MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.1084371
11 (3) 112-128 (2023) - Research Article
ISSN: 2147-6268

Global Solution and Blow-up for a Thermoelastic System of p-Laplacian Type with Logarithmic Source

Carlos Alberto Raposo da Cunha*, Adriano Pedreira Cattai, Octavio Paulo Vera Villagran, Ganesh Chandra Gorain and Ducival Carvalho Pereira

Abstract
This manuscript deals with global solution, polynomial stability and blow-up behavior at a finite time for the nonlinear system

$$\begin{cases} u'' - \Delta_p u + \theta + \alpha u' = |u|^{p-2}\, u \ln |u| \\ \theta' - \Delta\theta = u' \end{cases}$$

where Δ_p is the nonlinear p-Laplacian operator, $2 \leq p < \infty$. Taking into account that the initial data is in a suitable stability set created from the Nehari manifold, the global solution is constructed by means of the Faedo-Galerkin approximations. Polynomial decay is proven for a subcritical level of initial energy. The blow-up behavior is shown on an instability set with negative energy values.

Keywords: *Global solution; blow-up; thermoelastic system of p-Laplacian type; logarithmic source.*

AMS Subject Classification (2020): *Primary: 35A01 ; Secondary: 35B40; 74F05; 93D20.*

**Corresponding author*

1. Introduction

A thermoelastic system is the result of the coupling of a hyperbolic equation with a parabolic equation. As is well known, these systems describe the elastic and thermal behavior of elastic, heat-conducting media, especially the interactions between elastic stresses and temperature differences. The pioneering work on thermoelasticity without p -Laplacian was presented by C. M. Dafermos [1] in 1968. Since then, a great interest has been aroused in different contexts and nowadays there are many results on global and local solutions, stability, and burst behavior of solutions in thermoelasticity theory. We can cite [2–11] with references therein.

Received : *08-03-2022,* ***Accepted :*** *29-07-2022*

*(**Cite as** "C.A. Raposo da Cunha, A.P. Cattai, O.P.V. Villagran, G.C. Gorain, D.C. Pereira, Global Solution and Blow-up for a Thermoelastic System of p-Laplacian Type with Logarithmic Source, Math. Sci. Appl. E-Notes, 11(3) (2023), 112-128")*

Nonlinear hyperbolic problems have always been much studied by mathematicians and physicists. From the mathematical point of view, in [12] was investigated the initial boundary value problem of a nonlinear wave equation with weak and strong damping terms and logarithmic term, and in [13] the viscoelastic wave equation with a strong damping and nonlinearity logarithmic source was considered. In physics, the nonlinear logarithmic source $|u|^{p-2}u\ln|u|$ arises in inflation cosmology, supersymmetric led theories, quantum mechanics, nuclear physics, and fluid mechanics, [14–17].

Regarding global solution for wave equation of p-Laplacian type without an additional dissipation term

$$u'' - \Delta_p u = 0, \tag{1.1}$$

for $n = 1$, M. Derher [18] proved the local in time existence of solution and showed by a generic counter-example that the global in time solution can not be expected. Adding a strong damping $-\Delta u'$ in (1.1) the well-posedness and asymptotic behavior was studied by J. M. Greenberg [19]. In fact, the strong damping plays an important role on the existence and stability for p-Laplacian wave equation see for instance for $n \geq 2$ [20–27]. Nevertheless, if the strong damping is replaced by a weaker damping u', then global existence and uniqueness are only know for $n = 1; 2$, see [28]. For the intermediary damping given by $(-\Delta)^{\alpha}u'$, with $0 < \alpha \leq 1$, in [29] was proved the global solution depending on the growth of a forcing term. The background of these problems are in physics, especially in solid mechanics. The p-Laplacian problem for the electromagnetic effects in high-temperature Type II superconductors is considered in [30] where authors presented an extension of previous work on relaxation schemes applied to degenerate parabolic problems. Global boundedness of weak solution in an attraction–repulsion chemotaxis system with p-Laplacian diffusion was considered in [31]. In [32], the entire blow-up solutions for a quasilinear p-Laplacian Schrödinger elliptic equation with a non-square diffusion term. By using the dual approach and some new iterative techniques, the difficulty due to the non-square diffusion term and the p-Laplacian operator is overcome and the nonexistence and existence of entire blow-up solutions are established.

Thermoelastic problems involving the p-Laplacian are becoming the new object of research. The following thermoelastic system which contains corner-edge Laplacian and p-Laplacian type operators with potential function

$$u'' - \Delta_{p,\mathbb{K}}u - \varepsilon V(\tilde{x})u + \theta = |u|^{\alpha-1}u,$$
$$\theta' - \Delta_{\mathbb{K}}u = u',$$

with $\alpha > 1$ was studied in [33] where $\mathbb{K}$ is the stretched manifold with respect to the manifold K with corner-edge singularity and $\tilde{x} \in \mathbb{K}$. The operator $\Delta_{p,\mathbb{K}} + \varepsilon V(\tilde{x})$ with $p \neq 2$ arises from a diversity of physical phenomena, like in reaction-diffusion problems, in nonlinear elasticity, in non-Newtonian fluids and petroleum extraction. In [34] the relationship with non-Newtonian Mechanics was considered. Authors present a full classification of the short-time behavior of the interfaces and local solutions to the nonlinear parabolic p-Laplacian type reaction-diffusion equation of non-Newtonian elastic filtration

$$u' - \left(|u_x|^{p-2}u_x\right)_x + bu^{\beta} = 0,\ 1 < p < 2,\ \beta > 0.$$

In [35] was studied the problem for a parabolic equation involving fractional p-Laplacian with logarithmic nonlinearity. For $2 \leq p < \infty$ the existence of a global solution for the thermoelastic system of p-Laplacian type given by

$$\left\{ \begin{array}{rcl} u'' - \Delta_p u + \theta & = & |u|^{r-1}u, \\ \theta' - \Delta\theta & = & u'. \end{array} \right. \tag{1.2}$$

has been proven in [36]. Later, in [37], by employing the potential well theory, authors discuss the properties of finite-time blow-up and give the lower and upper bounds of blow-up time to the solutions.

Regarding the model (1.2) in this manuscript, we analyze the competition between the weak damping $\alpha u',\ \alpha > 0$ and the logarithmic source $|u|^{p-2}u\ln|u|$. To our goal we consider the following system

$$u'' - \Delta_p u + \theta + \alpha u' = |u|^{p-2}u\ln|u|\,,\ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.3}$$

$$\theta' - \Delta\theta = u',\ \ (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.4}$$

$$u(x,0) = u_0(x),\ \ u'(x,0) = u_1(x),\ \ \theta(x,0) = \theta_0(x),\ \ x \in \Omega, \tag{1.5}$$

$$u(x,t) = \theta(x,t) = 0 \ \text{ on } \ \partial\Omega \times [0,\infty). \tag{1.6}$$

This paper is organized as follows. In the Section 2, we introduce the notation and some technical lemmas. Section 3 deals with the potential well, we introduce some notations and the stability set for the problem. In the section 4 we introduce a suitable Galerkin basis necessary to deal with the operator p-Laplacian. In the section 5 we prove the existence of global solution by Faedo-Galerkin method. In section 6 we prove the polynomial. Finally in section 7 we prove the blow-up in finite time for initial data in the instability set.

