Eurasian Journal of Science Engineering and Technology

Research

FORWARD & INVERSE KINEMATICS SOLUTION OF 6-DOF ROBOTS THOSE HAVE OFFSET & SPHERICAL WRISTS

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ABSTRACT

One of the critical design decisions that arise during the design of an industrial robot is the function of the joints to be used and their location. Of course, for the designed robot to provide the expected performance, the selection of the motors and gear reducers that will create these joints will primarily affect the determination of these joints. However, both the examination of the existing industrial robots and the fact that the motor and gear reducer information that can be supplied can be easily obtained with today's technology, these joints can be determined quickly as a result of a short investigation. Along with the mechanical design, robot control unit design also has stages that progress in parallel and depend on or affect the mechanical design decisions. One of the most important of these is that especially inverse kinematics calculations can be performed analytically, enabling the robot control unit to make decisions and give commands in real-time.

This article aims to publish the geometric calculation of inverse kinematics of commonly used exampled configuration of 6-DOF industrial robot that has offset and spherical joint along with interactive calculation tables and sheets. There are numerous articles published on the same subject but none of them has provided any verification supplement so far. Thus, it is also aimed to confirm and verify the calculations given in this article by providing convenient tables and sheets and to create a solid foundation for future studies.

Keywords: Inverse kinematics, Forward kinematics, 6-DOF Industrial robot, Spherical wrist, Offset wrist, Analytical solution, Geometric solution.

1. INTRODUCTION

In simple words, kinematics is the study of motion while ignoring the causes of that. Any component of that motion is considered to be rigid. Connections among those components are called joints. There are many joint types but in the end, they all can be expressed with two base joint types; prismatic (sliding) and rotating. Each base joint in the kinematic body defines the DOF[1] of that mechanism. For industrial robots, usually, a rotating joint type is used and if the robot has 6 DOF, it generally means it has 6 rotating joints.

Forward kinematics is to compute the position of end-effector[2] by using specified joint parameters (for rotating joint, it is rotation angle). The solution of any DOF robotic manipulator has to be done in advance and is straightforward theoretical calculation [1], [2], [3], [4], [5], [6], [7], [8], [9]. A method called Denavit Hartenberg[3] convention widely used solution for both Forward & Inverse solutions.

Inverse kinematics is exactly the opposite of forward kinematics, which is to compute joint angles by using a specified end-effector position. The solution is also diverse with numerous approaches. Some most well know theoretical ones [3], [6] are:

Algebraic Solution: Forward kinematics calculations like approach. It has several advantages, such as a) being the most robust solution for real-time calculations, b) it is a straightforward math calculation scheme that may not need DH convention, and c) it is feasible for 3 DOF robot kinematics. On the other hand, it requires an algebraic calculation of the inverse transformation matrix. It introduces the following disadvantages; a) not feasible for almost all 3+ DOF real-life robotics kinematics, b) the solution gets much more complicated when offsets exist and DOFs are increased.

Geometric Solution: Dividing mechanism into several plane geometry problems. When compared with Algebraic, this method also requires extended trigonometric knowledge besides conventional math. Advantages of the solution can be; a) as with algebraic, the solution does not strictly depend on robot structure. b) As in forward kinematics solution, DH parameters representation is also the de-facto standard for the geometric solution. c) the solution is recognized as suitable for real-time

[1] Degrees of Freedom

[2] Located at the end of the robotic arm, to interact with the environment

[3] Abbreviated as D-H or DH method and expressed with DH table

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Recieved: 04.03.2022 Accepted: 05.05.2022
doi: 10.55696/ejset.1082648

calculations. Disadvantages might be; a) avoidance might be possible but, almost all solutions include singularities, b) nonlinearities (multiple solutions) exist due to the use of local coordinates.

Quaternion Solution: This is the newest solution approach that uses an alternative artificial imaginary complex space called Quaternion. Advantages are a) introducing compact formulation; b) reduced number of equations, c) gimbal lock and other singularities are avoided. Drawbacks might be; a) rotations only mechanisms can be handled. b) needs more math to comprehend; c) difficult to interpret and d) less intuitive.

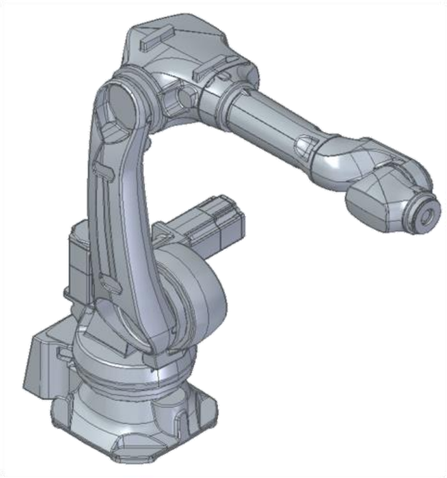


Figure 1. An example robot CAD model

Since our kinematic mechanism includes rotations for each DOF, the quaternion solution seems to be the best fit, but for intuition and easy interpretation, the geometric solution approach will be exercised in this paper. In reality, exampled robot joints are in 3D space, but for the sake of solution simplicity, they all are projected onto a single plane. Shifts between joints are called offsets and the exampled robot has two offsets; the first one is between 1st & 2nd joints and the second one is between 3rd & 4th joints [10]. The 4th, 5th & 6th joints altogether forms spherical wrist [1]. 6-DOF exampled robot will be divided into two 3-DOF mechanisms; the first one compromises 1st, 2nd, and 3rd joints, and the second one compromises spherical wrist joints.

Due to the nature of the inverse kinematics solution, each joint angle could have multiple values. This nature is well documented and investigated in many literature [11], [12]. This article and its supplements will only reveal just the single solution which covers the widest range.

