NATURENGS
MTU Journal of Engineering and Natural Sciences
https://dergipark.org.tr/tr/pub/naturengs
DOI: 10.46572/naturengs.1082785

Research Article

Pluriharmonic Conformal Bi-Slant Riemannian Maps

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(**Received:** 04.03.2022; **Accepted:** 20.05.2022)

ABSTRACT: In this study, the notion of pluriharmonic map was applied to conformal bi-slant Riemannian maps from a Kaehler manifold to a Riemannian manifold to examine its geometric properties. Such relations between pluriharmonic map, horizontally homothetic map, and totally geodesic map were obtained.

Keywords: Riemannian map, Conformal Riemannian map, Conformal bi-slant Riemannian map, Pluriharmonic map.

1. INTRODUCTION

The notion of submersion was introduced by O'Neill [1] and Gray [2]. Submersion theory between almost Hermitian manifolds was studied by Watson [3]. Then, Fischer studied the theory of submersion in various types and generalized it to Riemannian maps [4]. Riemannian maps between Riemannian manifolds generalize isometric immersions and Riemannian submersions. Let $\Phi: (\mathrm{M}_1, \mathrm{g}_1) \to (\mathrm{M}_2, \mathrm{g}_2)$ be a smooth map between Riemannian manifolds such that $0 < rank\Phi < min\,\{dim(M_1), dim(M_2)\}$. Then, the tangent bundle of TM_1 of M_1 has the following decomposition:

$$TM_1 = ker\Phi_* \oplus (ker\Phi_*)^\perp.$$

Since $rank\Phi < min\,\{dim(M_1), dim(M_2)\}$, we have $(range\Phi_*)^\perp$. Hence, the tangent bundle of TM_2 of M_2 has the following decomposition:

$$TM_2 = range\Phi_* \oplus (range\Phi_*)^\perp.$$

A smooth map $\Phi: (\mathrm{M}_1^m, \mathrm{g}_1) \to (\mathrm{M}_2^m, \mathrm{g}_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $\Phi^h_{*p_1}: \left(ker\Phi_{*p_1}\right)^\perp \longrightarrow (range\Phi_*)$ is a linear isometry. Therefore, the Riemannian map satisfies the equation

$$g_1(X, Y) = g_2\big(\Phi_*(X), \Phi_*(Y)\big)$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$. Hence, isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $\mathrm{ker}\Phi_* = \{0\}$ and $(\mathrm{range}\Phi_*)^\perp = \{0\}$ [4]. Moreover, Şahin and Yanan examined conformal Riemannian maps [5-8], see also [9]. We say

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that $\Phi:(\mathrm{M}^{\mathrm{m}},\mathrm{g}_{\mathrm{M}})\longrightarrow(\mathrm{N}^{\mathrm{n}},\mathrm{g}_{\mathrm{N}})$ is a conformal Riemannian map at $p\in M$ if $0<rank\Phi_{*p}\leq min\{m,n\}$ and Φ_{*} maps the horizontal space $\left(\ker(\Phi_{*\mathrm{p}})^{\perp}\right)$ conformally onto $range(\Phi_{*p})$, i.e., there exists a number $\lambda^{2}(\mathrm{p})\neq 0$ such that

$$g_{N}\left(\Phi_{*p}(X),\Phi_{*p}(Y)\right)=\lambda^{2}(p)g_{M}(X,Y)$$

for $X,Y\in\Gamma((ker\Phi_{*})^{\perp})$. Also, Φ is called conformal Riemannian if Φ is conformal Riemannian at each $p\in M$. Here, λ is the dilation of Φ at a point $p\in M$ and it is a continuous function as $\lambda: M\rightarrow[0,\infty)$ [10]. One can see more research on curvature relations for conformal bi-slant submersions and the relation between submersion theory and bi-slant structure, which is studied by Aykurt Sepet [11,12].

An even-dimensional Riemannian manifold (M,g_{M},J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^{2}=-I$ where I denotes the identity transformation of TM and

$$g_{M}(X,Y)=g_{M}(JX,JY),\forall X,Y\in\Gamma(TM).$$

Let (M,g_{M},J) be an almost Hermitian manifold and its Levi-Civita connection ∇ concerning g_{M}. If J is parallel concerning ∇, i.e.

$$(\nabla_{X}J)Y=0,$$

we say M is a Kaehler manifold [13].

Let $\Phi:(M,g_{M},J)\longrightarrow(N,g_{N})$ be a map from a complex manifold (M,g_{M},J) to a Riemannian manifold (N,g_{N}). Then Φ is called a pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_{*})(X,Y)+(\nabla\Phi_{*})(JX,JY)=0$$

for $X,Y\in\Gamma(TM)$ [14].