2. Preliminaries

The duality pairing between the space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$ will be denoted using the form $\langle\cdot,\cdot\rangle_p$. According to Poincaré's inequality, the standard norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla\cdot\|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\|\cdot\|_{W_0^{1,p}(\Omega)}=\|\nabla\cdot\|_p$. We denote $\|\cdot\|_{L^2(\Omega)}=|\cdot|_2$ and the usual inner product by $(\cdot,\cdot)$.

Let B be a Banach space and $u:[0,T]\to B$ a mensurable function. We denote by

$$L^p(0,T;B)=\left\{u\;:\;\left(\int_0^T ||u(t)||_B^p\,dt\right)^{1/p}<\infty,\ \text{if}\ 1\le p<\infty\right\},$$

$$L^\infty(0,T;B)=\left\{u\;:\;\sup_{t\in(0,T)}\operatorname{ess}||u(t)||_B<\infty,\ \text{if}\ p=\infty\right\}.$$

The p-Laplacian operator is given by $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right)$. $\Delta_p u$ can be extended to a monotone, bounded, hemicontinuos and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega),\qquad \langle-\Delta_p u,v\rangle_p=\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v\,\mathrm{d}x.$$

We assume that the parameter p satisfies the following assumptions.

(H): $p\ge 2$ if $n=1,2$ and $2\le p\le\dfrac{2n-2}{n-2}$ if $n\ge 3$.

By (H) we have

$$W_0^{1,2(p-1)}(\Omega)\hookrightarrow H_0^1(\Omega)\hookrightarrow L^2(\Omega).$$

Now, we present some results that will be used in this manuscript.

Lemma 2.1 (Kim [38], Lemma 1.4). *Let u_m be a sequence of functions such that as $m\to\infty$*

$$u^m\overset{*}{\rightharpoonup}u \text{ in } L^\infty(0,T;H^\beta(\Omega)),\quad \text{weakly star,}$$
$$u_t^m\rightharpoonup u_t \text{ in } L^2(0,T;H^\alpha(\Omega)),\quad \text{weakly,}$$

where $-1\le\alpha<\beta\le1$. Then, we have

$$u^m\rightharpoonup u \text{ in } C([0,T]\,;\,H^\eta(\Omega)),\quad \text{for any}\ \ \eta<\beta.$$

Lemma 2.2 (Lions [39], Lemma 1.3). *Let $Q=\Omega\times(0,T),\ T>0$ a bounded open set of $\mathbb{R}^n\times\mathbb{R}$ and $g_m,g\,:\,Q\to\mathbb{R}$ functions of $L^p(0,T;L^p(\Omega))=L(Q)$, $1<p<\infty$ such that $||g_m||_{L^p(Q)}\le C$, $g_m\to g$ a.e. in Q. Then*

$$g_m\rightharpoonup g\ \text{ in }\ L^p(0,T;L^p(\Omega))\ \text{as}\ m\to\infty.$$

Lemma 2.3 (Lions-Aubin [39], Theorem 5.1). *Let $T>0$, $1<p_0,p_1<\infty$. Consider $B_0\subset B\subset B_1$ Banach spaces, B_0,B_1 reflexives, B_0 with compact immersion in B. Define $W=\{u\mid u\in L^{p_0}(0,T;B_0)\,,\,u'\in L^{p_1}(0,T;B_1)\}$ equipped with the norm $||u||_W=||u||_{L^{p_0}(0,T;B_0)}+||u||_{L^{p_1}(0,T;B_1)}$. Then, W has compact immersion in $L^{p_0}(0,T;B)$.*

Lemma 2.4 (Martinez [40]). *Let $E:(0,\infty)\to(0,\infty)$ be a nonincreasing function and $\phi:[0,\infty)\to[0,\infty)$ an increasing C^1 function such that $\phi(0)=0$ and $\phi(t)\to\infty$ as $t\to\infty$.*

Assume that there exist $\sigma>-1$ and $\omega>0$ such that

$$\int_S^\infty E^{1+\sigma}(t)\phi'(t)\,dt \le \frac{1}{\omega}E^\sigma(0)E(S),\ 0\le S<\infty.$$

Then

$$\begin{aligned}
&E(t)=0\ \forall t\ge \frac{E(0)^\sigma}{\omega|\sigma|},\ \ \text{if } -1<\sigma<0,\\
&E(t)\le E(0)\big(\frac{1+\sigma}{1+\omega\phi(t)}\big)^{1/\sigma}\ \forall t\ge 0,\ \ \text{if } \sigma>0,\\
&E(t)\le E(0)e^{1-\omega\phi(t)}\ \forall t\ge 0,\ \ \text{if } \sigma=0.
\end{aligned}$$

Lemma 2.5 (Levine [41], Qin-Rivera [42]). *Suppose that $\phi(t)\in C^2[0,\infty)$ is a positive function satisfying*

$$\phi(t)\phi''(t)-(1+\gamma)(\phi'(t))^2\ge -2C_1\phi(t)\phi'(t)-C_2(\phi(t)^2,$$

being $C_1,C_2\ge 0$ and $\gamma>0$ are constants. If

$$C_1+C_2\ge 0,\phi(0)>0,\phi'(0)+\gamma_2\frac{1}{\gamma}\phi(0)>0,$$

then

$$\lim_{t\to T_-}\ \phi(t)=+\infty,$$

where

$$T\le \frac{1}{2\sqrt{C_1^2+\gamma C_2}}\ln\left[\frac{\gamma_1\phi(0)+\gamma\phi'(0)}{\gamma_2\phi(0)+\gamma\phi'(0)}\right],$$

and

$$\gamma_1=-C_1+\sqrt{C_1^2+\gamma C_2},\quad \gamma_2=-C_1-\sqrt{C_1^2+\gamma C_2}.$$

3. The potential well

In this section we use the potential theory, a power full tool in the study of the global existence of solution to partial differential equation. See Payne-Sattinger [43]. It is well-known that the energy of a PDE system, in some sense, splits into the kinetic and the potential energy.

The energy of the problem (1.3)-(1.6) is given by

$$E(t)=\frac{1}{2}\int_\Omega |u'(t)|^2\,\mathrm{d}x+\frac{1}{p^2}\int_\Omega |u(t)|^p\,\mathrm{d}x+\frac{1}{2}\int_\Omega |\theta(t)|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x-\frac{1}{p}\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x.$$

Mutiplying (1.3) by u', (1.4) by θ, performing integration by parts and using (1.6) we obtain

$$\frac{d}{\mathrm{d}t}E(t)=-\alpha||u'(t)||_2^2-||\nabla\theta(t)||_2^2. \tag{3.1}$$

We introduce the functional

$$J(u(t))=\frac{1}{p^2}\int_\Omega |u(t)|^p\,\mathrm{d}x+\frac{1}{p}\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x-\frac{1}{p}\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x.$$

The Nehari functional associated with $J(u(t))$ is $I:\ W_0^{1,p}(\Omega)\cap W_0^{1,2(p-1)}(\Omega)\to\mathbb{R}$ defined by

$$I(u(t))=\int_\Omega |\nabla u(t)|^p\,\mathrm{d}x-\int_\Omega |u(t)|^p\ln|u(t)|\,\mathrm{d}x. \tag{3.2}$$