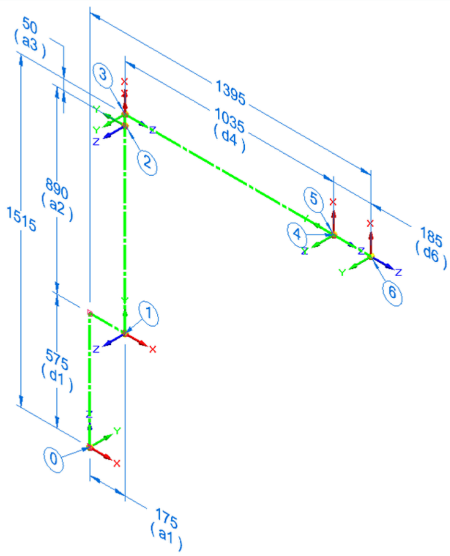


Figure 2. Schematic of robot shown at Figure 1

2. EXAMPLED ROBOT AND ITS SCHEMATIC REPRESENTATION

A 6-DOF industrial robot is chosen. Below find its CAD model (Figure 1), and schematics (Figure 2&3) to be used to define its home position. Initial solution parameters & variables will be extracted from these schematics as well.

3. ROBOT VARIABLES, PARAMETERS & DENAVIT HARTENBERG TABLE

A very commonly used 6-DOF robot has the following variables, parameters, parameters and corresponding Denavit-Hartenberg (DH) table. In this section, these known and unknown will be exposed using the schematics in previous section 2.

3.1 Variables

This 6-DOF robot manipulator has six links and all of them are revolute joints. By changing the angle of these revolute links, a robot can reach and fulfill its function as expected. Therefore, variables of the kinematics system of the introduced robot manipulator are these revolute joint angles and their starting value accepted as 0 (zero) at home position, and it should be bigger than -180° and less than or equal to 180°. In this article and its supplements;

— θ denotes joint angle,
— Subscript n (where n = 1...6) denotes revolute joints,
— A number (1...6) subscript denotes a revolute joint number which is starts from the robot base and goes to the tip,
— Subscript v denotes variable,
— Subscript h denotes home offset joint angle at home position,
— Subscript f denotes joint angle for forward kinematics calculation which includes both variable and home offset part,
— Subscript i denotes joint angle for inverse kinematics calculation which includes both variable and home offset part.

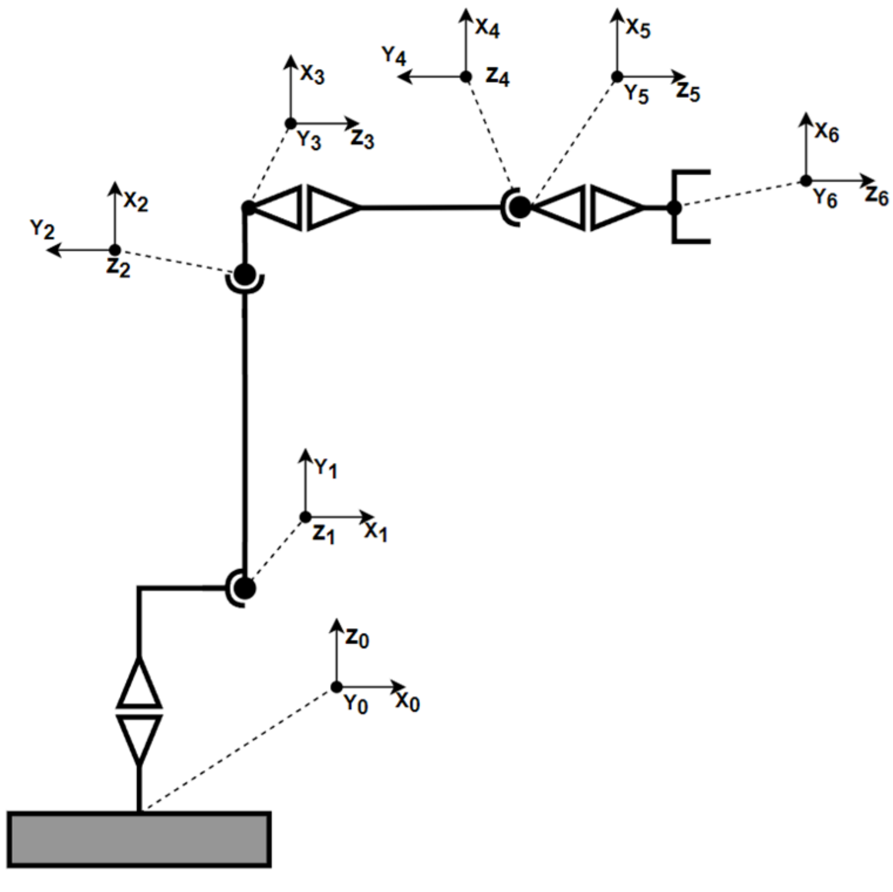


Figure 3: 2D Representation of schematics on Figure 2

3.2 Parameters & Denavit-Hartenberg Table

For the specified home position in schematic Figure 2&3, joint angle offset (θ), twist angle (α), joint offset (d), and joint length (a) should be defining column parameters of the Denavit-Hartenberg (DH) table. Each row of the DH table defines mentioned parameters of each joint. Therefore, we have 6 (row) joints by 4 (column) parameters table. Definition of each parameter is;

θ: Angle about the previous Z, from old X to new X.

α: Angle about new X, from old Z-axis to new Z-axis.

d: Offset distance along the previous Z, from old joint to new joint center.

a: Length along new X, from old joint to new joint center.