Here, we recall some basic definitions of conformal Riemannian maps from a Kaehler manifold to a Riemannian manifold.

Let $\Phi:(M,g_{M},J)\longrightarrow(N,g_{N})$ be a conformal Riemannian map between a Kaehler manifold (M,g_{M},J) and a Riemannian manifold (N,g_{N}).

1. If the map Φ satisfies the following condition:

   $$J(ker\Phi_{*})\subset(ker\Phi_{*})^{\perp},$$

   Then Φ is called a conformal anti-invariant Riemannian map [6].

2. If the following conditions are satisfying:
   i. There exists a subbundle of $ker\Phi_{*}$ such that $J(D_{1})=D_{1}$,
   ii. There exists a complementary subbundle D_{2} to D_{1} in $ker\Phi_{*}$ such that $J(D_{2})\subset(ker\Phi_{*})^{\perp}$,

   We say that Φ is a conformal semi-invariant Riemannian map [7].

3. If for any non-zero vector $X \in \Gamma(ker\Phi_*)$ at a point $p \in M$; the angle $\theta(X)$ between the space $ker\Phi_*$ and JX is a constant, i.e. it is independent of the choice of the tangent vector $X \in \Gamma(ker\Phi_*)$ and the choice of the point $p \in M$, we say that Φ is a conformal slant Riemannian map. In this case, the angle θ is called the slant angle of the conformal slant Riemannian map [8].
4. If the vertical distribution $ker\Phi_*$ of Φ admits two orthogonal complementary distributions D_θ and $D_\perp$ such that D_θ is slant and $D_\perp$ is anti-invariant, i.e., we have

$$ker\Phi_* = D_\theta \oplus D_\perp.$$

Hence, Φ is called a conformal hemi-slant Riemannian map and the angel θ is called hemi-slant angle of the conformal Riemannian map [15].

5. At last, Φ is called a conformal semi-slant Riemannian map if there is a distribution $D_1 \subset ker\Phi_*$ such that

$$ker\Phi_* = D_1 \oplus D_2, J(D_1) = D_1$$

and the angle $\theta = \theta(X)$ between JX and the space $(D_2)_p$ is constant for nonzero $X \in (D_2)_p$ and $p \in M$, where D_2 is the orthogonal complement distribution of D_1 in $ker\Phi_*$. The angel θ is called semi-slant angle of the map [16].

Therefore, we define conformal bi-slant Riemannian maps from a Kaehler manifold to a Riemannian manifold. Some geometric properties of conformal bi-slant Riemannian maps are examined via pluriharmonic map.

2. MATERIAL AND METHODS

This section gives several definitions and results for the study for conformal bi-slant Riemannian maps. Let $\Phi: (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(X, Y) = \nabla_X^\Phi \Phi_*(Y) - \Phi_*(\nabla_X Y)$$

for $X, Y \in \Gamma(TM)$. The second fundamental form $(\nabla\Phi_*)$ is symmetric. Note that Φ is said to be totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$ [17]. Here, we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_X Y = h\nabla_{hX} vY + v\nabla_{hX} hY,$$

$$\mathcal{T}_X Y = h\nabla_{vX} vY + v\nabla_{vX} hY$$

for $X, Y \in \Gamma(TM)$ with the Levi-Civita connection ∇ of g_M. Here, we denote by v and h the projections on the vertical distribution $ker\Phi_*$ and the horizontal distribution $(ker\Phi_*)^\perp$, respectively. For any $X \in \Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. Note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution [1]. In addition, by definitions of O'Neill's tensor fields, we have

$$\nabla_U V = \mathcal{T}_U V + v\nabla_U V,$$

$$\nabla_U X = h\nabla_U X + \mathcal{T}_U X,$$

$$\nabla_X V = \mathcal{A}_X V + v\nabla_X V,$$

$$\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$ and $U, V \in \Gamma(ker\Phi_*)$ [18].

If a vector field X on M is related to a vector field X' on N, we say X is a projectable vector field. If X is both a horizontal and a projectable vector field, we say X is a basic vector field on M [19]. When we mention a horizontal vector field throughout this study, we always consider a basic vector field. On the other hand, let $\Phi: (M^m, g_M) \longrightarrow (N^n, g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$(\nabla\Phi_*)(X,Y) \mid_{range\Phi_*} = X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) - g_M(X,Y)\Phi_*\big(grad(\ln\lambda)\big)$$

where $X, Y \in \Gamma((ker\Phi_*)^\perp)$ [10]. Hence, we obtain $\nabla_X^\Phi \Phi_*(Y)$ as

$$\nabla_X^\Phi \Phi_*(Y) = \Phi_*(h\nabla_X Y) + X(\ln\lambda)\Phi_*(Y) + Y(\ln\lambda)\Phi_*(X) - g_M(X,Y)\Phi_*\big(grad(\ln\lambda)\big) + (\nabla\Phi_*)^\perp(X,Y)$$

where $(\nabla\Phi_*)^\perp(X,Y)$ is the component of $(\nabla\Phi_*)(X,Y)$ on $(range\Phi_*)^\perp$ for $X, Y \in \Gamma((ker\Phi_*)^\perp)$ [6].