Associated with the $J(\lambda u(t))$ we have the well known *Nehari Manifold* given by

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \left[\frac{\mathrm{d}}{\mathrm{d}\,\lambda} I(\lambda u(t)) \right]_{\lambda=1} = 0 \right\}$$
$$= \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x = \int_\Omega |u(t)|^p \ln |u(t)| \,\mathrm{d}x \right\}.$$

Now, we introduce the potential well (stable set)

$$\mathcal{W}_1 = \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x > \int_\Omega |u(t)|^p \ln |u(t)| \,\mathrm{d}\,x \right\} \cup \{0\}.$$

and the unstable set

$$\mathcal{W}_2 = \left\{ u(t) \in W_0^{1,p}(\Omega) \cap W_0^{1,2(p-1)}(\Omega)/\{0\} : \int_\Omega |\nabla u(t)|^p \,\mathrm{d}x < \int_\Omega |u(t)|^p \ln |u(t)| \,\mathrm{d}\,x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [44],

$$d \stackrel{\text{def}}{=} \inf_{u(t) \in W_0^{1,p}(\Omega)/\{0\}} \sup_{0 \le \lambda} J(\lambda u(t)).$$

It is well-known that under H the depth of the well d is a strictly positive constant, see [[45], Theorem 4.2], and

$$d = \inf_{u(t) \in \mathcal{N}} J(u(t)).$$

The source term induces a potential energy in the system that act in opposed to effect of the stabilizing mechanism. In this sense, it is possible that the energy from the source term destabilize all the system and produce a blow-up a finite time. For provide a global solution, the stability set $\mathcal{W}_1$ create a valley or a well of the depth d, see Y. Ye [27], where the potential energy of the solution can never escape the potential well.

We will prove that $\mathcal{W}_1$ is invariant set for sub-critical initial energy.

Proposition 3.1. *Let $u_0 \in \mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. If $E(0) < d$ then $u(t) \in \mathcal{W}_1$.*

Proof. Let $T > 0$ be the maximum existence time. From (3.1) we get

$$E(t) \le E(0) < d, \text{ for all } t \in [0, T).$$

and then,

$$\frac{1}{2} \int_\Omega |u'(t)|^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\theta(t)|^2 \,\mathrm{d}x + J(u(t)) < d, \text{ for all } t \in [0, T),$$

that is,

$$E(t) < d, \text{ for all } t \in [0, T). \tag{3.3}$$

Arguing by contradiction, we suppose that there exists a first $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) > 0$ for all $0 \le t < t_0$, that is,

$$\int_\Omega |\nabla u(t_0)|^p \,\mathrm{d}x = \int_\Omega |u(t_0)|^p \ln |u(t_0)| \,\mathrm{d}x.$$

From the definition of $\mathcal{N}$, we have that $u(t_0) \in \mathcal{N}$, which leads to

$$J(u(t_0)) \ge \inf_{u(t) \in \mathcal{N}} J(u(t)) = d.$$

By definition of $E(t)$,

$$\frac{1}{2} \int_\Omega |u'(t_0)|^2 \,\mathrm{d}x + \frac{1}{2} \int_\Omega |\theta(t_0)|^2 \,\mathrm{d}x + J(u(t_0)) \ge d, \text{ it holds that, } E(t_0) \ge d,$$

which contradicts with (3.3). Then $u(t) \in \mathcal{W}_1$ for all $t \in [0, T)$. □

4. Galerkin basis

From Sobolev immersion, we have

$$W_0^{\nu,q}(\Omega) \hookrightarrow W_0^{\nu-k,q_k}(\Omega), \ \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $\nu - k = 1$, and $q = 2$, we get

$$\nu = 1 + \frac{n}{2} - \frac{n}{p} = 1 + \frac{n(p-2)}{2p} > 0$$

and we obtain a Hilbert Space $H_0^\nu(\Omega)$ such that

$$H_0^\nu(\Omega) = W_0^{\nu,2}(\Omega) \hookrightarrow W_0^{1,p}(\Omega).$$

Let s an integer for which $s > \nu$. We have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

According to the Rellich-Kondrachov theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, so is also the immersion $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$. From spectral theory, there exists an operator defined by

$$\{H_0^s(\Omega), \ L^2(\Omega), \ (\!(\cdot,\cdot)\!)_{H_0^s(\Omega)}\}$$

and a sequence of eigenvectors $(v_j)_{j\in\mathbb{N}}$ of this operator such that

$$(\!(v_j, v)\!)_{H_0^s(\Omega)} = \lambda_j(v_j, v), \ \text{for all } v \in H_0^s(\Omega)$$

with $\lambda_j > 0, \ \lambda_j \leq \lambda_{j+1}, \ \text{ and } \ \lambda_j \to +\infty \ \text{ as } \ j \to +\infty$. Moreover $(v_j)_{j\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\Omega)$ and $\left(w_j = \frac{v_j}{\sqrt{\lambda_j}}\right)_{j\in\mathbb{N}}$ is a complete orthonormal system in $H_0^s(\Omega)$. Then $(w_j)_{j\in\mathbb{N}}$ yields a "Galerkin basis" for both $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$.

5. Global solution

Theorem 5.1. *Consider $E(0) < d$. Given $u_0 \in \mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$, there exist functions $u, \theta\colon \Omega \times (0,T) \to \mathbb{R}$ in the class*

$$\begin{aligned} u &\in L^\infty(0,T; W_0^{1,p}(\Omega)),\\ u' &\in L^\infty(0,T; L^2(\Omega)),\\ \theta &\in L^\infty(0,T; H_0^1(\Omega)), \end{aligned}$$

such that, for all $\phi \in W_0^{1,p}(\Omega)$, $\psi \in L^2(\Omega)$

$$\frac{d}{dt}(u',\phi) + \langle -\Delta_p u, \phi\rangle_p + (\theta,\phi) = (|u|^{p-2}u\ln|u|, \phi) \ \textit{in } D'(0,T), \tag{5.1}$$

$$\frac{d}{dt}(\theta,\psi) + (-\Delta\theta, \psi) = (u',\psi) \ \textit{in } D'(0,T), \tag{5.2}$$

$$u(x,0) = u_0(x), \ \ u'(x,0) = u_1(x), \ \ \theta(x,0) = \theta_0(x) \ \ a.e. \ in \ \Omega. \tag{5.3}$$

Proof. Let's use the Galerkin basis obtained in the previous section. For each $m \in \mathbb{N}$, let us put

$$V_m = \text{Span}\{w_1, w_2, \ldots, w_m\}.$$

We search for functions

$$u_m(t) = \sum_{j=1}^m f_{jm}(t)w_j, \ \ \theta_m(t) = \sum_{j=1}^m g_{jm}(t)w_j,$$

such that any $\phi, \psi \in V_m$, $u_m(t)$ and $\theta_m(t)$ satisfies the following approximate problem

$$\frac{d}{\mathrm{d}t}(u_m'(t),\phi)+\langle -\Delta_p u_m(t),\phi\rangle_p + (\theta_m(t),\phi) = (|u_m(t)|^{p-2}u_m(t)\ln|u_m(t)|,\phi), \tag{5.4}$$

$$\frac{d}{\mathrm{d}t}(\theta_m(t),\psi) + (-\Delta\theta_m(t),\psi) = (u_m'(t),\psi), \tag{5.5}$$

with the initial conditions $u_m(0) = u_{0m}$, $u_m'(0) = u_{1m}$ and $\theta_m(0) = \theta_{0m}$, where u_{0m}, u_{1m} and θ_{0m} are choose so that

$$u_{0m} \to u_0 \in W_0^{1,p}(\Omega),\ u_{1m} \to u_1 \ in\ L^2(\Omega) \ \text{ and } \ \theta_{0m} \to \theta_0 \ in\ H_0^1(\Omega). \tag{5.6}$$