In addition to that while placing the local coordinates in Figure 2 following rules apply;

- Z-axis is always the joint axis on which the joint rotates about if it is a rotational joint or moves along if it is a translational joint.
- X-axis must be perpendicular and intersect both new Z & old Z
- Y-axis's placement follows the right-hand rule based on the X & Z axes

By using the above transformation parameters definitions, constraints between axes and the schematic representation of the robot manipulator shown in Figure 2&3, D-H parameters shown in Table 1 can be derived.

The most crucial steps for constructing the D-H table are the placement and orientation of local coordinates so that the complexity of our solution is minimized.

Table 1: D-H Parameters of robot manipulator that's schematics shown in Figure 2&3

	$\theta°_h$	**α°**	**d(mm)**	**a(mm)**
1st Joint	0	90	575	175
2nd Joint	90	0	0	890
3rd Joint	0	90	0	50
4th Joint	0	-90	1035	0
5th Joint	0	90	0	0
6th Joint	0	0	185	0

When variables and parameters merged, the θ, α, d and a values for each joint would be as follows;

$$
\begin{array}{llll}
\theta_{1f} = \theta_{1v} + \theta°_{1h} & \alpha_{1f} = 90° & d_{1f} = 575 & a_{1f} = 175 \\
\theta_{2f} = \theta_{2v} + \theta°_{2h} & \alpha_{2f} = 0° & d_{2f} = 0 & a_{2f} = 890 \\
\theta_{3f} = \theta_{3v} + \theta°_{3h} & \alpha_{3f} = 90° & d_{3f} = 0 & a_{3f} = 50 \\
\theta_{4f} = \theta_{4v} + \theta°_{4h} & \alpha_{4f} = -90° & d_{4f} = 1035 & a_{4f} = 0 \\
\theta_{5f} = \theta_{5v} + \theta°_{5h} & \alpha_{5f} = 90° & d_{5f} = 0 & a_{5f} = 0 \\
\theta_{6f} = \theta_{6v} + \theta°_{6h} & \alpha_{6f} = 0° & d_{6f} = 185 & a_{6f} = 0
\end{array}
$$

In a nutshell, for forward kinematics calculation, the rotation angle at each joint is a variable. There are also 4 D-H parameters for each joint. Therefore, for the 6 DOF mechanism robot, there will be 6 variables and 6x4=24 variables to calculate the end-effector's 3 (x,y,z) cartesian location and 3 angular orientation. For forward kinematics calculation, these six total unknowns always stay constant, whether the mechanism has 2, 3 or 10 joints.

An increased number of DOF on robotics kinematics has almost no effect on forward kinematics, but this cannot be said for inverse kinematics calculation. Depending on the number of additional DOF and each joint's formation, the chosen method may not be feasible.

4. FORWARD KINEMATICS CALCULATION

In 1955, Denavit and Hartenberg proposed a matrix method to construct the coordinate system connected to each link in the robot's joint chains to describe the translational and rotational relationship between adjacent links. This robot kinematics model is based on the D-H coordination system. Transformations between two consecutive joints can be written by substituting the parameters in the parameter table in their corresponding place in the matrix called "A_n", where;

$$A_{nf} = \begin{bmatrix} \cos\theta_{nf} & -\sin\theta_{nf}\cos\alpha_{nf} & \sin\theta_{nf}\sin\alpha_{nf} & a_{nf}\cos\theta_{nf} \\ \sin\theta_{nf} & \cos\theta_{nf}\cos\alpha_{nf} & \cos\theta_{nf}\sin\alpha_{nf} & a_{nf}\sin\theta_{nf} \\ 0 & \sin\alpha_{nf} & \cos\alpha_{nf} & d_{nf} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When known joint parameters are put into their corresponding place in this matrix, then the following matrices are obtained for each joint starting from the base (1) to the tip (6);

$$A_{1f} = \begin{bmatrix} \cos\theta_{1f} & 0 & \sin\theta_{1f} & a_{1f}\cos\theta_{1f} \\ \sin\theta_{2f} & 0 & -\cos\theta_{1f} & a_{1f}\sin\theta_{1f} \\ 0 & 1 & 0 & d_{1f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{2f} = \begin{bmatrix} \cos\theta_{2f} & -\sin\theta_{2f} & 0 & a_{2f}\cos\theta_{2f} \\ \sin\theta_{2f} & \cos\theta_{2f} & 0 & a_{2f}\sin\theta_{2f} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{3f} = \begin{bmatrix} \cos\theta_{3f} & 0 & \sin\theta_{3f} & a_{3f}\cos\theta_{3f} \\ \sin\theta_{3f} & 0 & -\cos\theta_{3f} & a_{3f}\sin\theta_{3f} \\ 0 & 1 & 0 & d_{3f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{4f} = \begin{bmatrix} \cos\theta_{4f} & 0 & -\sin\theta_{4f} & 0 \\ \sin\theta_{4f} & 0 & \cos\theta_{4f} & 0 \\ 0 & -1 & 0 & d_{4f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{5f} = \begin{bmatrix} \cos\theta_{5f} & 0 & \sin\theta_{5f} & 0 \\ \sin\theta_{5f} & 0 & -\cos\theta_{5f} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{6f} = \begin{bmatrix} \cos\theta_{6f} & -\sin\theta_{6f} & 0 & 0 \\ \sin\theta_{6f} & \cos\theta_{6f} & 0 & 0 \\ 0 & 0 & 1 & d_{6f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For abbreviation and simplicity following notation substitutions will be used throughout this article;

$$C_n = \cos\theta_n \qquad S_n = \sin\theta_n$$

$$C_{ab} = C_a C_b - S_a S_b \qquad S_{ab} = C_a S_b + S_a C_b$$

The total transformation matrix from the robot base to the hand is as follows;