3. RESULTS AND DISCUSSION

In this section, we define conformal bi-slant Riemannian maps, give their decomposition and study some theorems for conformal bi-slant Riemannian maps by applying the notion of pluriharmonic map on certain distributions. Therefore, we want to obtain relations among geometric structures.

Definition 3.1. Let (M, g_M, J) be a Kaehler manifold and (N, g_N) be a Riemannian manifold. Then, a conformal Riemannian map $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ is called a conformal bi-slant Riemannian map if and only if D_1 and D_2 are slant distributions with their slant angles θ_1 and θ_2, respectively, such that

$$ker\Phi_* = D_1 \oplus D_2.$$

Here, if the slant angles satisfy that $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$, Φ is called a proper conformal bi-slant Riemannian map [20].

We explain decompositions of distributions for the conformal bi-slant Riemannian map Φ.

Suppose that Φ is a conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). For any $U \in \Gamma(ker\Phi_*)$, we have

$$U = PU + QU,$$

where $PU \in \Gamma(D_1)$ and $QU \in \Gamma(D_2)$. On the other hand, we have

$$JU = \psi U + \phi U,$$

for $U \in \Gamma(ker\Phi_*)$ where $\psi U \in \Gamma(ker\Phi_*)$ and $\phi U \in \Gamma((ker\Phi_*)^\perp)$. Also, for any $X \in \Gamma((ker\Phi_*)^\perp)$, we write

$$JX = BX + CX,$$

where $BX \in \Gamma(ker\Phi_*)$ and $CX \in \Gamma((ker\Phi_*)^\perp)$. Therefore, the horizontal distribution $(ker\Phi_*)^\perp$ can be decomposed as

$$(ker\Phi_*)^\perp = \phi D_1 \oplus \phi D_2 \oplus \mu,$$

where μ is the orthogonal complementary distribution of $\phi D_1 \oplus \phi D_2$ in $(ker\Phi_*)^\perp$ [20].

We have the following theorem same as conformal bi-slant submersions.

Theorem 3.2. Let Φ be a conformal bi-slant Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, we have

$$\psi^2 U_i = -(cos^2 \theta_i) U_i$$

for $U_i \in \Gamma(D_i)$, $i = 1,2$ [12].

Recall that, Φ is said to be a horizontally homothetic map if $h\big(grad(ln\,\lambda)\big) = 0$. It means that horizontal part of the gradient vector field of the dilation λ is equal to zero [19]. On the other hand, Φ is said to be totally geodesic map if $(\nabla\Phi_*)(E, F) = 0$ for $E, F \in \Gamma(TM)$ [5].

Firstly, we derive new notions by using pluriharmonic map, see [7,8]. Hence, let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is called a D_1- pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(U_1, V_1) + (\nabla\Phi_*)(JU_1, JV_1) = 0$$

for $U_1, V_1 \in \Gamma(D_1)$.

Theorem 3.3. Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal bi-slant Riemannian map between a Kaehler manifold (M, g_M, J) and a Riemannian manifold (N, g_N). If Φ is a D_1- pluriharmonic map, then the following two assertions imply the third assertion,

i. D_1 defines a totally geodesic foliation on M,
ii. The map Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(\phi U_1, \phi V_1) = 0$,
iii. $h\nabla_{\mathcal{U}_1}\phi\psi V_1 + \phi\mathcal{T}_{\mathcal{U}_1}\phi V_1 + Ch\nabla_{U_1}\phi V_1 = \mathcal{T}_{\psi\mathcal{U}_1}\psi V_1 + \mathcal{A}_{\phi\mathcal{U}_1}\psi V_1 + \mathcal{A}_{\phi\mathcal{V}_1}\psi U_1$

for $U_1, V_1 \in \Gamma(D_1)$.

