Turk. J. Math. Comput. Sci.
15(2)(2023) 375–381
© MatDer
DOI : 10.47000/tjmcs.1082849

Some Curvature Conditions on 3-Dimensional Quasi-Sasakian Manifolds Admitting Conformal Ricci Soliton

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Received: 04-03-2022 • Accepted: 14-03-2023

Abstract. In this paper, we examine 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton. We give some theorems for W_0^* flat, $\xi - W_0^*$ flat and $\phi - W_0^*$ semisymmetric 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton. Also we study conformal Ricci soliton on a 3-dimensional quasi-Sasakian manifold satisfying the conditions $W_0^*(\xi, X).S = 0$ and $R(\xi, X).W_0^* = 0$.

2010 AMS Classification: 53A40, 53D10

Keywords: Ricci soliton, conformal Ricci soliton, quasi-Sasakian manifold.

1. Introduction

Quasi-Sasakian manifold is a natural generalization of Sasakian manifold whose notion was introduced by Blair [4] to unify Sasakian and cosympletic structures. The properties of quasi-Sasakian manifolds have been studied by various authors such as Gonzalez and Chinea [15], Kanemaki [18, 19], De and Sarkar [9], De et al. [10], Turan et al. [26] and many others. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [21] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [22].

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0,$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

The concept of conformal Ricci flow [12] was developed by Fischer during 2003-2004 which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on M is defined by the equation [12]

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg$$

and $r(g) = -1$, where M is considered as a smooth closed connected oriented $n-$manifold, p is a scalar non-dynamical field (time dependent scalar field), $r(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

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The notion of conformal Ricci soliton equation was introduced by Basu and Bhattacharyya [1] in 2015 and the equation is given by

$$\pounds_V g + 2S = \left(2\lambda - \left(p + \frac{2}{n}\right)\right) g, \tag{1.1}$$

where λ is constant. The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

Also, W_0^* curvature tensor with respect to Levi-Civita connection is defined by [23]

$$W_0^*(X,Y)Z = R(X,Y)Z + \frac{1}{2}\left(S(Y,Z)X - g(X,Z)QY\right). \tag{1.2}$$

The authors studied to improve the topic of solitons in [2, 3, 5–8, 13, 14, 16, 24, 27, 28]. Some recent studies about this topic are given in [11, 17, 25].

This paper is organized as follows: After preliminaries, we give some basic information about the conformal Ricci soliton and quasi-Sasakian manifolds. Then, we give some theorems for W_0^* flat, $\xi - W_0^*$ flat and $\phi - W_0^*$ semisymmetric 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton in the following sections. Finally we give conformal Ricci soliton on a 3-dimensional quasi-Sasakian manifold satisfying the conditions $W_0^*(\xi, X) \cdot S = 0$ and $R(\xi, X) \cdot W_0^* = 0$.

2. Preliminaries

Let M be a connected almost contact metric manifold of dimension $(2n+1)$ with an almost contact metric structure (ϕ, ξ, η, g) and g is a Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$). A three dimensional almost contact metric manifold M is quasi-Sasakian if and only if [20]

$$\nabla_X \xi = -\beta \phi X.$$

for a certain function β on M, such that $\xi\beta = 0$ where ∇ is the Levi-Civita connection of M. Clearly, a 3-dimensional quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. If $\beta = constant$, then the manifold reduces to a β-Sasakian manifold and $\beta = 1$ gives the Sasakian structure. Throughout in the paper, we are using the fact that $\beta = constant$.
In a 3-dimensional quasi-Sasakian manifold, we have [21]

$$(\nabla_X \phi) Y = -\beta \left(g(X,Y)\xi - \eta(Y)X\right),$$

$$(\nabla_X \eta) Y = -\beta g(\phi X, Y),$$

$$\begin{aligned} R(X,Y)Z \;=\; & \left(\frac{r}{2} - 2\beta^2\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ & + \left(3\beta^2 - \frac{r}{2}\right)\left(\begin{array}{c} g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \end{array}\right), \end{aligned} \tag{2.2}$$

$$S(X,Y) = \left(\frac{r}{2} - \beta^2\right) g(X,Y) - \left(3\beta^2 - \frac{r}{2}\right)\eta(X)\eta(Y),$$

for all X, Y, Z on M.