$$T_f = A_{1f}A_{2f}A_{3f}A_{4f}A_{5f}A_{6f} = \begin{bmatrix} n_{xf} & o_{xf} & a_{xf} & p_{xf} \\ n_{yf} & o_{yf} & a_{yf} & p_{yf} \\ n_{zf} & o_{zf} & a_{zf} & p_{zf} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In which;

$$n_{xf} = C_{6f}\left(C_{5f}\left(C_{1f}C_{23f}C_{4f} + S_{1f}S_{4f}\right) - C_{1f}S_{23f}S_{5f}\right) + S_{6f}(S_{1f}C_{4f} - C_{1f}C_{23f}S_{4f})$$

$$n_{yf} = C_{6f}\left(C_{5f}\left(S_{1f}C_{23f}C_{4f} + C_{1f}S_{4f}\right) - S_{1f}S_{23f}S_{5f}\right) - S_{6f}(C_{1f}C_{4f} + S_{1f}C_{23f}S_{4f})$$

$$n_{zf} = C_{6f}\left(C_{23f}S_{5f} + S_{23f}C_{4f}C_{5f}\right) - S_{23f}S_{4f}S_{6f}$$

$$o_{xf} = S_{6f}\left(C_{1f}S_{23f}S_{5f} - C_{5f}\left(S_{1f}C_{23f}C_{4f} + S_{1f}S_{4f}\right)\right) + C_{6f}(S_{1f}C_{4f} - C_{1f}C_{23f}S_{4f})$$

$$o_{yf} = S_{6f}\left(S_{1f}S_{23f}S_{5f} - C_{5f}\left(S_{1f}C_{23f}C_{4f} + C_{1f}S_{4f}\right)\right) - C_{6f}(C_{1f}C_{4f} + S_{1f}C_{23f}S_{4f})$$

$$o_{zf} = -S_{6f}\left(C_{23f}S_{5f} + S_{23f}C_{4f}C_{5f}\right) - S_{23f}S_{4f}C_{6f}$$

$$a_{xf} = S_{5f}\left(C_{1f}C_{23f}C_{4f} + S_{1f}S_{4f}\right) + C_{1f}S_{23f}C_{5f}$$

$$a_{yf} = S_{5f}\left(S_{1f}C_{23f}C_{4f} - C_{1f}S_{4f}\right) + S_{1f}S_{23f}C_{5f}$$

$$a_{zf} = S_{23f}C_{4f}S_{5f} - C_{23f}C_{5f}$$

$$p_{xf} = d_{6f}(S_{5f}\left(C_{1f}C_{23f}C_{4f} + S_{1f}S_{4f}\right) + C_{1f}S_{23f}C_{5f}) + C_{1f}(a_{1f} + a_{2f}C_{2f} + a_{3f}C_{23f} + d_{4f}S_{23f})$$

$$p_{yf} = d_{6f}(S_{5f}\left(S_{1f}C_{23f}C_{4f} + C_{1f}S_{4f}\right) + S_{1f}S_{23f}C_{5f}) + S_{1f}(a_{1f} + a_{2f}C_{2f} + a_{3f}C_{23f} + d_{4f}S_{23f})$$

$$p_{zf} = a_{2f}S_{2f} + d_{1f} + a_{3f}S_{23f} - d_{4f}C_{23f} + d_{6f}(S_{23f}C_{4f}S_{5f} - C_{23f}C_{5f})$$

The lowest row in the resulting 4x4 transformation matrix is considered as the ineffective row. The top left 3x3 matrix is the rotation matrix, and the 3x1 matrix from top to bottom in the far right column is the translation matrix.

Accordingly, the rotation matrix from the base coordinate system of the robot to the tip of the robot;

$$T_{Rf} = \begin{bmatrix} n_{xf} & o_{xf} & a_{xf} \\ n_{yf} & o_{yf} & a_{yf} \\ n_{zf} & o_{zf} & a_{zf} \end{bmatrix}$$

Where;

$[n_{xf}\ n_{yf}\ n_{zf}]$: the unit vector indicates the direction of the X-axis at the robot end tip to the base coordinate system.

$[o_{xf}\ o_{yf}\ o_{zf}]$: the unit vector indicates the direction of the Y-axis at the robot end tip to the base coordinate system.

$[a_{xf}\ a_{yf}\ a_{zf}]$: the unit vector indicates the direction of the Z-axis at the robot end tip to the base coordinate system.

The unit vector shows the direction of the X, Y and/or Z-axis at the robot tip according to the base coordinate system. However, the expectation is to express all of these in angular form rather than vectorial. Below find how to express tip rotation in *ZY′Z″* Euler and *XY′Z″* Tait-Bryan angles. The matrices used to calculate these angles are available from [13].

ZY′Z″ Euler angles of the robot tip coordinate axis to the base coordinate axis (subscript *e* denotes Euler);

$$Z_{ef} = \arctan2\left(\frac{a_{yf}}{a_{xf}}\right)$$

$$Y'_{ef} = \arctan2\left(\frac{\sqrt{1-a_{zf}^2}}{a_{zf}}\right)$$

$$Z''_{ef} = \arctan2\left(\frac{o_{zf}}{-n_{zf}}\right)$$

XY′Z″ Tait-Bryan angles of the robot tip coordinate axis with respect to the base coordinate axis (subscript *t* denotes Tait-Bryan);

$$X_{tf} = \arctan2\left(\frac{-a_{yf}}{a_{zf}}\right)$$

$$Y'_{tf} = \arctan2\left(\frac{a_{xf}}{\sqrt{1-a_{xf}^2}}\right)$$

$$Z''_{ef} = \arctan2\left(\frac{-o_{xf}}{n_{xf}}\right)$$

The translation matrix from the base coordinate system of the robot to the tip of the robot;

$$T_{Tf} = \begin{bmatrix} p_{xf} \\ p_{yf} \\ p_{zf} \end{bmatrix}$$

The transformation matrix that both includes translation and *ZY′Z″* euler angles can be represented as;

$$T_{ef} = \begin{bmatrix} p_{xf} \\ p_{yf} \\ p_{zf} \\ Z_{ef} \\ Y'_{ef} \\ Z''_{ef} \end{bmatrix}$$

This concludes the forward kinematics calculation. In proceeding inverse kinematics calculations above derived T_{ef} matrix will be taken as input and θ_n joint rotations will be derived.