Proof. Firstly, we know that Φ is a D_1- pluriharmonic map, then we have

$$(\nabla\Phi_*)(U_1, V_1) + (\nabla\Phi_*)(JU_1, JV_1) = 0 \quad (1)$$

for $U_1, V_1 \in \Gamma(D_1)$. Since M is a Kaehler manifold by using the notion of second fundamental form of a map and its restriction to the horizontal distribution, we get

$$0 = (\nabla\Phi_*)(U_1, V_1) + (\nabla\Phi_*)(JU_1, JV_1) \tag{2}$$

$$\begin{aligned} 0 = &-\Phi_*\big(\nabla_{U_1}V_1\big) - \Phi_*\big(\nabla_{\psi U_1}\psi V_1 + \nabla_{\phi U_1}\psi V_1 + \nabla_{\phi V_1}\psi U_1\big) \\ &+(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) + \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) \\ &+\phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big) \end{aligned} \tag{3}$$

for $U_1, V_1 \in \Gamma(D_1)$. Then, from O'Neill's tensor fields by using Eq. (3), we get

$$\begin{aligned} 0 = &\,\Phi_*\big(J\nabla_{U_1}\psi V_1 + \nabla_{U_1}\phi\psi V_1\big) - \Phi_*\big(\mathcal{T}_{\psi U_1}\psi V_1 + \mathcal{A}_{\phi U_1}\psi V_1 + \mathcal{A}_{\phi V_1}\psi U_1\big) \\ &+(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) + \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) \\ &+\phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big) \end{aligned} \tag{4}$$

$$\begin{aligned} 0 = &\,\Phi_*\big(\nabla_{U_1}\psi^2 V_1 + \nabla_{U_1}\phi\psi V_1\big) + \Phi_*\big(J\mathcal{T}_{U_1}\phi V_1 + Jh\nabla_{U_1}\phi V_1\big) \\ &-\Phi_*\big(\mathcal{T}_{\psi U_1}\psi V_1 + \mathcal{A}_{\phi U_1}\psi V_1 + \mathcal{A}_{\phi V_1}\psi U_1\big) \\ &+(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) + \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) \\ &+\phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big). \end{aligned} \tag{5}$$

From Theorem 3.2. in Eq. (5), we obtain

$$\begin{aligned} cos^2\theta_1\,\Phi_*\big(\nabla_{U_1}V_1\big) = &\,\Phi_*\big(h\nabla_{U_1}\phi\psi V_1\big) + \Phi_*\big(\phi\mathcal{T}_{U_1}\phi V_1 + Ch\nabla_{U_1}\phi V_1\big) \\ &-\Phi_*\big(\mathcal{T}_{\psi U_1}\psi V_1 + \mathcal{A}_{\phi U_1}\psi V_1 + \mathcal{A}_{\phi V_1}\psi U_1\big) \\ &+(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) + \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) \\ &+\phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big). \end{aligned} \tag{6}$$

Now, consider that i. and ii. are satisfied in Eq. (6). Since D_1 defines a totally geodesic foliation on M and Φ is a horizontally homothetic map, we have $\Phi_*\big(\nabla_{U_1}V_1\big) = 0$, $(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) = 0$ and $\phi U_1(ln\,\lambda)\Phi_*(\phi V_1) + \phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big) = 0$ for $U_1, V_1 \in \Gamma(D_1)$, respectively. Hence, one can clearly see the proof of iii. from Eq. (6). If ii. and iii. are satisfied in Eq. (6), we get

$$cos^2\theta_1\,\Phi_*\big(\nabla_{U_1}V_1\big) = 0. \tag{7}$$

So, easily we say that D_1 defines a totally geodesic foliation on M for $U_1, V_1 \in \Gamma(D_1)$. The proof of i. is completed. Suppose that i. and iii. are satisfied in Eq. (6), we obtain

$$\begin{aligned} 0 = &\,(\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1) + \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) + \phi V_1(ln\,\lambda)\Phi_*(\phi U_1) \\ &-g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big). \end{aligned} \tag{8}$$

In Eq. (8), if we separate components as to which one belongs to $range\Phi_*$ or its orthogonal complement distribution $(range\Phi_*)^{\perp}$, we obtain $0 = (\nabla\Phi_*)^{\perp}(\phi U_1, \phi V_1)$. Hence, we get

$$0 = \phi U_1(ln\,\lambda)\Phi_*(\phi V_1) + \phi V_1(ln\,\lambda)\Phi_*(\phi U_1) - g_M(\phi U_1, \phi V_1)\Phi_*\big(grad(ln\,\lambda)\big). \quad (9)$$

