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COMBATING MULTICOLLINEARITY: A NEW TWO-PARAMETER APPROACH

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ABSTRACT

The ordinary least square (OLS) estimator is the Best Linear Unbiased Estimator (BLUE) when all linear regression model assumptions are valid. The OLS estimator, however, becomes inefficient in the presence of multicollinearity. Various one and two-parameter estimators have been proposed to circumvent the problem of multicollinearity. This paper presents a new twoparameter estimator called Liu-Kibria Lukman Estimator (LKL) estimator. The proposed estimator is compared theoretically and through Monte Carlo simulation with existing estimators such as the ordinary least square, ordinary ridge regression, Liu, Kibria-Lukman, and Modified Ridge estimators. The results show that the proposed estimator performs better than existing estimators considered in this study under some conditions, using the mean square error criterion. A real-life application to Portland cement and Longley datasets supported the theoretical and simulation results by giving the smallest mean square error compared to the existing estimators.

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Keywords: LKL Estimator, OLS Estimator, Monte Carlo Simulation, Multicollinearity, Mean Square Error

ÇOKLU İÇ İLİŞKİ İLE MÜCADELE: YENİ İKİ PARAMETRE YAKLAŞIMI

ÖZ

Regresyon analizinde tüm varsayımlarının sağlanması durumunda en küçük kareler (EKK) tahmin edicisi, en iyi doğrusal yansız tahmin edicidir. Fakat, EKK tahmin edicisi çoklu iç ilişki durumunda etkinliğini kaybetmektedir. Çoklu iç ilişki problemini çözmek için tek ve iki parametreli bazı tahmin ediciler önerilmiştir. Bu çalışmada, Liu-Kibria Lukman tahmin edicisi olarak adlandırılan yeni iki parametreli bir tahmin edici önerilmiştir. Önerilen yeni tahmin edici EKK, Ridge, Liu, Kibria-Lukman, Modifiye Ridge tahmin edicileri ile teorik olarak ve Monte Carlo simülasyon çalışması ile karşılaştırılmıştır. Yapılan çalışmalarda ele alınan tahmin ediciler için hata kareler ortalaması kriterine göre önerilen tahmin edicinin daha iyi performansa sahip olduğu gösterilmiştir. Portland çimento ve Longley veri setleri için yapılan gerçek veri uygulamasında elde edilen sonuçlar ile teorik karşılaştırma ve simülasyon çalışmasının sonuçları desteklenmiştir.

Anahtar Kelimeler: LKL Tahmin Edicisi, EKK Tahmin Edicisi, Monte Carlo Simulasyon, Çoklu İç İlişki, Hata Kareler Ortalaması

1. INTRODUCTION

Consider a multiple linear regression model define in matrix form:

$$y = X\beta + \varepsilon, \tag{1}$$

where y is an $n \times 1$ vector of observations, β is a $p \times 1$ vector of unknown regression coefficients, X is an $n \times p$ observed matrix of the regression, and ε is an $n \times 1$ vector of random errors, which is distributed as multivariate normal with mean 0 and covariance matrix $\sigma^2 I_n, I_n$ being an identity matrix of order n. The OLS estimator of β is obtained as:

$$\hat{\beta} = (X'X)^{-1}X'y \tag{2}$$

The OLSE suffers a setback when there is strong intercorrelation among the regressor variables. The condition is called multicollinearity, often encountered in social sciences (Chatterjee and Hadi, 2006). Handling this problem has attracted and is still attracting many researchers' attention due to the problem of parameter estimation and hypothesis testing if an Ordinary Least Square (OLS) estimator is employed. This problem's consequences include large standard error, non-conformity of the sign of regression coefficient estimates with the prior, imprecise estimates, and insignificance of the true regression coefficients (Chatterjee et al., 2000, Maddala, 2002, Greene, 2003, Lukman and Ayinde, 2017). To overcome this problem, various regression estimators have been suggested in works of literature by researchers as alternative methods to OLSE, which includes: estimator based on Principal Component Analysis Regression (Massy, 1965, Marquardt, 1970, Naes and Marten, 1988), the Ridge Regression Estimator (Hoerl, 1962, Hoerl and Kennard, 1975), estimator based on Partial Least Squares (Helland, 1988, Helland, 1990, Phatak ad Jony, 1997), a modified new two-parameter estimator (Lukman et al., 2019) and a New Ridge-Type Estimator (Kibria and Lukman, 2020). Some recently proposed estimators for circumventing the problem of multicollinearity are (Qasim et al., 2021, Ahmad and Aslam, 2020, Owolabi et al., 2022a, Owolabi et al., 2022b, Dawouda et al., 2022).