5. INVERSE KINEMATICS CALCULATION

In the previous session, we have found the end-effector location in cartesian space by inputting joint rotation angles. In real-world robots operates exactly the opposite; the robot controller needs to know the joint angles for a given end-effector's location. This is done by inverse kinematics calculation which will be explained in this session. Input variables of inverse calculation will be the T_{ef} matrix' cell values derived in the previous session, but all subscripts will be changed from *f* to *i* and therefore our input matrix will be;

$$T_{ei} = \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{zi} \\ Z_{ei} \\ Y'_{ei} \\ Z"_{ei} \end{bmatrix}$$

Rotational angles of each joint will be calculated by using the values in matrix T_{ei}, To do so transformation matrix will be constructed from T_{ei} by going reverse.

The translation part of the transformation matrix is evident; [p_{xi} p_{yi} p_{zi}] vertically forms the top of the rightmost column of the transformation matrix. In order to form the rotation matrix, it will be sufficient to substitute the rotation values given as input in the relevant euler rotation matrix. In reference [13] corresponding rotation matrix calculation for *ZY'Z''* euler angles is already stated. Following substitutions, notations and abbreviations are used to form the rotation matrix from euler angles. As an addition only *i* index is added as a subscript to indicate the inverse calculation;

$$\begin{array}{ccc} \alpha_i = Z_{ei} & \beta_i = Y'_{ei} & \gamma_i = Z"_{ei} \\ c_{1i} = \cos(\alpha_i) & c_{2i} = \cos(\beta_i) & c_{3i} = \cos(\gamma_i) \\ s_{1i} = \sin(\alpha_i) & s_{2i} = \sin(\beta_i) & s_{3i} = \sin(\gamma_i) \end{array}$$

Using these abbreviations on the rotation part (3x3 one at upper left) in the transformation matrix;

$$T_i = \begin{bmatrix} c_{1i}c_{2i}c_{3i} - s_{1i}s_{3i} & -c_{3i}s_{1i} - c_{1i}c_{2i}s_{3i} & c_{1i}s_{2i} & p_{xi} \\ c_{1i}s_{3i} - c_{2i}c_{3i}s_{1i} & c_{1i}c_{3i} - c_{2i}s_{1i}s_{3i} & s_{1i}s_{2i} & p_{yi} \\ -c_{3i}s_{2i} & s_{2i}s_{3i} & c_{2i} & p_{zi} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It was also shown in forward kinematics calculation, the abbreviated notation of this transformation matrix was as follows;

$$T_f = \begin{bmatrix} n_{xi} & o_{xi} & a_{xi} & p_{xi} \\ n_{yi} & o_{yi} & a_{yi} & p_{yi} \\ n_{zi} & o_{zi} & a_{zi} & p_{zi} \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_{1i}A_{2i}A_{3i}A_{4i}A_{5i}A_{6i}$$

In the matrix, the translation values are already determined, rotational values are as follows;

$$\begin{array}{lll} n_{xi} = c_{1i}c_{2i}c_{3i} - s_{1i}s_{3i} & o_{xi} = -c_{3i}s_{1i} - c_{1i}c_{2i}s_{3i} & a_{xi} = c_{1i}s_{2i} \\ n_{yi} = c_{1i}s_{3i} - c_{2i}c_{3i}s_{1i} & o_{yi} = c_{1i}c_{3i} - c_{2i}s_{1i}s_{3i} & a_{yi} = s_{1i}s_{2i} \\ n_{zi} = -c_{3i}s_{2i} & o_{zi} = s_{2i}s_{3i} & a_{zi} = c_{2i} \end{array}$$

The schematic representation of the robot manipulator with dimensions is shown in **Hata! Başvuru kaynağı bulunamadı.**. The process of finding the angles of each joint of the robot manipulator can be divided into two steps as;

1) find the first three joint angles and then,
2) find the next three spherical joint angles by making use of the first three angles.

In **Hata! Başvuru kaynağı bulunamadı.**, a geometric explanation brought to finding the first joint angle θ_{1i}.

$$P_{04} = P_{06} - P_{46} \ where, P_{06} = \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{zi} \end{bmatrix}, \ P_{46} = d_{6i} \begin{bmatrix} a_{xi} \\ a_{yi} \\ a_{zi} \end{bmatrix}$$

$$P_{04} = \begin{bmatrix} x_{05} \\ y_{05} \\ z_{05} \end{bmatrix} = \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{zi} \end{bmatrix} - d_{6i} \begin{bmatrix} a_{xi} \\ a_{yi} \\ a_{zi} \end{bmatrix} = \begin{bmatrix} p_{xi} - d_{6i}a_{xi} \\ p_{yi} - d_{6i}a_{yi} \\ p_{zi} - d_{6i}a_{zi} \end{bmatrix}$$

Thus;

$$\theta_{1i} = \theta_{1v} = \text{arctan2}\left(\frac{p_{yi}-d_{6i}a_{yi}}{p_{xi}-d_{6i}a_{xi}}\right) \tag{1}$$