For $\phi U_1 \in \Gamma(\phi D_1)$, since Φ is a conformal map, we get from Eq. (9),

$$0 = \phi U_1(ln\,\lambda)g_N\big(\Phi_*(\phi V_1), \Phi_*(\phi U_1)\big)$$
$$+\phi V_1(ln\,\lambda)g_N\big(\Phi_*(\phi U_1), \Phi_*(\phi U_1)\big)$$
$$-g_M(\phi U_1, \phi V_1)g_N\left(\Phi_*\big(grad(ln\,\lambda)\big), \Phi_*(\phi U_1)\right) \quad (10)$$

$$0 = \lambda^2\phi U_1(ln\,\lambda)g_M(\phi V_1, \phi U_1)$$
$$+\lambda^2\phi V_1(ln\,\lambda)g_M(\phi U_1, \phi U_1)$$
$$-\lambda^2\, g_M(\phi U_1, \phi V_1)\phi U_1(ln\,\lambda) \quad (11)$$

$$0 = \lambda^2\phi V_1(ln\,\lambda)g_M(\phi U_1, \phi U_1). \quad (12)$$

In Eq. (12), we have $\phi V_1(ln\,\lambda) = 0$. This means, the dilation λ is a constant on ϕD_1. On the other hand, if we take $U_1 = V_1$, $\phi U_2 \in \Gamma(\phi D_2)$ and $U_3 \in \Gamma(\mu)$ from Eq. (9), we get

$$0 = -\lambda^2\phi U_2(ln\,\lambda)g_M(\phi U_1, \phi U_1), \quad (13)$$

$$0 = -\lambda^2 U_3(ln\,\lambda)g_M(\phi U_1, \phi U_1), \quad (14)$$

respectively. From Eq. (13) and Eq. (14), we get $\phi U_2(ln\,\lambda) = 0$ and $U_3(ln\,\lambda) = 0$, respectively. Hence, the dilation λ is a constant on ϕD_2 and μ. Therefore, the map Φ is a horizontally homothetic map. iii. is satisfied. The proof is completed.

Similarly, we have the following notion and theorem.

Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is called a D_2- pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(U_2, V_2) + (\nabla\Phi_*)(JU_2, JV_2) = 0$$

for $U_2, V_2 \in \Gamma(D_2)$.

Theorem 3.4. Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal bi-slant Riemannian map between a Kaehler manifold (M, g_M, J) and a Riemannian manifold (N, g_N). If Φ is a D_2- pluriharmonic map, then the following two assertions imply the third assertion,

i. D_2 defines a totally geodesic foliation on M,
ii. The map Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(\phi U_2, \phi V_2) = 0$,
iii. $h\nabla_{\mathcal{U}_2}\phi\psi V_2 + \phi\mathcal{T}_{\mathcal{U}_2}\phi V_2 + Ch\nabla_{\mathcal{U}_2}\phi V_2 = \mathcal{T}_{\psi\mathcal{U}_2}\psi V_2 + \mathcal{A}_{\phi\mathcal{U}_2}\psi V_2 + \mathcal{A}_{\phi\mathcal{V}_2}\psi U_2$

for $U_2, V_2 \in \Gamma(D_2)$.

Proof. The proof of the Theorem 3.4. can get similarly with Theorem 3.3.

Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is called a $(ker\Phi_*)^\perp$- pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(X,Y) + (\nabla\Phi_*)(JX,JY) = 0$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$.

Theorem 3.5. Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal bi-slant Riemannian map between a Kaehler manifold (M, g_M, J) and a Riemannian manifold (N, g_N). If Φ is a $(ker\Phi_*)^\perp$-pluriharmonic map, then one of the following assertions implies the other assertion,

i. $\mathcal{T}_{\mathcal{B}X}BY + \mathcal{A}_{\mathcal{C}X}BY + \mathcal{A}_{\mathcal{C}Y}BX = 0$,
ii. The map Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(X,Y) + (\nabla\Phi_*)^\perp(CX,CY) = 0$,

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$.

Proof. If Φ is a $(ker\Phi_*)^\perp$- pluriharmonic map, we have

$$(\nabla\Phi_*)(X,Y) + (\nabla\Phi_*)(JX,JY) = 0 \tag{15}$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$. Then, by using definition of second fundamental form of a map and its decomposition onto $range\Phi_*$ and $(range\Phi_*)^\perp$ in Eq. (15), we obtain

$$\begin{aligned} 0 &= (\nabla\Phi_*)^\perp(X,Y) + (\nabla\Phi_*)^\perp(CX,CY) - \Phi_*(\mathcal{T}_{\mathcal{B}X}BY + \mathcal{A}_{\mathcal{C}X}BY + \mathcal{A}_{\mathcal{C}Y}BX) \\ &+ X(ln\,\lambda)\Phi_*(Y) + Y(ln\,\lambda)\Phi_*(X) - g_M(X,Y)\Phi_*\big(grad(ln\,\lambda)\big) \\ &+ CX(ln\,\lambda)\Phi_*(CY) + CY(ln\,\lambda)\Phi_*(CX) - g_M(CX,CY)\Phi_*\big(grad(ln\,\lambda)\big) \end{aligned} \tag{16}$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$. If i. is satisfied in Eq. (16), we have $\mathcal{T}_{\mathcal{B}X}BY + \mathcal{A}_{\mathcal{C}X}BY + \mathcal{A}_{\mathcal{C}Y}BX = 0$. So, we get from Eq. (16),