The paper aims to propose a new two-parameter estimator for the regression parameter when the regressor variables of the model are linearly related. This estimator will serve as an alternative to the OLS estimator, and the performance of the proposed estimator will be compared with some existing estimators.

The other parts of this paper are organized as follows. The existing and proposed estimators with their properties are discussed in Section 2. The proposed estimator is compared with OLS, Ridge, Liu, Kibra-Lukman, and Modified Ridge Type regressions in Section 3. In Section 4, the biasing parameter was obtained. In Section 5, a simulation study and numerical examples are provided. Concluding remarks are provided in Section 6.

2. SOME EXISTING BIASED ESTIMATORS

The canonical form of Eq. (1) is rewritten as:

$$y = Z\alpha + \varepsilon \tag{3}$$

where $Z = XQ$, $\alpha = Q'\beta$ and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1, \dots, \lambda_p)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_p > 0$ are the ordered eigenvalues $X'X$. The ordinary least square estimator (OLSE) of equation (2) can be defined as:

$$\hat{\alpha} = \Lambda^{-1}X' + \varepsilon \tag{4}$$

and the Mean Square Error matrix (MSEM) of $\hat{\alpha}$ is defined as

$$MSEM(\hat{\alpha}) = \Lambda^{-1}\sigma^2 \tag{5}$$

The Ordinary Ridge Regression Estimator by Hoerl and Kennard (1970) is given as:

$$\hat{\alpha}_k = B_0X'y \tag{6}$$

where $B_0 = (\Lambda + kI)^{-1}$ and k is a non-negative biasing parameter. And the MSEM is given as:

$$MSEM(\hat{\alpha}_k) = \sigma^2 B_0\Lambda^{-1}B_0' + (B_0\Lambda - I)\alpha\alpha'(B_0\Lambda - I)' \tag{7}$$

The Liu estimator is defined as:

$$\hat{\alpha}_d = S\hat{\alpha} \tag{8}$$

where $S = (\Lambda + I)^{-1}(\Lambda + dI)$ and d is a biasing parameter of Liu Estimator. The MSEM of $\hat{\alpha}_d$ is defined as:

$$MSEM(\hat{\alpha}_d) = \sigma^2 S\Lambda^{-1}S + (S - I)\alpha\alpha'(S - I)' \tag{9}$$

The Kibria-Lukman (KL) estimator is defined as:

$$\hat{\alpha}_{KL} = M\hat{\alpha} \tag{10}$$

where $M = (\Lambda + kI)^{-1}(\Lambda - kI)$ and the MSEM of $\hat{\alpha}_{KL}$ is given as

$$MSEM(\hat{\alpha}_{KL}) = \sigma^2 M\Lambda^{-1}M + (M - I)\alpha\alpha'(M - I)' \tag{11}$$

The Modified Ridge Type (MRT) estimator is defined as:

$$\hat{\alpha}_{MRT} = R\hat{\alpha} \tag{12}$$

where $R = \Lambda(\Lambda + k(1 + d)I)^{-1}$ and the MSEM is given as

$$MSEM(\hat{\alpha}_{MRT}) = \sigma^2 R\Lambda^{-1}R + (R - I)\alpha\alpha'(R - I)' \tag{13}$$

The Modified New Two Parameter (MNTP) estimator is defined as:

$$\hat{\alpha}_{MNTP} = C_0\hat{\alpha}$$

where $C_0 = (\Lambda + I)^{-1}(\Lambda + dI)(\Lambda + kdI)^{-1}Z'$ and the MSEM is given as

$$MSEM(\hat{\alpha}_{MNTP}) = \sigma^2 C_0 {C_0}' + (C_0 Z - I)\alpha\alpha'(C_0 Z - I)' \quad (14)$$