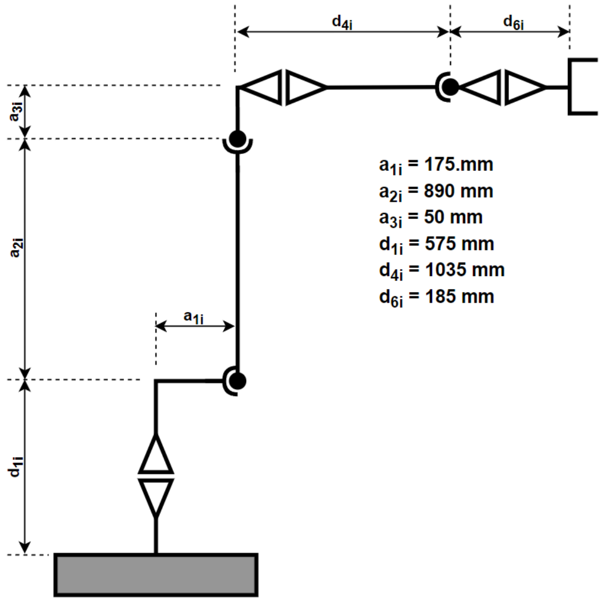


Figure 4: 2D Representation of robot manipulator with key dimensions

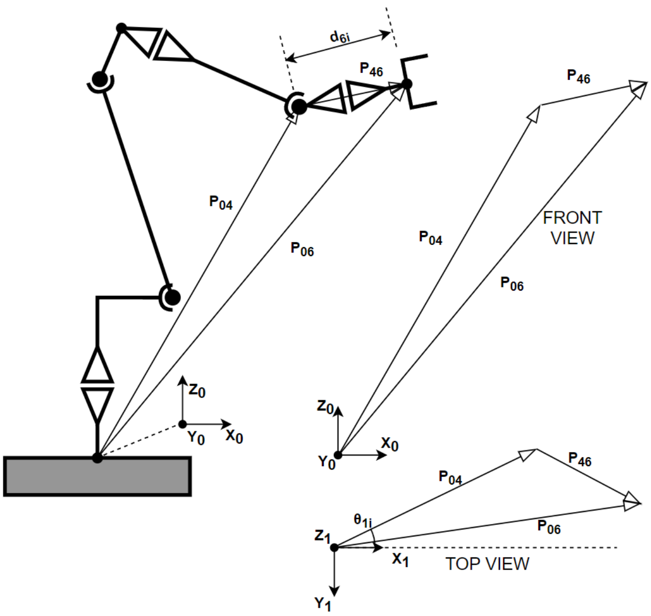


Figure 5: Graphical expression to derive θ_{1i}.

Again using the geometric solution shown in Figure 1 and cosine law (Figure 2), θ_{3i} can be derived as follows;

$$P_{01} = \begin{bmatrix} x_{01} \\ y_{01} \\ z_{01} \end{bmatrix} = \begin{bmatrix} a_{1i}C_{1i} \\ a_{1i}S_{1i} \\ d_{1i} \end{bmatrix}, P_{04} = \begin{bmatrix} x_{04} \\ y_{04} \\ z_{04} \end{bmatrix} = \begin{bmatrix} p_{xi} - d_{6i}a_{xi} \\ p_{yi} - d_{6i}a_{yi} \\ p_{zi} - d_{6i}a_{zi} \end{bmatrix}$$

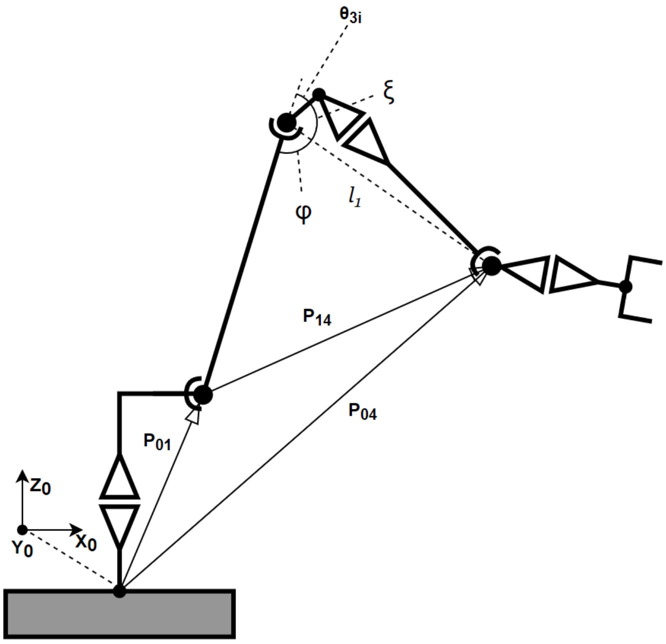
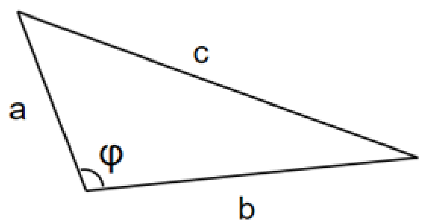


Figure 1: Graphical expression to derive θ_{3i}



$$\cos(\varphi) = \frac{a^2 + b^2 - c^2}{2 . a . b}$$

Figure 2: Law of cosine

Thus;

$$P_{14} = \begin{bmatrix} x_{14} \\ y_{14} \\ z_{14} \end{bmatrix} = P_{04} - P_{01} = \begin{bmatrix} p_{xi} - d_{6i}a_{xi} - a_{1i}C_{1i} \\ p_{yi} - d_{6i}a_{yi} - a_{1i}S_{1i} \\ p_{zi} - d_{6i}a_{zi} - d_{1i} \end{bmatrix}$$

and,

$$P_{14L} = \sqrt{|\, P_{14}^T \cdot P_{14} \,|}, \qquad l_1 = \sqrt{a_{3i}^2 + d_{4i}^2}$$