$$\begin{aligned} 0 &= (\nabla\Phi_*)^\perp(X,Y) + (\nabla\Phi_*)^\perp(CX,CY) \\ &+ X(ln\,\lambda)\Phi_*(Y) + Y(ln\,\lambda)\Phi_*(X) \\ &- g_M(X,Y)\Phi_*\big(grad(ln\,\lambda)\big) \\ &+ CX(ln\,\lambda)\Phi_*(CY) + CY(ln\,\lambda)\Phi_*(CX) \\ &- g_M(CX,CY)\Phi_*\big(grad(ln\,\lambda)\big). \end{aligned} \tag{17}$$

In Eq. (17), we know that $0 = (\nabla\Phi_*)^\perp(X,Y) + (\nabla\Phi_*)^\perp(CX,CY)$ since they belong to $(range\Phi_*)^\perp$. On the other hand, from elements of $range\Phi_*$ we examine horizontally homotheticness of the map. Hence, from Eq. (17) by using conformality of the map we have

$$\begin{aligned} 0 &= 2X(ln\,\lambda)g_N\big(\Phi_*(Y),\Phi_*(X)\big) + 2Y(ln\,\lambda)g_N\big(\Phi_*(X),\Phi_*(X)\big) \\ &- 2g_M(X,Y)g_N\left(\Phi_*\big(grad(ln\,\lambda)\big),\Phi_*(X)\right) \end{aligned} \tag{18}$$

$$0 = 2\lambda^2 Y(ln\,\lambda) g_M(X, X) \tag{19}$$

for $X = CX$ and $Y = CY$. Here, since $\lambda^2 \neq 0$ and $g_M(X, X) \neq 0$, we get $Y(ln\,\lambda) = 0$. It means that λ is a constant on horizontal distribution $(ker\Phi_*)^\perp$. Hence, the map Φ is a horizontally homothetic map. ii. is satisfied. If ii. is satisfied in Eq. (16), we have

$$0 = X(ln\,\lambda)\Phi_*(Y) + Y(ln\,\lambda)\Phi_*(X) - g_M(X, Y)\Phi_*\big(grad(ln\,\lambda)\big)$$

$$+CX(ln\,\lambda)\Phi_*(CY) + CY(ln\,\lambda)\Phi_*(CX) - g_M(CX, CY)\Phi_*\big(grad(ln\,\lambda)\big)$$

and

$$0 = (\nabla\Phi_*)^\perp(X, Y) + (\nabla\Phi_*)^\perp(CX, CY).$$

So, from Eq. (16), we obtain

$$0 = -\Phi_*(\mathcal{T}_{BX}BY + \mathcal{A}_{CX}BY + \mathcal{A}_{CY}BX). \tag{20}$$

Hence, Eq. (20) shows us that i. is satisfied. The proof is completed.

Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is called a $ker\Phi_*$ - pluriharmonic map if Φ satisfies the following equation:

$$(\nabla\Phi_*)(U, V) + (\nabla\Phi_*)(JU, JV) = 0$$

for $U, V \in \Gamma(ker\Phi_*)$.

Theorem 3.6. Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal bi-slant Riemannian map between a Kaehler manifold (M, g_M, J) and a Riemannian manifold (N, g_N). If Φ is a $ker\Phi_*$-pluriharmonic map, then the following two assertions imply the third assertion,

i. $ker\Phi_*$ defines a totally geodesic foliation on M,
ii. $\mathcal{A}_{\phi V}\psi PU + \mathcal{A}_{\phi U}\psi V + \mathcal{T}_{\psi U}\psi V + h\nabla_{\psi QU}\phi V = \phi\mathcal{T}_U\phi V + Ch\nabla_U\phi V + h\nabla_U\phi\psi V$,
iii. The map Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(\phi U, \phi V) = 0$

for $U, V \in \Gamma(ker\Phi_*)$.