The $\hat{\alpha}_{TSS}$ estimator is defined as:

$$\hat{\alpha}_{TSS} = F_0 \hat{\alpha}$$

where $F_0 = (\Lambda - kdI)(\Lambda + kI)^{-1}$ and the MSEM is given as

$$MSEM(\hat{\alpha}_{TSS}) = \sigma^2 F_0 \Lambda^{-1} {F_0}' + (F_0 - I)\alpha\alpha'(F_0 - I)' \quad (15)$$

The proposed estimator

Several works of literature have established that the OLSE is highly efficient in estimating the regression parameters in a linear regression model when the classical assumptions are not violated. However, in the presence of multicollinearity, OLSE will suffer a setback. Hoerl and Kennard (1970) and Liu (1993) proposed a ridge regression estimator and Liu regression estimator, respectively, to improve the estimation of parameters in the presence of multicollinearity. These two estimators are, however, one-parameter estimators.

The proposed Liu-Kibra-Lukman (LKL) estimator for α is obtained by replacing $\hat{\alpha}_{KL}$ with $\hat{\alpha}_{OLS}$ in the Liu estimator following a method similar to that proposed by Liu (1993), Yang and Chang (2010), and Kaçıranlar *et al.* (1999). The proposed estimator is obtained as follows:

$$\hat{\alpha}_{LKL} = CA\hat{\alpha} \quad (16)$$

where $C = (\Lambda + I)^{-1}(\Lambda + dI)$, $A = (\Lambda + kI)^{-1}(\Lambda - kI)$, d and k are the biasing parameters.

The expectation vector, bias vector, dispersion matrix, and mean squared error matrix (MSEM) of the proposed estimator are given as follows:

$$E(\hat{\alpha}_{LKL}) = CA\alpha \quad (17)$$

$$B(\hat{\alpha}_{LKL}) = (CA - I)\alpha \quad (18)$$

$$D(\hat{\alpha}_{LKL}) = \sigma^2 CA\Lambda^{-1}CA' \quad (19)$$

$$MSEM(\hat{\alpha}_{LKL}) = \sigma^2 CA\Lambda^{-1}CA' + (CA - I)\alpha\alpha'(CA - I)' \quad (20)$$

To prove the statistical properties of $\hat{\alpha}_{LKL}$, The following lemmas are useful.

Lemma 1: Let $n \times n$ matrices $M > 0$, $N > 0$ (or $N \geq 0$), then $N > M$ if and only if $\lambda_1(NM^{-1}) < 1$ where $\lambda_1(NM^{-1}) < 1$ is the largest eigenvalue of matrix NM^{-1} (Wang *et al.*, 2006).

Lemma 2: Let M be an $n \times n$ positive definite matrix, that is $M > 0$, and α be some vector, then $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1} \alpha \leq 1$ (Farebrother, 1976).

Lemma 3: Let $\hat{\alpha}_i = A_i y \;\; i = 1,2$ be two linear estimators of α. Suppose that $D = cov(\hat{\alpha}_1) - cov(\hat{\alpha}_2) > 0$, where $cov(\hat{\alpha}_i)\; i = 1,2$ denotes the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha, i = 1,2$. Consequently,

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_1' - b_2 b_2' > 0 \tag{21}$$

If and only if $b_2'[\sigma^2 D + b_1 b_1{}']^{-1} < 1$, where $MSEM(\hat{\alpha}_i) = cov(\hat{\alpha}_i) + b_i b_i{}'$ (Trenkler and Toutenburg, 1990).

3. COMPARISON AMONG THE ESTIMATORS

In this section, the MSEM of the proposed estimator $\hat{\alpha}_{LKL}$ is compared with other existing estimators theoretically.