Definition 2.1. A quasi-Sasakian manifold is said to be an $\eta-$Einstein manifold if its non-vanishing Ricci tensor S is of the form

$$S(X,Y) = a g(X,Y) + b\eta(X)\eta(Y), \tag{2.3}$$

where a and b are smooth functions on the manifold. If $b = 0$, then the manifold is said to be an Einstein manifold.

Now from the definition of Lie derivative, we get

$$\left(\pounds_\xi g\right)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi).$$

In view of the equation $g(X,\xi) = \eta(X)$ with $g(X,\phi Y) = -g(\phi X, Y)$, we find

$$\left(\pounds_\xi g\right)(X,Y) = 0.$$

Using above equation in (1.1), we obtain

$$S(X,Y) = wg(X,Y), \tag{2.4}$$

where $w = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right]$.

From (2.4), we arrive at

$$QX = wX,$$

$$S(X,\xi) = w\eta(X), \tag{2.5}$$

$$S(\xi,\xi) = w. \tag{2.6}$$

Thus, we can state the following.

Proposition 2.2. *If a 3-dimensional quasi-Sasakian manifold admits conformal Ricci soliton, then the manifold becomes an Einstein manifold.*

3. W_0^*-Flat 3-Dimensional Quasi-Sasakian Manifold Admitting Conformal Ricci Soliton

In view of (1.2), if M is a W_0^*-flat 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton, we have

$$R(X,Y)Z = \frac{1}{2}\left(g(X,Z)QY - S(Y,Z)X\right). \tag{3.1}$$

Taking inner product of (3.1) with ξ and in view of the equation $g(X,\phi Y) = -g(\phi X, Y)$, we get

$$g(R(X,Y)Z,\xi) = \frac{1}{2}\left(S(Y,\xi)g(X,Z) - S(Y,Z)\eta(X)\right).$$

From the equation (2.2), we find

$$2\beta^2\left(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right) = g(X,Z)\eta(Y) - S(Y,Z)\eta(X).$$

Taking $X = \xi$ in the above equation and using (2.6) with (2.1), we arrive at

$$S(Y,Z) = -2\beta^2 g(Y,Z) + \left(2\beta^2 + w\right)\eta(Y)\eta(Z).$$

So, by virtue of (2.3), we can state the following theorem.

Theorem 3.1. *A W_0^*-flat 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton is an $\eta-$Einstein manifold.*

4. $\phi - W_0^*$ Semi-Symmetric 3-Dimensional Quasi-Sasakian Manifold Admitting Conformal Ricci Soliton

Firstly we give the following definition:

Definition 4.1. A quasi-Sasakian manifold is said to be $\phi - W_0^*$ semi-symmetric if [7]

$$W_0^*(X,Y)\cdot\phi = 0 \tag{4.1}$$

for all X, Y on M.

Let M be a $\phi - W_0^*$ semi-symmetric quasi-Sasakian manifold admitting conformal Ricci soliton, the from (4.1) we have

$$\left(W_0^*(X,Y)\cdot\phi\right)Z = W_0^*(X,Y)\phi Z - \phi W_0^*(X,Y)Z.$$

From (1.2), we get

$$R(X,Y)\phi Z - \phi R(X,Y)Z + \frac{1}{2}\begin{pmatrix} S(Y,\phi Z)X - g(X,\phi Z)QY \\ -S(Y,Z)\phi X + g(X,Z)\phi QY \end{pmatrix} = 0. \tag{4.2}$$