Putting $\phi = w_i$, $\psi = w_i$, $i = 1, 2, \ldots, m$, and using

$$u_m''(t) = \sum_{j=1}^m f_{jm}''(t)w_j(x), \quad \Delta_p u_m(t) = \sum_{j=1}^m f_{jm}(t)\Delta_p w_j(x),$$
$$\theta_m'(t) = \sum_{j=1}^m g_{jm}'(t)w_j(x), \quad \Delta\theta_m(t) = \sum_{j=1}^m g_{jm}(t)\Delta w_j(x),$$

we observe that (5.4)-(5.5) leads to a system of ODEs in the variable t that has a local solution $u_m(t), \theta_m(t)$ in a interval $[0, t_m)$ by virtue of Carathéodory's theorem. In the next step we obtain a priori estimates for the solution $u_m(t), \theta_m(t)$ so that they can be extended to the whole interval $[0, T]$, $T > 0$.

5.1 A priori estimates

Replacing $\phi = u_m'(t)$, $\psi = \theta_m(t)$ in the approximate equation (5.4), (5.5) we get

$$(u_m''(t), u_m'(t)) + \langle -\Delta_p u_m(t), u_m'(t)\rangle_p + (\theta_m(t), u_m'(t)) = (|u_m(t)|^{p-2}u_m(t)\ln|u_m(t)|, u_m'(t)), \tag{5.7}$$
$$(\theta_m'(t), \theta_m(t)) + (-\Delta\theta_m(t), \theta_m(t)) = (u_m'(t), \theta_m(t)), \tag{5.8}$$

Let $z \in D(0, t_m)$. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality pairing between D' and D. So we have

$$\langle (u_m''(t), u_m'(t)), z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.9}$$

$$\langle \langle -\Delta_p u_m(t), u_m'(t)\rangle_p, z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\int_\Omega |\nabla u_m(t)|^p\,\mathrm{d}x, z\right\rangle, \tag{5.10}$$

$$\langle (u_m'(t), u_m'(t)), z\rangle = \left\langle \int_\Omega |u_m'(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.11}$$

$$\begin{aligned}\big\langle (|u_m(t)|^{p-2}u_m(t)\ln u_m(t), u_m'(t)), z\big\rangle &= \left\langle \frac{1}{p}\frac{d}{\mathrm{d}t}\int_\Omega |u_m(t)|^p \ln u_m(t)\,\mathrm{d}x, z\right\rangle \\ &\quad - \left\langle \frac{1}{p^2}\frac{d}{\mathrm{d}t}\int_\Omega |u_m(t)|^p\,\mathrm{d}x, z\right\rangle,\end{aligned} \tag{5.12}$$

$$\langle (\theta_m'(t), \theta_m(t)), z\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x, z\right\rangle, \tag{5.13}$$

$$\langle (-\Delta\theta_m(t), \theta_m(t))\,, z\rangle = \left\langle \int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x, z\right\rangle. \tag{5.14}$$

Replacing (5.9), (5.10), (5.11), (5.12), (5.13), (5.14) in (5.7) and (5.8) we obtain in $D'(0, t_m)$

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) = -\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x - \int_\Omega |u_m'(t)|^2\,\mathrm{d}x, \tag{5.15}$$

from where follows that the approximate energy

$$\begin{aligned}E_m(t) &= \frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x + \frac{1}{p^2}\int_\Omega |u_m(t)|^p\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u_m(t)|^p\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u_m(t)|^p \ln|u_m(t)|\,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(t)|^2\,\mathrm{d}x + J(u(t))\end{aligned}$$

satisfies

$$\begin{aligned}E_m(t) &\le E_m(0)\\ &= \frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(0)|^2\,\mathrm{d}x + J(u_m(0)).\end{aligned}$$

We have that $J(u_m(0)) < d$ in $\mathcal{W}_1$. By to convergence of initial data (5.6), there exists a constant $C > 0$ independent of t and m such that

$$\frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\theta_m(0)|^2\,\mathrm{d}x \le C.$$

With the estimate $E_m(t) \le E_m(0) \le C$ we can extend the approximate solutions $u_m(t)$, $\theta_m(t)$ to the interval $[0,T], T > 0$. By using (5.15) we deduce

$$\int_0^T\!\!\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega |u_m'(t)|^2\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\!\!\int_\Omega |\nabla\theta_m(t)|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega |u_m'(t)|^2\,\mathrm{d}x\,\mathrm{d}t + E_m(t) \le E_m(0) \le C. \tag{5.16}$$

To prove that (1.4)-(1.6) carrying a good energy structure in $\mathcal{W}_1$, we need show that the forcing term is $L^2(0,T;L^2(\Omega))$. Consider $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{x \in \Omega : |u_m(t)(x)| \le 1\} \text{ and } \Omega_2 = \{x \in \Omega : |u_m(t)(x)| > 1\}.$$

From

$$\int_\Omega ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x = \int_{\Omega_1} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x + \int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x.$$

We have

$$\int_{\Omega_1} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le |\Omega|. \tag{5.17}$$

Note that,

$$\begin{aligned}\int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x &= \int_{\Omega_2} |u_m(t)|^{2p-4}|u_m(t)|^2|\ln|u_m(t)||^2\,\mathrm{d}x\\ &\le \int_{\Omega_2} |u_m(t)|^{2p-4}|u_m(t)|^4|\ln|u_m(t)||^2\,\mathrm{d}x\\ &= \int_{\Omega_2} |u_m(t)|^{2p}|\ln|u_m(t)|^2\,\mathrm{d}x\\ &= \int_{\Omega_2} ||u_m(t)|^p\ln|u_m(t)||^2\,\mathrm{d}x.\end{aligned}$$

Taking into account that $u_m(t) \in \mathcal{W}_1$ we obtain

$$\int_{\Omega_2} ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le \int_\Omega |\nabla u|^p\,\mathrm{d}x. \tag{5.18}$$

From (5.17) and (5.18) we get

$$\int_\Omega ||u_m(t)|^{p-2}u_m(t)\ln|u_m(t)||^2\,\mathrm{d}x \le |\Omega| + \int_\Omega |\nabla u|^p\,\mathrm{d}x \le C. \tag{5.19}$$

Then we have

$$u_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{5.20}$$
$$u_m'(t) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{5.21}$$
$$u_m'(t) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{5.22}$$
$$|u_m(t)|^{p-2}u_m(t)\ln u_m(t) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{5.23}$$
$$-\Delta_p u_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,p'}(\Omega)), \tag{5.24}$$
$$\theta_m(t) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{5.25}$$
$$-\Delta\theta_m(t) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). \tag{5.26}$$

Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$ we can use the standard projection arguments as described in Lions [39], pages 75-76, to obtain an estimate for $u''_m(t)$. Let P_m be the orthogonal projection $P_m : L^2(\Omega) \to V_m$, that is

$$P_m h = \sum_{n=1}^{m} (h, w_j) w_j, \quad h \in L^2(\Omega).$$

Approximated problem (5.7) leads to

$$u''_m(t) = P_m \Delta_p u_m(t) - P_m \theta_m(t) - P_m u'_m(t) + P_m |u_m(t)|^{p-2} u_m(t) \ln |u_m(t)|.$$

As $-\Delta_p u_m(t) \in L^2(0, T; (W^{-1,p'}(\Omega))$, from estimates (5.23), (5.25) we obtain

$$u''_m(t) \quad \text{is bounded in} \quad L^\infty(0, T; W^{-1,p'}(\Omega)). \tag{5.27}$$

5.2 Passage to the limit

From (5.20)-(5.27) going to the suitable subsequence if necessary (which we continue to denote in the same way), there exist $u(t)$, $\theta(t)$ such that

$$\begin{aligned}
u_m(t) &\overset{*}{\rightharpoonup} u(t) \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega)), &(5.28)\\
u'_m(t) &\overset{*}{\rightharpoonup} u'(t) \text{ in } L^\infty(0, T; L^2(\Omega)), &(5.29)\\
u'_m(t) &\rightharpoonup u'(t) \text{ in } L^2(0, T; L^2(\Omega)), &(5.30)\\
-\Delta_p u_m(t) &\overset{*}{\rightharpoonup} \mathcal{X}_1(t) \text{ in } L^\infty(0, T; W^{-1,p'}(\Omega)), &(5.31)
\end{aligned}$$

$$\begin{aligned}
|u_m(t)|^{p-2} u_m(t) \ln u_m(t) &\rightharpoonup \mathcal{X}_2(t) \text{ in } L^2(0, T; L^2(\Omega)), &(5.32)\\
\theta_m(t) &\rightharpoonup \theta(t) \text{ in } L^2(0, T; L^2(\Omega)), &(5.33)\\
-\Delta \theta_m(t) &\overset{*}{\rightharpoonup} -\Delta \theta(t) \text{ in } L^\infty(0, T; L^2(\Omega). &(5.34)
\end{aligned}$$

Applying the Lions-Aubin compactness lemma, from (5.27), (5.28) and (5.29) we get

$$\begin{aligned}
u_m(t) &\to u(t) \text{ strongly in } L^2(0, T; L^2(\Omega)) \text{ and } a.e. \text{ in } Q, &(5.35)\\
u'_m(t) &\to u'(t) \text{ strongly in } L^2(0, T; L^2(\Omega)) \text{ and } a.e. \text{ in } Q. &(5.36)
\end{aligned}$$

We need to prove that $\mathcal{X}_1(t) = -\Delta_p u(t)$. The following elementary inequality

$$\left| |x|^{p-2} x - |y|^{p-2} y \right| \le C \left(|x|^{p-2} + |y|^{p-2} \right) |x - y| \tag{5.37}$$

is a consequence of the Mean Value Theorem. Using (5.37) and Hölder generalized inequality with

$$\frac{p-2}{2(p-1)} + \frac{1}{2} + \frac{1}{2(p-1)} = 1,$$

we deduce, for $z \in \mathcal{D}(0, T)$ and $v \in V_m$, that

$$\begin{aligned}
\left| \int_0^T \langle (-\Delta_p u_m(t)) - (-\Delta_p u(t)), v \rangle z(t) dt \right| &= \left| \int_0^T \int_\Omega \left(|\nabla u_m(t)|^{p-2} \nabla u_m(t) - |\nabla u(t)|^{p-2} \nabla u(t) \right) \nabla v \; dx z(t) \, dt \right| \\
&\le C |\theta|_\infty \int_0^T \int_\Omega \left(|\nabla u_m(t)|^{p-2} + |\nabla u(t)|^{p-2} \right) |\nabla u_m(t) - \nabla u(t)| |\nabla v| \, dx dt \\
&\le C_1 \int_0^T \left(\|\nabla u_m(t)\|_{2(p-1)}^{p-2} + \|\nabla u(t)\|_{2(p-1)}^{p-2} \right) |\nabla u_m(t) - \nabla u(t)| \, \|\nabla v\|_{2(p-1)} dt,
\end{aligned}$$

that leads to

$$\left| \int_0^T \langle (-\Delta_p u_m(t)) - (-\Delta_p u(t)), v \rangle_p \, z(t) \, \mathrm{d}t \right| \le C \int_0^T |\nabla u_m(t) - \nabla u(t)| \, \mathrm{d}t. \tag{5.38}$$

Now, from (5.28) and (5.29), by lemma 2.1 we have

$$u_m \to u \text{ in } C([0,T]\,;\, L^2(\Omega)).$$

whence

$$\nabla u_m(t) \to \nabla u(t) \text{ a. e. in } [0,T].$$

Therefore, by (5.31) and (5.38) we have $\mathcal{X}_1(t) = -\Delta_p u$, that is

$$-\Delta_p u_m(t) \rightharpoonup -\Delta_p u(t) \text{ in } L^2(0,T; W^{-1,p'}(\Omega)), \tag{5.39}$$

Now we will prove $\mathcal{X}_2(t) = |u(t)|^{p-2}u(t)\ln u(t)$. From (5.19) we have

$$|u_m|^{p-2}u_m \ln|u_m| \text{ is bounded in } L^2(0,T; L^2(\Omega)) = L^2(Q). \tag{5.40}$$

Using continuity of function $s \to |s|^{p-2}s\ln|s|$ and (5.35) we have

$$|u_m|^{p-2}u_m \ln|u_m| \to |u|^{p-2}u\ln|u| \text{ a.e. in } Q. \tag{5.41}$$

Then, by using Lions's lemma, (5.40) and (5.41) leads to

$$|u_m|^{p-2}u_m \ln|u_m| \rightharpoonup |u|^{p-2}u\ln|u| \text{ in } L^2(0,T; L^2(\Omega)). \tag{5.42}$$

Now, with the convergences (5.29), (5.39), (5.42), (5.33) and (5.34) we can pass to the limit in the approximate system and we get (5.1),(5.2). The verification of the initial data is a routine procedure. The prove of existence is complete. □

6. Polynomial decay for $E(0) < d$

In this section, we prove the $||u||_p^p$ decay polynomially for subcritical level of initial energy.

Theorem 6.1. *Let u_0 in the stability set $\mathcal{W}_1$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. If $E(0) < d$ then the weak solution $u(t)$ of the problem (1.3)-(1.6) decay polynomially. That is,*

$$\|u(t)\|_p^p \le \|u(0)\|_p^p \left[\frac{1+\sigma}{1+\omega t}\right]^{\frac{1}{\sigma}}$$

where $\sigma > \dfrac{1}{2}$, $\omega = \dfrac{[\|u(0)\|_p^p]^\sigma}{C}$, $C > 0$.