Where *L* subscript represents the length of the corresponding vector. Applying cosine law and well known trigonometric formula;

$$\varphi = \arccos\left(\frac{l_1^2 + a_{2i}^2 - P_{14L}}{2 l_1 a_2}\right), \qquad \zeta = \operatorname{arctan2}\left(\frac{d_{4i}}{a_{3i}}\right)$$

Therefore;

$$\theta_{3i} = \theta_{3v} = \varphi - \zeta - \pi \tag{2}$$

Once again using the geometric solution described in Figure 3, cosine law (Figure 2), and well know trigonometric formula θ_{2i} can be derived as;

$$P_{14} = \begin{bmatrix} x_{14} \\ y_{14} \\ z_{14} \end{bmatrix} = \begin{bmatrix} p_{xi} - d_{6i} a_{xi} - a_{1i} C_{1i} \\ p_{yi} - d_{6i} a_{yi} - a_{1i} S_{1i} \\ p_{zi} - d_{6i} a_{zi} - d_{1i} \end{bmatrix}$$

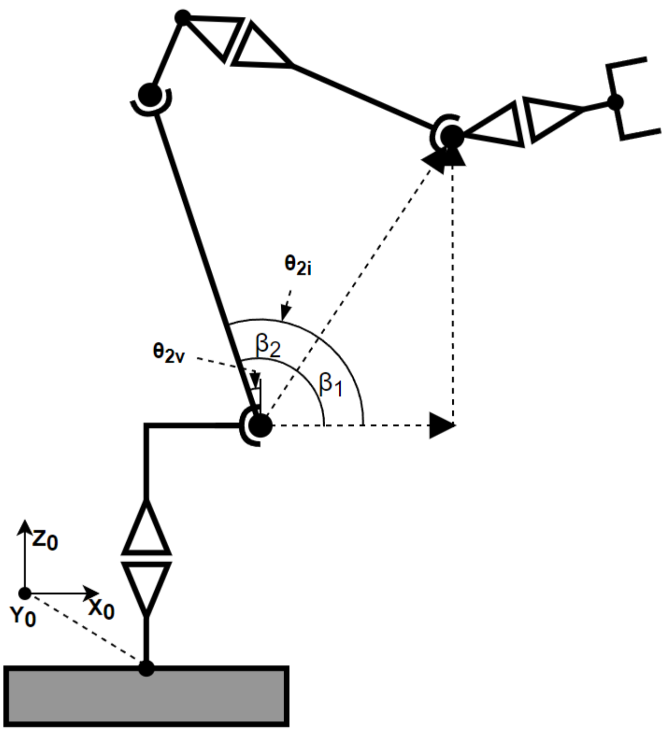


Figure 3: Graphical expression to derive θ_{2i}

From the above equation, β_1 and β_2 can be found as;

$$\beta_1 = \arctan2\left(\frac{z_{14}}{\sqrt{x_{14}^2+y_{14}^2}}\right)$$

$$\beta_2 = \arccos\left(\frac{a_{2i}^2-P_{14L}^2-l_1^2}{2a_2P_{14L}}\right)$$

From geometry;

$$\theta_{2i} = \theta_{2v} + \frac{\pi}{2} = \beta_1 + \beta_2 \tag{3}$$

The first phase of the inverse calculation is finalized by finding the first three joint rotational angles. The second and the last phase to find the rest of the three joints that make up the spherical wrist will be solved geometrically.

To calculate θ_{5i} we will assume $\theta_{4i} = 0$, or better, the absence of the 4th joint. Then the angle between the rotation vectors (R_z) will give us θ_{5i}.
Before doing that some equality declaration has to be stated to be the base for further calculations;

$$A_{1i} = \begin{bmatrix} C_{1i} & 0 & S_{1i} & a_{1i}C_{1i} \\ S_{1i} & 0 & -C_{1i} & a_{1i}S_{1i} \\ 0 & 1 & 0 & d_{1i} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{2i} = \begin{bmatrix} C_{2i} & -S_{2i} & 0 & a_{2i}C_{2i} \\ S_{2i} & C_{2i} & 0 & a_{2f}S_{2i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{3i} = \begin{bmatrix} C_{3i} & 0 & S_{3i} & a_{3i}C_{3i} \\ S_{3i} & 0 & -C_{3i} & a_{3i}S_{3i} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The following is also known;

$$C_{12} = C_1C_2 - S_1S_2 \qquad S_{12} = C_1S_2 + S_1C_2$$

For the sake of finding the 5th joint angle, it is assumed that the 4th joint is absent. 1st to 3rd joint transformation matrix will be also considered to be 1st to 4th joint transformation matrix. Since all first three joint angles know, A_{13} and consequently A_{14} can be derived;