By using of (2.2) with (2.4) in (4.2), we obtain

$$\left(\frac{\tau}{2}-2\beta^2\right)\begin{pmatrix} g(Y,\phi Z)X-g(X,\phi Z)Y \\ -g(Y,Z)\phi X+g(X,Z)\phi Y \end{pmatrix}+\left(3\beta^2-\frac{\tau}{2}\right)\begin{pmatrix} g(Y,\phi Z)\eta(X)\xi-g(X,\phi Z)\eta(Y)\xi \\ +\eta(Y)\eta(Z)\phi X-\eta(X)\eta(Z)\phi Y \end{pmatrix}$$
$$+\frac{1}{2}\begin{pmatrix} wg(Y,\phi Z)X-wg(X,\phi Z)Y \\ -wg(Y,Z)\phi X+wg(X,Z)\phi Y \end{pmatrix}=0. \tag{4.3}$$

Putting $Y=\xi$ in (4.3) and by use of (2.1), we find

$$\left(\beta^2+w\right)(g(X,\phi Z)\xi+\eta(Z)\phi X)=0. \tag{4.4}$$

Again, taking $Z=\xi$ in (4.4), we obtain

$$\left(\beta^2+w\right)\phi X=0.$$

So, we get the following.

Theorem 4.2. *If a 3-dimensional quasi-Sasakian manifold satisfies $W_0^*\cdot\phi=0$ and admits conformal Ricci soliton, then*

i) If $p=-\frac{2}{3}+\beta^2$, then $\lambda=0$ and Ricci soliton is steady,
ii) If $p>-\frac{2}{3}+\beta^2$, then $\lambda>0$ and Ricci soliton is shrinking,
iii) If $p<-\frac{2}{3}+\beta^2$, then $\lambda<0$ and Ricci soliton is expanding.

5. Conformal Ricci soliton on a 3-dimensional quasi-Sasakian manifold satisfying $W_0^*(\xi,X)\cdot S=0$

Assume that M is a 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton satisfying $W_0^*(\xi,X)\cdot S=0$. From this equation, we can write

$$S(W_0^*(\xi,X)Y,Z)+S(Y,W_0^*(\xi,X)Z)=0.$$

Using (1.2) with (2.2), we have

$$2\beta^2\begin{pmatrix} g(X,Y)S(\xi,Z)-S(X,Z)\eta(Y) \\ +g(X,Z)S(Y,\xi)-S(Y,X)\eta(Z) \end{pmatrix}+\frac{1}{2}\begin{pmatrix} S(X,Y)S(\xi,Z)-wS(X,Z)\eta(Y) \\ +S(X,Z)S(Y,\xi)-wS(Y,X)\eta(Z) \end{pmatrix}=0. \tag{5.1}$$

By use of (2.4) in (5.1), we arrive at

$$2\beta^2\left(S(X,Y)-wg(X,Y)\right)=0.$$

Thus, we can state the following.

Theorem 5.1. *If a 3-dimensional quasi-Sasakian manifold satisfies $W_0^*(\xi,X)\cdot S=0$ and admits conformal Ricci soliton, then the manifold is an Einstein manifold.*

6. Conformal Ricci Soliton on a 3-Dimensional Quasi-Sasakian Manifold Satisfying $R(\xi,X)\cdot W_0^*=0$

In this section, we consider a 3-dimensional quasi-Sasakian manifold admits conformal Ricci soliton and M is W_0^*-semisymmetric, i.e., $R(\xi,X)\cdot W_0^*=0$ holds on M. Thus, we have for all X,Y,Z,V on M

$$R(\xi,X)W_0^*(Y,Z)V-W_0^*(R(\xi,X)Y,Z)V-W_0^*(Y,R(\xi,X)Z)V-W_0^*(Y,Z)R(\xi,X)V=0. \tag{6.1}$$