Proof. As $\ln|u| \le |u|$, we have

$$\int_\Omega |u|^p \ln|u|\,\mathrm{d}x \le \int_\Omega |u|^{p+1}\,\mathrm{d}x = \|u\|_{p+1}^{p+1}.$$

By Hölder inequality we obtain

$$\|u\|_{p+1}^{p+1} \le \|u\|_p^{\nu(p+1)}\|u\|_q^{(1-\nu)(p+1)}, \;\; \nu \in (0,1).$$

Applying Young inequality

$$\|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|_p^{\nu(p+1)p} + \frac{C_0(\varepsilon)}{q}\|u\|_q^{q(1-\nu)(p+1)}$$

with $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $q < p$, and then,

$$\|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|_p^{\nu(p+1)p} + C(\varepsilon)\|u\|_q^{q(1-\nu)(p+1)}.$$

For $\nu = \dfrac{1}{2}$ we have

$$\int_\Omega |u|^p \ln|u|\,\mathrm{d}x \le \|u\|_{p+1}^{p+1} \le \frac{\varepsilon}{p}\|u\|^{[\frac{p+1}{2}]p} + C(\varepsilon)\|u\|_q^{[\frac{p+1}{2}]q}. \tag{6.1}$$

We define

$$L(t) = N\|u\|_q^{[\frac{p+1}{2}]q} + \|\nabla u\|_p^p - \int_\Omega |u|^p \ln|u| \,\mathrm{d}x. \tag{6.2}$$

As $u \in W_1$ we get $L(t) > 0$. By using (6.1) and Poincaré inequality in (6.2) we obtain

$$\begin{aligned} L(t) &\geq N\|u\|_q^{[\frac{p+1}{2}]q} + C_p\|u\|_p^p - \|u\|_{p+1}^{p+1} \\ &\geq N\|u\|_q^{[\frac{p+1}{2}]q} + C_p\|u\|_p^p - \frac{\varepsilon}{p}\|u\|_p^{[\frac{p+1}{2}]p} - C(\varepsilon)\|u\|_p^{[\frac{p+1}{2}]q} \\ &\geq (N - C(\varepsilon))\,\|u\|_q^{[\frac{p+1}{2}]q} + \|u\|_p^p\left(C_p - \frac{\varepsilon}{p}\right)\|u\|_p^{\frac{p+1}{2}} \end{aligned}$$

Choosing $N, \varepsilon > 0$ such that $C_p - \dfrac{\varepsilon}{p} > C > 0$ and $N - C(\varepsilon) > 0$ we have

$$L(t) \geq C\left[\|u\|_p^p\right]^{\frac{p+1}{2}}.$$

As $p > 2$,

$$\begin{aligned} \frac{p+1}{2} &= \frac{p}{2} + \frac{1}{2} = \frac{p}{2} + \frac{1}{2} - 1 + 1 = \frac{p}{2} - \frac{1}{2} + 1 \\ &= \sigma + 1, \;\; \sigma > \frac{1}{2}. \end{aligned}$$

Then

$$\|u\|_p^{p[\frac{p+1}{2}]} = \left[\|u\|_p^p\right]^{\sigma+1}, \;\; \sigma > \frac{1}{2} \; and$$

we obtain

$$L(t) \geq C\left[\|u\|_p^p\right]^{\sigma+1}, \;\; \sigma > \frac{1}{2}. \tag{6.3}$$

By other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \leq p^2\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq 0,$$

that is, $\|u(t)\|_p^p$ is nonincreasing function. Then $-\dfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \geq 0$. For each $\infty > T > S \geq 0$, let $t > 0$ such that $t \in (S, T)$ and define

$$A = \left\{t \in (S,T)\,;\; -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p > L(t)\right\}.$$

If $t \in (S, T)$ satisfy

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p \leq L(t)$$

consider $0 < \eta(t) < \infty$ such that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\eta(t) \geq L(t),$$

and take

$$\overline{A} = \left\{t \in (S,T)\,;\; -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\eta(t) \geq L(t)\right\}.$$

Let

$$\eta = \sup\{\eta(t);\; t \in \overline{A},\; 0 < \eta(t) < \infty\}.$$

Then $0 < \eta < \infty$ and

$$\begin{aligned} \int_S^T L(t)\,\mathrm{d}t &= \int_A L(t)\,\mathrm{d}t + \int_{\overline{A}} L(t)\,\mathrm{d}t \\ &\leq (1+\eta)\int_S^T -\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p\,\mathrm{d}t \\ &\leq (1+\eta)\|u(S)\|_p^p, \;\; \forall\, S \geq 0. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4)

$$\begin{aligned}\int_S^T \left[\|u(t)\|_p^p\right]^{\sigma+1} \mathrm{d}t &\le C^{-1} \int_S^T L(t)\,\mathrm{d}t \\ &\le C^{-1}(1+\eta)\|u(S)\|_p^p \\ &\le \frac{1}{\omega}\left[\|u(0)\|_p^p\right]^{\sigma}\|u(S)\|_p^p\end{aligned}$$

where $\omega = \dfrac{[\|u(0)\|_p^p]^\sigma}{C^{-1}(1+\eta)}$.

From Lemma 2.4, with $E(t) = \|u(t)\|_p^p$ and $\phi(t) = t$ we obtain

$$\|u(t)\|_p^p \le \|u(0)\|_p^p \left[\frac{1+\sigma}{1+\omega t}\right]^{\frac{1}{\sigma}}$$

where $\sigma > \dfrac{1}{2}$, $\omega > 0$, $C > 0$. ☐

7. Blow-up in finite time

As in section 3 we can prove that $\mathcal{W}_2$ is invariant for sub-critical initial energy, that is,

Proposition 7.1. *Let $u_0 \in \mathcal{W}_2$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$. If $E(0) < d$ then $u(t) \in \mathcal{W}_2$.*

Theorem 7.1. *Let u_0 in the instability set $\mathcal{W}_2$, $u_1 \in L^2(\Omega)$, $\theta_0 \in H_0^1(\Omega)$ and $r > 1$ a fixed real number. If $||u_0||_2^2 < \sqrt{r-1}(u_0, u_1)$ and $E(0) < d$ then the weak solution $u(t)$ of the problem (1.3)-(1.6) will blow up at finite time. Namely, the maximum existence time $T < \infty$ and*

$$\lim_{t \to T_-} ||u(t)||_p^p = +\infty,$$

where

$$T < \frac{1}{\sqrt{r-1}} \ln\left[\frac{(r-1)(u_0,u_1) + \sqrt{r-1}||u_0||_2^2}{(r-1)(u_0,u_1) - \sqrt{r-1}||u_0||_2^2}\right].$$

Proof. By contradiction, suppose that the solution $u(t) \in \mathcal{W}_2$ is global. That is, we let $T = \infty$. Let $\phi(t) = |u(t)|^2$. We have $\phi'(t) = 2(u(t), u'(t))$. Applying Hölder inequality we get

$$2(u(t), u'(t)) \le 2|u(t)|\,|u'(t)|$$

and

$$[\phi'(t)]^2 \le 4|u(t)|^2\,|u'(t)|^2$$

that leads to

$$[\phi'(t)]^2 \le 4\phi(t)|u'(t)|^2. \tag{7.1}$$

We have

$$(u''(t), u(t)) = -\|\nabla u(t)\|_p^p - \int_\Omega u(t)\theta(t)\,\mathrm{d}x - \frac{\alpha}{2}\frac{d}{\mathrm{d}t}|u(t)|^2 + \int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.$$