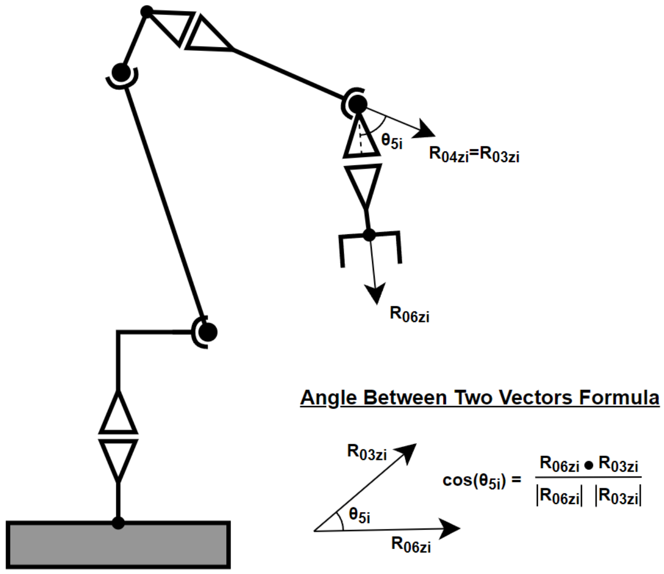


Figure 4: Graphical expression to derive θ_{5i}

$$A_{14i} = A_{13i} = A_{1i} \cdot A_{2i} \cdot A_{3i} = \begin{bmatrix} C_{1i}C_{23i} & S_{1i} & C_{1i}S_{23i} & C_{1i}(a_{1i} + a_{2i}C_{2i} + a_{3i}C_{23i}) \\ S_{1i}C_{23i} & -C_{1i} & S_{1i}S_{23i} & S_{1i}(a_{1i} + a_{2i}C_{2i} + a_{3i}C_{23i}) \\ S_{23i} & 0 & -C_{23i} & a_{2i}S_{2i} + d_{1i} + a_{3i}S_{23i} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From input values, the rotation vector about *z* (R_{6zi}) is known. From the above-mentioned assumption that is R_{4zi} is equal to R_{3zi}, R_{4zi} is known. Then the angle between two vectors formula as shown on Figure 4 will be applied to find the θ_{5i} unknown.

$$R_{6zi} = \begin{bmatrix} a_{xi} \\ a_{yi} \\ a_{zi} \end{bmatrix}, \qquad R_{3zi} = R_{4zi} = \begin{bmatrix} C_{1i}S_{23i} \\ S_{1i}S_{23i} \\ -C_{23i} \end{bmatrix}$$

Since both R_{6zi} and R_{4zi} are unit vectors, matrix multiplication will give the cosine of the angle between two vectors;

$$\theta_{5i} = \theta_{5v} = \arccos\left(\overrightarrow{R_{6zi}} \cdot \overrightarrow{R_{3zi}}\right) \tag{4}$$

4th and 6th joints are only the joint angles left which are unknown to us.

Since both A_{13i} & A_{16i} are known, we can derive A_{46i} as follows;

$$A_{46i} = A_{13i}^{-1} \cdot A_{16i} = \begin{bmatrix} I_x & J_x & K_x & L_x \\ I_y & J_y & K_y & L_y \\ I_z & J_z & K_z & L_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where all *I*, *J*, *K*, and *L* are calculated and known values.

On the other hand, the A_{46i} matrix can also be derived symbolically by multiplying the following matrices;

$$A_{46i} = A_{4i} \cdot A_{5i} \cdot A_{6i} = \begin{bmatrix} E_x & F_x & G_x & H_x \\ E_y & F_y & G_y & H_y \\ E_z & F_z & G_z & H_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where;

$$E_x = C_{4i}C_{5i}C_{6i} - S_{4i}S_{6i} \quad F_x = -(C_{4i}C_{5i}S_{6i} + S_{4i}C_{6i}) \quad G_x = C_{4i}S_{5i} \quad H_x = C_{4i}S_{5i}d_{6i}$$
$$E_y = S_{4i}C_{5i}C_{6i} + C_{4i}S_{6i} \quad F_y = -S_{4i}C_{5i}S_{6i} + C_{4i}C_{6i} \quad G_y = S_{4i}S_{5i} \quad H_y = S_{4i}S_{5i}d_{6i}$$
$$E_z = -S_{5i}C_{6i} \quad F_z = S_{5i}S_{6i} \quad G_z = C_{5i} \quad H_z = C_{5i}d_{6i} + d_{4i}$$

From the previous symbolic matrix, it can be easily noticed that the inverse tangent (*arctan2*) of matrix elements 2.3 (H_y) over 1.3 (H_x) gives θ_{4i} and the inverse tangent of elements 3.2 (F_z) over 3.1 (E_z) gives θ_{6i}.

$$\theta_{4i} = \theta_{4v} = \mathrm{arctan2}\left(\frac{H_y}{H_x}\right) = \mathrm{arctan2}\left(\frac{L_y}{L_x}\right) \tag{5}$$

and,

$$\theta_{6i} = \theta_{6v} = \mathrm{arctan2}\left(\frac{F_z}{E_z}\right) = \mathrm{arctan2}\left(\frac{J_z}{I_z}\right) \tag{6}$$

Thus, all unknown angles for 6 joints are found by using the geometric solution approach.

6. CONCLUSION

Open source coding for forward and inverse kinematics solution might be obtained in low or high-level languages, but using the code might be painful for many reasons such as;

— compiler may need to be purchased
— coding language might be learned
— debugging is hard if there is any bug in the code
— coding should be fully understood before adapting or extending

These difficulties most likely lead to moving away from those who want to dive into robotics and progress at a rapid pace. With this article and provided supplemental calculation sheets (SMath & Excel) [14], a solid foundation is thought to be established for understanding both forward & especially inverse kinematics solutions.

For both easy understanding and solution simplicity, just a single solution is introduced. Inverse kinematics solution covers the widest but limited range. For instance, the 5th joint cannot derive to negative rotational angles. Angles on the 1st and 2nd quadrants of the unit circle are covered because this joint angle is derived from the *arccos* function.

Applicability of the chosen inverse calculation approach can be exercised with supplied interactive table sheet. Compared to quaternion, it is not a singularity-free solution. In case these singularities are in the robot's working space, additional algorithms may be needed to resolve them.

If desired, other inverse kinematics solution variants can be easily developed. Previously mentioned [11], [12] references are a great starting point.

SIMILARTY RATE: 10%

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