Using (2.2) in (6.1), we get

$$\beta^2\begin{pmatrix} g(X,W_0^*(Y,Z)V)\xi-\eta(W_0^*(Y,Z)V)X \\ -g(X,Y)W_0^*(\xi,Z)V+\eta(Y)W_0^*(X,Z)V \\ -g(X,Z)W_0^*(Y,\xi)V+\eta(Z)W_0^*(Y,X)V \\ -g(X,V)W_0^*(Y,Z)\xi+\eta(V)W_0^*(Y,Z)X \end{pmatrix}=0. \tag{6.2}$$

Putting $V=\xi$ in (6.2) and using (2.1), we have

$$\beta^2\begin{pmatrix} g(X,W_0^*(Y,Z)\xi)\xi-\eta(W_0^*(Y,Z)\xi)X \\ -g(X,Y)W_0^*(\xi,Z)\xi+\eta(Y)W_0^*(X,Z)\xi \\ -g(X,Z)W_0^*(Y,\xi)\xi+\eta(Z)W_0^*(Y,X)\xi \\ -\eta(X)W_0^*(Y,Z)\xi+W_0^*(Y,Z)X \end{pmatrix}=0. \tag{6.3}$$

Taking inner product with ξ in (6.3) and using (2.1), we obtain

$$\beta^2 \begin{pmatrix} g(X, W_0^*(Y,Z)\xi) - \eta(W_0^*(Y,Z)\xi)\eta(X) \\ -g(X,Y)\eta(W_0^*(\xi,Z)\xi) + \eta(Y)\eta(W_0^*(X,Z)\xi) \\ -g(X,Z)\eta(W_0^*(Y,\xi)\xi) + \eta(Z)\eta(W_0^*(Y,X)\xi) \\ -\eta(X)\eta(W_0^*(Y,Z)\xi) - \eta(W_0^*(Y,Z)X) \end{pmatrix} = 0.$$

Now, using (1.2) in above equation, we arrive at

$$\beta^2 \begin{pmatrix} \left(\beta^2 + \frac{w}{2}\right)(g(X,Y)\eta(Z) - g(X,Z)\eta(Y)) \\ +\beta^2 (g(X,Z)\eta(Y) - g(Y,Z)\eta(X)) \\ +\frac{1}{2}(S(X,Z)\eta(Y) - wg(X,Y)\eta(Z)) \end{pmatrix} = 0. \tag{6.4}$$

Finally, putting $Y = \xi$ in (6.4) and using (1.2), we obtain

$$S(X,Z) = wg(X,Z).$$

Thus, we give the following theorem.

Theorem 6.1. *If a 3-dimensional quasi-Sasakian manifold satisfies $R(\xi, X) \cdot W_0^* = 0$ and admits conformal Ricci soliton, then the manifold is an Einstein manifold.*

7. $\xi - W_0^*$ Flat 3-dimensional Quasi-Sasakian manifold admitting conformal Ricci soliton

Definition 7.1. A quasi-Sasakian manifold is said to be $\xi - W_0^*$ flat if

$$W_0^*(X,Y) \cdot \xi = 0$$

for all X, Y on M.

Assume that M is a $\xi - W_0^*$ flat 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton, then from (1.2) we have

$$R(X,Y)\xi = \frac{1}{2}(g(X,\xi)QY - S(Y,\xi)X). \tag{7.1}$$

Taking inner product with Z in (7.1) and using (2.1) with (2.2), we obtain

$$\beta^2 (g(X,Z)\eta(Y) - g(Y,Z)\eta(X)) = \frac{1}{2}(S(Y,Z)\eta(X) - g(X,Z)S(Y,\xi)). \tag{7.2}$$

Putting $X = \xi$ in (7.2) and in view of (2.5), we arrive at

$$S(Y,Z) = -2\beta^2 g(Y,Z) + \left(2\beta^2 + w\right)\eta(Y)\eta(Z),$$

which gives the following theorem.

Theorem 7.2. *A $\xi - W_0^*$ flat 3-dimensional quasi-Sasakian manifold admitting conformal Ricci soliton is an $\eta-$Einstein manifold.*

Now, we will give an example of a three dimensional quasi-Sasakian manifold.

