4}w_{6i-5}}$$

$$y_{6n-1} = z_{6n}w_{6n} \tag{3.45}$$

$$= \frac{z_0 \left(\frac{y_{-1}P_{18n+3} + z_0P_{18n}}{y_{-1}P_{18n} + z_0P_{18n-3}} \right)}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}}$$

$$z_{6n-1} = x_{6n}u_{6n} \tag{3.46}$$

$$= \frac{x_0 \left(\frac{z_{-1}P_{6n+1} + x_0P_{6n}}{z_{-1}P_{6n} + x_0P_{6n-1}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}}$$

Similarly, using the equalities (3.44), (3.45) and (3.46) in (3.35), (3.36) and (3.37), we get

$$x_{6n-2} = y_{6n-1}v_{6n-1} \tag{3.47}$$

$$= \frac{z_0 \left(\frac{y_{-1}P_{18n+3} + z_0P_{18n}}{y_{-1}P_{18n} + z_0P_{18n-3}} \right) \left(\frac{x_{-1}P_{12n} - y_0P_{12n-2}}{x_{-1}P_{12n-2} - y_0P_{12n-4}} \right)}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}}$$

$$y_{6n-2} = z_{6n-1}w_{6n-1} \tag{3.48}$$

$$= \frac{x_0 \left(\frac{z_{-1}P_{6n+1} + x_0P_{6n}}{z_{-1}P_{6n} + x_0P_{6n-1}} \right) \left(\frac{y_{-1}P_{18n} + z_0P_{18n-3}}{y_{-1}P_{18n-3} + z_0P_{18n-6}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}}$$

$$z_{6n-2} = x_{6n-1}u_{6n-1} \tag{3.49}$$

$$= \frac{y_0 \left(\frac{x_{-1}P_{12n+2} - y_0P_{12n}}{x_{-1}P_{12n} - y_0P_{12n-2}} \right) \left(\frac{z_{-1}P_{6n} + x_0P_{6n-1}}{z_{-1}P_{6n-1} + x_0P_{6n-2}} \right)}{\prod_{i=1}^n v_{6i}u_{6i-1}w_{6i-2}v_{6i-3}u_{6i-4}w_{6i-5}}$$

Using the equalities (3.41), (3.42) and (3.43) in (3.35), (3.36) and (3.37), we obtain

$$x_{6n+1} = \frac{z_{6n}}{u_{6n+1}} \tag{3.50}$$

$$= \frac{z_0 \left(\frac{z_{-1}P_{6n+1} + x_0P_{6n}}{z_{-1}P_{6n+2} + x_0P_{6n+1}} \right)}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}}$$

$$y_{6n+1} = \frac{x_{6n}}{v_{6n+1}} \quad (3.51) \quad z_{6n+1} = \frac{y_{6n}}{w_{6n+1}} \quad (3.52)$$

$$= \frac{x_0 \left(\frac{x_{-1}P_{12n+2} - y_0P_{12n}}{x_{-1}P_{12n+4} - y_0P_{12n+2}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}} = \frac{y_0 \left(\frac{y_{-1}P_{18n+3} + z_0P_{18n}}{y_{-1}P_{18n+6} + z_0P_{18n+3}} \right)}{\prod_{i=1}^n v_{6i}u_{6i-1}w_{6i-2}v_{6i-3}u_{6i-4}w_{6i-5}}$$

Similarly, using the equalities (3.50), (3.51) and (3.52) in (3.35), (3.36) and (3.37), we get

$$x_{6n+2} = \frac{z_{6n+1}}{u_{6n+2}} = \frac{y_0 \left(\frac{y_{-1}P_{18n+3} + z_0P_{18n}}{y_{-1}P_{18n+6} + z_0P_{18n+3}} \right) \left(\frac{z_{-1}P_{6n+2} + x_0P_{6n+1}}{z_{-1}P_{6n+3} + x_0P_{6n+2}} \right)}{\prod_{i=1}^n v_{6i}u_{6i-1}w_{6i-2}v_{6i-3}u_{6i-4}w_{6i-5}} \quad (3.53)$$