Note that,

$$\begin{aligned}\phi''(t) &= 2|u'(t)|^2 + 2(u''(t), u(t)) \\ &= 2|u'(t)|^2 - 2\|\nabla u(t)\|_p^p - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2 + 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.\end{aligned}$$

By using

$$I(u(t)) = \|\nabla u(t)\|_p^p - \int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x$$

we get

$$\phi''(t) = 2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2 + 2\int_\Omega |u(t)|^p \ln|u(t)|\,\mathrm{d}x.$$

that is

$$\phi''(t) = 2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x - \alpha\frac{d}{\mathrm{d}t}|u(t)|^2.$$

Let $r > 0$ be a real number. By using (7.1) we obtain

$$\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 \geq \phi(t)\left(2|u'(t)|^2 - 2I(u(t)) - 2\int_\Omega u(t)\theta(t)\,\mathrm{d}x\right) - \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2 - (r+3)\phi(t)|u'(t)|^2.$$

Applying Young inequality we get

$$\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 \geq \phi(t)\left[-(r+1)|u'(t)|^2 - 2I(u(t)) - |u(t)|^2 - |\theta(t)|^2\right] - \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2. \tag{7.2}$$

From,

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|\theta(t)|^2 + J(u(t)).$$

we get

$$\begin{aligned}\frac{1}{2}|u'(t)|^2 &= -\frac{1}{2}|\theta(t)|^2 + E(t) - J(u(t)),\\ &\leq -\frac{1}{2}|\theta(t)|^2 + E(0) - J(u(t)),\\ &\leq -\frac{1}{2}|\theta(t)|^2 + d - J(u(t)).\end{aligned}$$

Then,

$$-(r+1)|u'(t)|^2 \geq (r+1)|\theta(t)|^2 + 2(r+1)(J(u(t)) - d). \tag{7.3}$$

By using (7.3) in (7.2) we obtain

$$\begin{aligned}\phi(t)\phi''(t) - \frac{r+3}{4}(\phi'(t))^2 \geq{}& \phi(t)\big[(r+1)|\theta(t)|^2 - |\theta(t)|^2\big]\\ &+ \phi(t)\big[2(r+1)(J(u(t)) - d)\big] + \phi(t)\big[-2I(u(t))\big]\\ &- \alpha\phi(t)\frac{d}{\mathrm{d}t}|u(t)|^2 - \phi(t)|u(t)|^2.\end{aligned}$$

Now, observe that $\big[(r+1)|\theta(t)|^2 - |\theta(t)|^2\big] > 0$, $-2I(u(t)) > 0$ in $\mathcal{W}_2$, and $J(u(t)) - d > 0$ because

$$d = \inf_{u\in\mathcal{N}} J(u).$$

Namely, we have

$$\phi(t)\phi''(t) - (1+\gamma)(\phi(t))^2 \geq -2c_1\phi(t)\phi'(t) - c_2(\phi(t))^2,$$

where $c_1 = \dfrac{\alpha}{2}$, $c_2 = 1$, $\gamma = \dfrac{r-1}{4}$. By $\sqrt{r-1}(u_0, u_1) > |u_0|^2$, $c_1 + c_2 > 0$, $\phi(0) > 0$ we get $\phi'(0) + \gamma_2\gamma^{-1}\phi(0) > 0$, for $\gamma_1 = \dfrac{\sqrt{r-1}}{2}$ and $\gamma_2 = -\dfrac{\sqrt{r-1}}{2}$.

Finally, from Lemma 2.5 we concludes that

$$\lim_{t\to T_-} \|u(t)\|_p^p \geq c\lim_{t\to T_-} |u(t)|^2 = +\infty,$$

where

$$T < \frac{1}{\sqrt{r-1}}\ln\left[\frac{(r-1)(u_0,u_1) + \sqrt{r-1}|u_0|^2}{(r-1)(u_0,u_1) - \sqrt{r-1}|u_0|^2}\right],$$

which contradicts $T = \infty$. Then $u(t)$ blows up in finite time. $\square$

8. Final comment

In recent years, results on global well-posedness, local well-posedness, blow-up, and asymptotic behavior of thermoelastic system have been studied. However, when considering the p-Laplacian operator, few results are known. We analyze the competition between the logarithmic source and the stabilization power given by the temperature difference. We show the existence of a global solution and the polynomial decay in a suitable stability set created from the Nehari Manifold. On the other hand, we prove the blow-up in finite time out of the stability set. We hope that the results presented here will be a font of inspiration for future research related to the topic.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable.

References

[1] Dafermos, C. M.: *On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity*. Arch. Rational Mech. Anal. **29**, 241-271 (1968).

[2] Chen, W.: *Cauchy problem for thermoelastic plate equations with different damping mechanisms*. Commun. Math. Sci. **18**, 429-457 (2020).

[3] Fareh, A., Messaoudi, S. A.: *Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy*. Appl. Math. Comput. **293**, 493-507 (2017).

[4] Feng, B.: *On a thermoelastic laminated Timoshenko beam: well posedness and stability*. Complexity **2020**, 5139419 (2020).

[5] Kafini, M., Messaoudi, S. A., Mustafa, M. I.: *Energy decay result in a Timoshenko-type system of thermoelasticity of type III with distributive delay*. J. Math. Phys. **54**, 101503 (2013).

[6] Lasiecka, I., Pokojovy, M., Wan, X.: *Global existence and exponential stability for a nonlinear thermoelastic Kirchhoff-Love plate*. Nonlinear Anal. Real World Appl. **38**, 184-221 (2017).

[7] Lebeau, G., Zuazua, E.: *Decay rates for the three-dimensional linear system of thermoelasticity*. Arch. Ration. Mech. Anal. **148**, 179-231 (1999).

[8] Nonato, C., Raposo, C. A., Feng, B.: *Exponential stability for a thermoelastic laminated beam with nonlinear weights and time-varying delay*. Asymptot. Anal. **2021**, 1-29 (2021).

[9] Racke, R., Ueda, Y.: *Nonlinear thermoelastic plate equations - global existence and decay rates for the Cauchy problem*. Journal of Differential Equations **263**, 8138-8177 (2017).

[10] Raposo, C. A., Villagran, O. P. V., Ferreira, J., Pişkin, E.: *Rao-Nakra sandwich beam with second sound*. Partial Differ. Equ. Appl. Math. **4**, 100053 (2021).

[11] Rivera, J. M.: *Energy decay rates in linear thermoelasticity*. Funkcial. Ekvac. **35**, 19-30 (1992).

[12] Lian, W., Xu, R.: *Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term*. Adv. Nonlinear Anal. **9**, 613-632 (2020).

[13] Ha, T. G., Park, S. H.: *Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity*. Adv. Differ. Equ. **2020**, 235 (2020).

[14] Barrow, J. D., Parsons, P.: *Inflationary models with logarithmic potentials*. Phys. Rev. D. **52**, 5576-5587 (1995).

[15] Enqvist, K., McDonald, J.: *Q-balls and baryogenesis in the MSSM*. Phys. Lett. **425**, 309-321 (1998).