Example 7.3. Let us consider the three-dimensional manifold $M = \left\{(x,y,z) \in R^3, (x,y,z) \neq (0,0,0)\right\}$, where (x,y,z) are the standard coordinates in R^3. The vector fields

$$E_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}$$

are linearly independent of each point of M. Let $\bar{g}$ be the Riemannian metric tensor defined by

$$\bar{g}(E_1,E_3) = \bar{g}(E_2,E_3) = \bar{g}(E_1,E_2) = 0, \quad \bar{g}(E_1,E_1) = \bar{g}(E_2,E_2) = \bar{g}(E_3,E_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = \bar{g}(Z,E_3)$ for any $Z \in \Gamma(TM)$. Let φ be the (1,1)-tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0.$$

Then, using the condition of the linearity of φ and $\bar{g}$, we obtain $\eta(E_3) = 1$,

$$\varphi^2 Z = -Z + \eta(Z)E_3,$$
$$\bar{g}(\varphi Z, \varphi W) = \bar{g}(Z, W) - \eta(Z)\eta(W),$$

for all $Z, W \in \Gamma(TM)$. Thus, for $\xi = E_3$, $M(\varphi, \xi, \eta, \bar{g})$ defines an almost contact metric manifold [9].

Now, let ∇ be the Levi-Civita connection with respect to the Riemannian metric $\bar{g}$. Then, we obtain

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = -E_2, \quad [E_2, E_3] = 0.$$

The Riemannian connection ∇ of the metric $\bar{g}$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y),$$

which is known as Kozsul's formula. Taking $E_3 = \xi$ and using the above formula it can be calculated as

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_1 = -\frac{1}{2}E_3, \quad \nabla_{E_1} E_3 = -\frac{1}{2}E_2,$$
$$\nabla_{E_2} E_3 = \frac{1}{2}E_1, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_1} E_2 = \frac{1}{2}E_3,$$
$$\nabla_{E_3} E_1 = -\frac{1}{2}E_2, \quad \nabla_{E_3} E_2 = \frac{1}{2}E_1, \quad \nabla_{E_3} E_3 = 0.$$

From the about represantations, one can easily see that $(\varphi, \xi, \eta, \bar{g})$ satisfies the formula $\nabla_X \xi = -\beta\phi X$. Hence, $M(\varphi, \xi, \eta, \bar{g})$ is a three dimensional quasi-Sasakian manifold with the structure function with $\beta = -\frac{1}{2}$.

Using the above relations we have the components of the curvature tensor ass follows

$$R(E_1, E_2)E_3 = 0, \quad R(E_2, E_3)E_3 = -\frac{1}{4}E_2, \quad R(E_1, E_3)E_3 = -\frac{1}{4}E_1,$$
$$R(E_1, E_2)E_2 = \frac{3}{4}E_1, \quad R(E_3, E_2)E_2 = -\frac{1}{4}E_3, \quad R(E_1, E_3)E_2 = 0,$$
$$R(E_1, E_2)E_1 = \frac{3}{4}E_2, \quad R(E_2, E_3)E_1 = 0, \quad R(E_3, E_1)E_1 = \frac{3}{4}E_3.$$

Now, we see that

$$S(E_1, E_1) = g(R(E_1, E_2)E_2, E_1) + g(R(E_1, E_3)E_3, E_1) = \frac{1}{2},$$
$$S(E_2, E_2) = g(R(E_2, E_1)E_1, E_2) + g(R(E_2, E_3)E_3, E_2) = \frac{1}{2},$$
$$S(E_3, E_3) = g(R(E_3, E_1)E_1, E_3) + g(R(E_3, E_2)E_2, E_3) = \frac{1}{2}$$

and

$$S(E_i, E_j) = 0, \quad (i \neq j).$$

Therefore, from (2.4) we obtain

$$\lambda = \frac{1}{2}\left(p + \frac{2}{3}\right).$$

So, one can verify our result from above equations.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript. All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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