3.1. Comparison between $\hat{\alpha}$ and $\hat{\alpha}_{LKL}$

The difference between $MSEM(\hat{\alpha}) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$$MSEM(\hat{\alpha}) - MSEM(\hat{\alpha}_{LKL}) = \sigma^2\Lambda^{-1} - (\sigma^2 CA\Lambda^{-1}CA + (CA - I)\alpha\alpha'(CA - I)') \tag{22}$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.1: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(\Lambda^{-1} - CA\Lambda^{-1}CA)]\alpha(CA - I) < 1 \tag{23}$$

Proof:

$$D(\hat{\alpha}) - D(\hat{\alpha}_{LKL}) = \sigma^2(\Lambda^{-1} - CA\Lambda^{-1}CA) = \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k)^2}\right\}_{i=1}^{p} \tag{24}$$

$\Lambda^{-1} - CA\Lambda^{-1}CA$ will be pd if and only if $(\lambda_i + 1)^2(\lambda_i + k)^2 - (\lambda_i + d)^2(\lambda_i - k)^2 > 0$. By lemma 3, the proof is completed.

3.2. Comparison between $\hat{\alpha}_k$ and $\hat{\alpha}_{LKL}$

The difference between $MSEM(\hat{\alpha}_k) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$MSEM(\hat{\alpha}_k) - MSEM(\hat{\alpha}_{LKL}) = \sigma^2 B_0 \Lambda^{-1} B_0 - \sigma^2 CA\Lambda^{-1}CA + (B_0\Lambda - I)\alpha\alpha'(B_0\Lambda - I)' - (CA - I)\alpha\alpha'(CA - I)'$ (25)

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.2: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_k$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_k) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(\mathrm{B}_0\Lambda^{-1}B_0 - CA\Lambda^{-1}CA) + (B_0\Lambda - I)\alpha\alpha'(B_0\Lambda - I)]\alpha(CA - I) < 1 \quad (26)$$

Proof: Considering the dispersion matrix difference between $\hat{\alpha}_k$ and $\hat{\alpha}_{LKL}$,

$$D_d = \sigma^2 B_0\Lambda^{-1}B_0 - CA\Lambda^{-1}CA$$

$$= \sigma^2(\lambda + kI)^{-1}\lambda(\lambda + kI)^{-1}$$

$$-\sigma^2(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}\lambda^{-1}(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}$$

$$= \sigma^2(\lambda + I)^{-1}(\lambda + kI)^{-1}[\lambda^2(\lambda + 1)^2 - (\lambda + dI)^2(\lambda - kI)^2]\lambda^{-1}(\lambda + I)^{-1}(\lambda + kI)^{-1}$$

$$= \sigma^2(\lambda + I)^{-1}(\lambda + kI)^{-1}[2\lambda^3 + 2\lambda^3 k + 4\lambda^2 kd + 2\lambda kd^2 + \lambda^2 - 2\lambda^3 d - \lambda^2 d^2 - \lambda^2 k^2 - 2\lambda dk^2 - k^2 d^2]\lambda^{-1}(\lambda + I)^{-1}(\lambda + kI)^{-1}$$

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

3.3. Comparison between $\hat{\alpha}_d$ and $\hat{\alpha}_{LKL}$

The difference between $MSEM(\hat{\alpha}_d) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$MSEM(\hat{\alpha}_d) - MSEM(\hat{\alpha}_{LKL}) = \sigma^2 S\Lambda^{-1}S - \sigma^2 CA\Lambda^{-1}CA + (S - I)\alpha\alpha'(S - I)' - (CA - I)\alpha\alpha'(CA - I)'$ (27)

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.3: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_d$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_d) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(S\Lambda^{-1}S - CA\Lambda^{-1}CA) + (S - I)\alpha\alpha'(S - I)]\alpha(CA - I) < 1 \quad (28)$$

Proof: Considering the dispersion matrix difference between $D(\hat{\alpha}_d)$ and $D(\hat{\alpha}_{LKL})$,

$$D_d = \sigma^2 S\Lambda^{-1}S - \sigma^2 CA\Lambda^{-1}CA'$$

$$= \sigma^2(\lambda + I)^{-1}(\lambda + dI)^1\lambda^{-1}(\lambda + I)^{-1}(\lambda + dI)^1 - \sigma^2(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)$$

$$= \lambda^{-1}(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}$$

$= \sigma^2(\lambda + I)^{-1}(\lambda + kI)^{-1}[4\lambda k]\lambda^{-1}(\lambda + I)^{-1}(\lambda + kI)^{-1}$ (29)