Proof. If Φ is a $ker\Phi_*$- pluriharmonic map, we have

$$(\nabla\Phi_*)(U, V) + (\nabla\Phi_*)(JU, JV) = 0 \tag{21}$$

for $U, V \in \Gamma(ker\Phi_*)$. By using decomposition theorems for conformal bi-slant Riemannian maps in Eq. (21), we get

$$0 = (\nabla\Phi_*)(U, V) + (\nabla\Phi_*)(J(PU + QU), \phi V + \psi V)$$

$$0 = (\nabla\Phi_*)(\psi PU, \psi V) + (\nabla\Phi_*)(\phi V, \psi PU)$$

$$+(\nabla\Phi_*)(\phi U, \psi V) + (\nabla\Phi_*)(\psi QU, \phi V + \psi V)$$

$$+(\nabla\Phi_*)(\phi U, \phi V) + (\nabla\Phi_*)(U, V). \tag{22}$$

Then, from definition of the second fundamental form of a map in Eq. (22), we get

$$
\begin{aligned}
0 = & -\Phi_*\big(\mathcal{T}_{\psi U}\psi V + \mathcal{A}_{\phi V}\psi PU + \mathcal{A}_{\phi U}\psi V + h\nabla_{\psi QU}\phi V\big) \\
& +(\nabla\Phi_*)^{\perp}(\phi U, \phi V) + (\nabla\Phi_*)^{\top}(\phi U, \phi V) \\
& +\Phi_*\big(J(\mathcal{T}_U\phi V + h\nabla_U\phi V)\big) + \Phi_*(\nabla_U J\psi V).
\end{aligned} \tag{23}
$$

Now, by using Theorem 3.2. and horizontal restriction of the second fundamental form of a map in Eq. (23), we obtain

$$
\begin{aligned}
0 = & -\Phi_*\big(\mathcal{T}_{\psi U}\psi V + \mathcal{A}_{\phi V}\psi PU + \mathcal{A}_{\phi U}\psi V + h\nabla_{\psi QU}\phi V\big) \\
& +(\nabla\Phi_*)^{\perp}(\phi U, \phi V) + \phi U(ln\,\lambda)\Phi_*(\phi V) \\
& +\phi V(ln\,\lambda)\Phi_*(\phi U) - g_M(\phi U, \phi V)\Phi_*\big(grad(ln\,\lambda)\big) \\
& +\Phi_*(\phi\mathcal{T}_U\phi V + Ch\nabla_U\phi V) + \Phi_*(-cos^2\theta\,\nabla_U V + h\nabla_U\phi\psi V) \\
cos^2\theta\,\Phi_*(\nabla_U V) = & -\Phi_*\big(\mathcal{T}_{\psi U}\psi V + \mathcal{A}_{\phi V}\psi PU + \mathcal{A}_{\phi U}\psi V + h\nabla_{\psi QU}\phi V\big) \\
& +(\nabla\Phi_*)^{\perp}(\phi U, \phi V) + \phi U(ln\,\lambda)\Phi_*(\phi V) \\
& +\phi V(ln\,\lambda)\Phi_*(\phi U) - g_M(\phi U, \phi V)\Phi_*\big(grad(ln\,\lambda)\big) \\
& +\Phi_*(\phi\mathcal{T}_U\phi V + Ch\nabla_U\phi V + h\nabla_U\phi\psi V).
\end{aligned} \tag{24}
$$

In Eq. (24), if i. and ii. are satisfied we have $\nabla_U V = 0$ and $\mathcal{A}_{\phi V}\psi PU + \mathcal{A}_{\phi U}\psi V + \mathcal{T}_{\psi U}\psi V + h\nabla_{\psi QU}\phi V = \phi\mathcal{T}_U\phi V + Ch\nabla_U\phi V + h\nabla_U\phi\psi V$, respectively. So, we get

$$
\begin{aligned}
0 = & (\nabla\Phi_*)^{\perp}(\phi U, \phi V) + \phi U(ln\,\lambda)\Phi_*(\phi V) \\
& +\phi V(ln\,\lambda)\Phi_*(\phi U) - g_M(\phi U, \phi V)\Phi_*\big(grad(ln\,\lambda)\big).
\end{aligned} \tag{25}
$$

Similarly, we obtain $(\nabla\Phi_*)^{\perp}(\phi U, \phi V) = 0$, clearly. Hence, Eq. (25) turns into

$$
0 = \phi U(ln\,\lambda)\Phi_*(\phi V) + \phi V(ln\,\lambda)\Phi_*(\phi U) - g_M(\phi U, \phi V)\Phi_*\big(grad(ln\,\lambda)\big). \tag{26}
$$