[16] Gorka, P.: *Logarithmic Klein-Gordon equation*. Acta Phys. Polon. B. **40**, 59-66 (2009).

[17] Zloshchastiev, K. G.: *Applications of wave equations with logarithmic nonlinearity in fluid mechanics*. J. Phys. Conf. Ser. **1**, 012-051 (2018).

[18] Dreher, M.: *The wave equation for the p-Laplacian*. Hokkaido Math. J. **36**, 21-52 (2007).

[19] Greenberg, J. M., MacCamy, R. C., Vizel, V. J.: *On the existence, uniqueness, and stability of solution of the equation* $\sigma^{'}(u_x)u_{xx} + \lambda u_{xtx} = \rho_0 u_{tt}$. J. Math. Mech. **17**, 707-728 (1968).

[20] Ang, D. D., Dinh, A. P. N.: *Strong solutions of a quasilinear wave equation with nonlinear damping*. SIAM J. Math. Anal. **19**, 337-347 (1988).

[21] Benaissa, A., Mokeddem, S.: *Decay estimates for the wave equation of p-Laplacian type with dissipation of m-Laplacian type*. Math. Methods Appl. Sci. **30**, 237-247 (2007).

[22] Biazutti, A. C.: *On a nonlinear evolution equation and its applications*. Nonlinear Anal. Theory Methods Appl. **24**, 1221-1234 (1995).

[23] D'Ancona, P., Spagnolo, S.: *On the life span of the analytic solutions to quasilinear weakly hyperbolic equations*. Indiana Univ. Math. J. **40**, 71-99 (1991).

[24] Ma, T. F., Soriano, J. A.: *On weak solutions for an evolution equation with exponential nonlinearities*. Nonlinear Analysis: Theory, Methods & Applications **37**, 1029-1038 (1999).

[25] Pei, P., Rammaha, M. A., Toundykov, D.: *Weak solutions and blow-up for wave equations of p-Laplacian type with supercritical sources*. J. Math. Phys. **56**, 081503 (2015).

[26] Rammaha, M., Toundykov, D., Wilstein, Z.: *Global existence and decay of energy for a nonlinear wave equation with p-Laplacian damping*. Discrete Contin. Dyn. Syst. **32** 4361-4390 (2012).

[27] Ye, Y.: *Global existence and asymptotic behavior of solutions for a class of nonlinear degenerate wave equations*. Differential Equations and Nonlinear Mechanics. **2007**, 19685 (2007).

[28] Chueshov, I., Lasiecka, I.: *Existence, uniqueness of weak solution and global attractors for a class of nonlinear $2D$ Kirchhoff-Boussinesq models*. Discrete Contin. Dyn. Syst. **15**, 777-809 (2006).

[29] Gao, H., Ma, T. F.: *Global solutions for a nonlinear wave equation with the p–Laplacian operator*. Electronic Journal of Qualitative Theory of Differential Equations. **11**, 1-13 (1999).

[30] Choi, H., Kim, H., Laforest, M.: *Relaxation model for the -Laplacian problem with stiffness*. J. Comput. Appl. Math. **344**, 173-189 (2018).

[31] Li, Y.: *Global boundedness of weak solution in an attraction–repulsion chemotaxis system with p-Laplacian diffusion*. Nonlinear Analysis: Real World Applications. **51**, 102933 (2020).

[32] Zhang, X., Liu, L., Wu, Y., Cui, Y.: *Entire blow-up solutions for a quasilinear p-Laplacian Schrödinger equation with a non-square diffusion term*. Appl. Math. Lett. **74**, 85-93 (2017).

[33] Kalleji, M. K.: *Weighted Hardy–Sobolev inequality and global existence result of thermoelastic system on manifolds with corner-edge singularities*. Math. Methods Appl. Sci. (2021). https://doi.org/10.1002/mma.7916

[34] Abdulla, U. G., Jeli, R.: *Evolution of interfaces for the nonlinear parabolic p-Laplacian-type reaction-diffusion equations. II. Fast diffusion vs. absorption*. European Journal of Applied Mathematics. **31**, 385-406 (2020).

[35] Boudjeriou, T.: *Stability of solutions for a parabolic problem involving fractional p-Laplacian with logarithmic nonlinearity*. Mediterr. J. Math. **17** (2020). https://doi.org/10.1007/s00009-020-01584-6

[36] Raposo, C. A., Ribeiro, J. O., Cattai, A. P.: *Global solution for a thermoelastic system with p-Laplacian*. Appl. Math. Lett. **86**, 119-125 (2018).

[37] Ding, H., Zhou, J.: *Global existence and blow-up for a thermoelastic system with p-Laplacian*. Appl. Anal. (2021). https://doi.org/10.1080/00036811.2021.1941906.

[38] Kim, J. U.: *A boundary thin obstacle problem for a wave equation*. Commun. Partial Differ. Equ. **14**, 1011-1026 (1989).

[39] Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod-Gauthier. Paris (1969).

[40] Martinez, P.: *A new method to obtain decay rate estimates for dissipative systems*. ESAIM Control, Optimisation and Calculus of Variations. **4**, 419-444 (1999).

[41] Levine, H. A.: *Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$*. Trans. Amer. Math. Soc. **192**, 1–21 (1974).

[42] Qin, Y., Rivera, J. M.: *Blow-up of solutions to the Cauchy problem in nonlinear one-dimensional thermoelasticity*. J. Math. Anal. Appl. **292**, 160–193 (2004).

[43] Payne, L. E., Sattinger, D. H.: *Saddle points and instability of nonlinear hyperbolic equations*. Israel Journal of Mathematics. **22**, 273-303 (1975).

[44] Ambrosetti, A., Rabinowitz, P. H.: *Dual variational methods in critical point theory and applications*. Journal of Functional Analysis. **14**, 349-381 (1973).

[45] Willem, M.: Minimax Theorems. Progress in Nonlinear Differential Equations and their Applications. Birkhöuser Boston Inc. Boston (1996).

Affiliations

CARLOS ALBERTO RAPOSO DA CUNHA
ADDRESS: Federal University of Bahia, Dept. of Mathematics, 40.170-110, Salvador-Brazil.
E-MAIL: carlos.raposo@ufba.br
ORCID ID: 0000-0001-8014-7499

ADRIANO PEDREIRA CATTAI
ADDRESS: State University of Bahia, Dept. of Mathematics, 41150-000, Salvador-Brazil.
E-MAIL: cattai@uneb.br
ORCID ID: 0000-0002-6171-6585

OCTAVIO PAULO VERA VILLAGRAN
ADDRESS: Tarapaca University, Dept. of of Mathematics, Arica-Chile.
E-MAIL: opverav@academicos.uta.cl
ORCID ID: 0000-0001-7304-0976

GANESH CHANDRA GORAIN
ADDRESS: J. K. College, Dept. of of Mathematics, 723101, Purulia-India.
E-MAIL: goraing@gmail.com
ORCID ID: 0000-0002-5326-3635

Ducival Carvalho Pereira
Address: State University of Pará, Dept. of Mathematics, 41150-000, Pará-Brazil.
e-mail: ducival@uepa.br
ORCID ID: 0000-0003-4511-0185