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

3.4. Comparison between $\hat{\alpha}_{KL}$ and $\hat{\alpha}_{LKL}$

The difference between $MSEM(\hat{\alpha}_{KL}) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$MSEM(\hat{\alpha}_{KL}) - MSEM(\hat{\alpha}_{LKL}) = \sigma^2 M\Lambda^{-1}M - \sigma^2 CA\Lambda^{-1}CA + (M - I)\alpha\alpha'(M - I)' - (CA - I)\alpha\alpha'(CA - I)'$ (30)

Let $k > 0$ and $0 < d < 1$. Thus,

Theorem 3.4: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_{KL}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{KL}) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(M\Lambda^{-1}M - CA\Lambda^{-1}CA) + (M - I)\alpha\alpha'(M - I)]\alpha(CA - I) < 1 \quad (31)$$

Proof: Considering the dispersion matrix difference between $D(\hat{\alpha}_{KL})$ and $D(\hat{\alpha}_{LKL})$,

$D_d = \sigma^2 M\Lambda^{-1}M - \sigma^2 CA\Lambda^{-1}CA'$

$= \sigma^2(\lambda - kI)(\lambda + kI)^{-1}\lambda^{-1}(\lambda - kI)(\lambda + kI)^{1} - \sigma^2(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}\lambda^{-1}(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}$

$= \sigma^2(\lambda + I)^{-1}(\lambda + kI)^{-1}[(\lambda + 1)^2(\lambda - kI)^2 - (\lambda + dI)^2(\lambda - kI)^2]\lambda^{-1}(\lambda + 1)^{-1}(\lambda + kI)^{-1}$

$= \sigma^2(\lambda + I)^{-1}(\lambda + kI)^{-1}[2\lambda + 1 - 2\lambda d - d^2]\lambda^{-1}(\lambda + I)^{-1}(\lambda + kI)^{-1}$ (32)

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

3.5. Comparison between $\hat{\alpha}_{MRT}$ and $\hat{\alpha}_{LKL}$

The difference between $MSEM(\hat{\alpha}_{MRT}) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$MSEM(\hat{\alpha}_{MRT}) - MSEM(\hat{\alpha}_{LKL}) = \sigma^2 R\Lambda^{-1}R - \sigma^2 CA\Lambda^{-1}CA + (R - I)\alpha\alpha'(R - I)' - (CA - I)\alpha\alpha'(CA - I)'$ (33)

Theorem 3.5: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_{MRT}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{MRT}) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(R\Lambda^{-1}R - CA\Lambda^{-1}CA) + (R - I)\alpha\alpha'(R - I)]\alpha(CA - I) < 1 \quad (34)$$

Proof: Considering the dispersion matrix difference between $D(\hat{\alpha}_{MRT})$ and $D(\hat{\alpha}_{LKL})$,

$D_d = \sigma^2 R\Lambda^{-1}R - \sigma^2 CA\Lambda^{-1}CA'$

$= \sigma^2(\lambda + k(1+d)I)^{-1}\lambda(\lambda + k(1+d)I)^{-1} - \sigma^2(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}\lambda^{-1}(\lambda - kI)(\lambda + dI)(\lambda + I)^{-1}(\lambda + kI)^{-1}$

$$= \sigma^2 diag\left\{\frac{\lambda_i}{\lambda_i + k(1+d))^2} - \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \tag{35}$$

will be pdf if and only if $\lambda_i^2(\lambda_i + 1)^2(\lambda_i + k)^2 - (\lambda_i + d)^2(\lambda_i - k)^2\big(\lambda_i + k(1+d)\big)^2 > 0$. For $0 < d < 1$ and $k > 0$, it was observed that $\lambda_i^2(\lambda_i + 1)^2(\lambda_i + k)^2 - (\lambda_i + d)^2(\lambda_i - k)^2\big(\lambda_i + k(1+d)\big)^2 > 0$. By lemma 3, the proof is completed.