For $\phi V \in \Gamma((ker\Phi_*)^{\perp})$, from the conformality of the map we get

$$
\begin{aligned}
0 = & \phi U(ln\,\lambda)g_N\big(\Phi_*(\phi V), \Phi_*(\phi V)\big) + \Phi V(ln\,\lambda)g_N\big(\Phi_*(\phi U), \Phi_*(\phi V)\big) \\
& -g_M(\phi U, \phi V)g_N\Big(\Phi_*\big(grad(ln\,\lambda)\big), \Phi_*(\phi V)\Big) \\
0 = & \lambda^2\phi U(ln\,\lambda)g_M(\phi V, \phi V).
\end{aligned} \tag{27}
$$

In Eq. (27), since $\lambda^2 \neq 0$ and $g_M(\phi V, \phi V) \neq 0$, we get $\phi U(ln\,\lambda) = 0$. It means that λ is a constant on $(ker\Phi_*)^{\perp}$. Hence, the map Φ is a horizontally homothetic map. iii. is proved. The other cases of the proof could be seen clearly. The proof is completed.

Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a map from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is called a *mixed* - pluriharmonic map if Φ satisfies the following equation:

$$
(\nabla\Phi_*)(X, U) + (\nabla\Phi_*)(JX, JU) = 0
$$

for $X \in \Gamma((ker\Phi_*)^\perp)$ and $U \in \Gamma(ker\Phi_*)$.

Theorem 3.7. Let $\Phi: (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal bi-slant Riemannian map between a Kaehler manifold (M, g_M, J) and a Riemannian manifold (N, g_N). If Φ is a $mixed$ - pluriharmonic map, then one of the following assertions imply the other assertion,

i. $\mathcal{A}_X U = -\mathcal{T}_{\mathcal{B}X}\psi U - \mathcal{A}_{\mathcal{C}X}\psi U - \mathcal{A}_{\phi U}BX$,

ii. The map Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(CX, \phi U) = 0$

for $X \in \Gamma((ker\Phi_*)^\perp)$ and $U \in \Gamma(ker\Phi_*)$.

Proof. If Φ is a $mixed$ - pluriharmonic map, by direct calculations we obtain

$$0 = (\nabla\Phi_*)(X, U) + (\nabla\Phi_*)(JX, JU)$$

$$\begin{aligned} 0 = &-\Phi_*(\mathcal{A}_X U) - \Phi_*\big(\mathcal{T}_{\mathcal{B}X}\psi U + \mathcal{A}_{\mathcal{C}X}\psi U + \mathcal{A}_{\phi U}BX\big) \\ &+(\nabla\Phi_*)^\perp(CX, \phi U) + CX(ln\,\lambda)\Phi_*(\phi U) + \phi U(ln\,\lambda)\Phi_*(CX) \\ &-g_M(CX, \phi U)\Phi_*\big(grad(ln\,\lambda)\big) \end{aligned} \quad (28)$$

for $X \in \Gamma((ker\Phi_*)^\perp)$ and $U \in \Gamma(ker\Phi_*)$. In Eq. (28), if i. is satisfied we get

$$\begin{aligned} 0 = &(\nabla\Phi_*)^\perp(CX, \phi U) + CX(ln\,\lambda)\Phi_*(\phi U) + \phi U(ln\,\lambda)\Phi_*(CX) \\ &-g_M(CX, \phi U)\Phi_*\big(grad(ln\,\lambda)\big). \end{aligned} \quad (29)$$

In Eq. (29), we get easily $(\nabla\Phi_*)^\perp(CX, \phi U) = 0$. Hence, from Eq. (29) we get

$$0 = CX(ln\,\lambda)\Phi_*(\phi U) + \phi U(ln\,\lambda)\Phi_*(CX) - g_M(CX, \phi U)\Phi_*\big(grad(ln\,\lambda)\big). \quad (30)$$

For $CX, \phi U \in \Gamma((ker\Phi_*)^\perp)$, from Eq. (30) we obtain

$$0 = \lambda^2\phi U(ln\,\lambda)g_M(CX, CX) \quad (31)$$

and

$$0 = \lambda^2 CX(ln\,\lambda)g_M(\phi U, \phi U), \quad (32)$$

respectively. From Eq. (31) and Eq. (32), we say λ is a constant on horizontal distribution. Hence, the map Φ is a horizontally homothetic map. ii. is proved. The converse of this situation is clear. The proof is completed.

4. CONCLUSIONS

Throughout this study, we obtained geometric relations by using derivations of the notion of pluriharmonic map as $D_1, D_2, (ker\Phi_*)^\perp, ker\Phi_*$ and mixed - pluriharmonic map onto conformal bi-slant Riemannian maps.

Declaration of Competing Interest

The author declares that they have no known competing financial interests or personal relationships that could influence the work reported in this paper.

Author Contribution

Şener Yanan contributed 100% at every stage of the article.

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