$$y_{6n+2} = \frac{x_{6n+1}}{v_{6n+2}} = \frac{z_0 \left(\frac{z_{-1}P_{6n+1} + x_0P_{6n}}{z_{-1}P_{6n+2} + x_0P_{6n+1}} \right) \left(\frac{x_{-1}P_{12n+4} - y_0P_{12n+2}}{x_{-1}P_{12n+6} - y_0P_{12n+4}} \right)}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}} \quad (3.54)$$

$$z_{6n+2} = \frac{y_{6n+1}}{w_{6n+2}} = \frac{x_0 \left(\frac{x_{-1}P_{12n+2} - y_0P_{12n}}{x_{-1}P_{12n+4} - y_0P_{12n+2}} \right) \left(\frac{y_{-1}P_{18n+6} + z_0P_{18n+3}}{y_{-1}P_{18n+9} + z_0P_{18n+6}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}} \quad (3.55)$$

Similarly, using the equalities (3.53), (3.54) and (3.55) in (3.35), (3.36) and (3.37), we have

$$x_{6n+3} = \frac{z_{6n+2}}{u_{6n+3}} = \frac{x_0 \left(\frac{x_{-1}P_{12n+2} - y_0P_{12n}}{x_{-1}P_{12n+4} - y_0P_{12n+2}} \right) \left(\frac{y_{-1}P_{18n+6} + z_0P_{18n+3}}{y_{-1}P_{18n+9} + z_0P_{18n+6}} \right) \left(\frac{z_{-1}P_{6n+3} + x_0P_{6n+2}}{z_{-1}P_{6n+4} + x_0P_{6n+3}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}} \quad (3.56)$$

$$y_{6n+3} = \frac{z_{6n+2}}{u_{6n+3}} = \frac{y_0 \left(\frac{x_{-1}P_{12n+6} - y_0P_{12n+4}}{x_{-1}P_{12n+8} - y_0P_{12n+6}} \right) \left(\frac{y_{-1}P_{18n+3} + z_0P_{18n}}{y_{-1}P_{18n+6} + z_0P_{18n+3}} \right) \left(\frac{z_{-1}P_{6n+2} + x_0P_{6n+1}}{z_{-1}P_{6n+3} + x_0P_{6n+2}} \right)}{\prod_{i=1}^n u_{6i}w_{6i-1}v_{6i-2}u_{6i-3}w_{6i-4}v_{6i-5}} \quad (3.57)$$

$$z_{6n+3} = \frac{y_{6n+2}}{w_{6n+3}} = \frac{z_0 \left(\frac{z_{-1}P_{6n+1} + x_0P_{6n}}{z_{-1}P_{6n+2} + x_0P_{6n+1}} \right) \left(\frac{x_{-1}P_{12n+4} - y_0P_{12n+2}}{x_{-1}P_{12n+6} - y_0P_{12n+4}} \right) \left(\frac{y_{-1}P_{18n+9} + z_0P_{18n+6}}{y_{-1}P_{18n+12} + z_0P_{18n+9}} \right)}{\prod_{i=1}^n w_{6i}v_{6i-1}u_{6i-2}w_{6i-3}v_{6i-4}u_{6i-5}} \quad (3.58)$$

Thus, by examining the solutions of difference equation systems in the literature, it was investigated the relationships integer sequences. In this study, the closed-form solutions of three-dimensional difference equation systems were found with an analytical approach and the relations between these solutions with pell number sequences were investigated.

4. Conclusions and Recommendations

In this work, we have successfully established in a constructive way the closed-form solution of system of rational difference equation

$$x_n = \frac{z_{n-1}z_{n-3}}{x_{n-2} + 2z_{n-3}},$$

$$y_n = \frac{x_{n-1}x_{n-3}}{-y_{n-2} + 6x_{n-3}},$$

$$z_n = \frac{y_{n-1}y_{n-3}}{z_{n-2} + 14y_{n-3}},$$

Where $n \in N_0$ and initial values $x_{-3}, x_{-2}, x_{-1}, y_{-3}, y_{-2}, y_{-1}, z_{-3}, z_{-2}, z_{-1}$ are nonzero real numbers. We correlated their solutions with pell numbers. We write the e-ISSN: 2148-2683

solutions in terms of Pell numbers with the help for Pell’s Binet formula.

The results in this article can be extended to a similar system of difference equations with the help of the Binet formula. It can be generalized to be expressed by number sequences such as Pell-Lucas, Jacopsthal, Fibonacci, Horadam number sequences.

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