3.6. Comparison between $\hat{\boldsymbol{\alpha}}_{MNTP}$ and $\hat{\boldsymbol{\alpha}}_{LKL}$

The difference between $MSEM(\hat{\alpha}_{MNTP}) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$$MSEM(\hat{\alpha}_{MNTP}) - MSEM(\hat{\alpha}_{LKL}) = (\sigma^2 C_0 C_0' + (C_0 Z - I)\alpha\alpha'(C_0 Z - I)') - (\sigma^2 CA\Lambda^{-1}(CA)' + (CA - I)\alpha\alpha'(CA - I)') \tag{36}$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.6: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_{MNTP}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{MNTP}) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'[\sigma^2(C_0 C_0' - CA\Lambda^{-1}CA) + (C_0 Z - I)\alpha\alpha'(C_0 Z - I)]\alpha(CA - I) < 1 \tag{37}$$

Proof:

$D(\hat{\alpha}_{MNTP}) - D(\hat{\alpha}_{LKL}) = \sigma^2(C_0 C_0' - CA\Lambda^{-1}(CA)')$

$$= \sigma^2 diag\left\{\frac{(\lambda_i + d)^2\lambda_i}{(\lambda_i+1)^2(\lambda_i+k)^2} - \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}\right\}_{i=1}^{p} \tag{38}$$

$C_0 C_0' - CA\Lambda^{-1}(CA)'$ will be pd if and only if $\lambda_i^2(\lambda_i + d)^2 - (\lambda_i + d)^2(\lambda_i - k)^2 > 0$. By lemma 3, the proof is completed.

3.7. Comparison between $\hat{\boldsymbol{\alpha}}_{TSS}$ and $\hat{\boldsymbol{\alpha}}_{LKL}$

The difference between $MSEM(\hat{\alpha}_{TSS}) - MSEM(\hat{\alpha}_{LKL})$ is given as follows:

$MSEM(\hat{\alpha}_{TSS}) - MSEM(\hat{\alpha}_{LKL}) = \left(\sigma^2 F_0 \Lambda^{-1} F_0' + \left(F_0 - I_p\right)\alpha\alpha'\left(F_0 - I_p\right)'\right) -$
$(\sigma^2 CA\Lambda^{-1}(CA)' + (CA - I)\alpha\alpha'(CA - I)')$ (39)

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 3.7: The estimator $\hat{\alpha}_{LKL}$ is superior to the estimator $\hat{\alpha}_{TSS}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{TSS}) - MSEM(\hat{\alpha}_{LKL}) > 0$ if and only if

$$\alpha'(CA - I)'\left[\sigma^2(F_0\Lambda^{-1}F_0' - CA\Lambda^{-1}(CA)') + \left(F_0 - I_p\right)\alpha\alpha'\left(F_0 - I_p\right)\right]\alpha(CA - I) < 1 \quad (40)$$

Proof:

$D(\hat{\alpha}_{TSS}) - D(\hat{\alpha}_{LKL}) = \sigma^2(F_0\Lambda^{-1}F_0' - CA\Lambda^{-1}(CA)')$

$$= \sigma^2 diag\left\{\frac{(\lambda_i - kd)^2}{\lambda_i^2(\lambda_i + k)^2} - \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k)^2}\right\}_{i=1}^{p} \quad (41)$$

$F_0\Lambda^{-1}F_0' - CA\Lambda^{-1}(CA)'$ will be pd if and only if $(\lambda_i + 1)^2(\lambda_i - kd)^2 - (\lambda_i + d)^2(\lambda_i - k)^2 > 0$. By lemma 3, the proof is completed.

4. SELECTION OF BIASING PARAMETERS k AND d

Following different authors such as Dorugade (2016), Saleh *et al.* (2019), Lukman *et al.* (2019), and Aslam and Ahmad (2020), among others, the optimal values of k and d are determined for the new estimator. In determining the optimal value of *d*, *k* is fixed. The optimal value of the *d* can be considered to be those *d* that minimize

$$MSEM(\hat{\alpha}_{LKL}) = \sigma^2 CA\Lambda^{-1}CA + (CA - I)\alpha\alpha'(CA - I)'$$
$$f(k, d) = MSEM(\hat{\alpha}_{LKL}) = tr[MSEM(\hat{\alpha}_{LKL})]$$

$$f(k, d) = \sigma^2 \sum_i^p \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k)^2} + \sum_i^p \frac{[k(2\lambda_i) + d + 1) + \lambda_i(1 - d)]^2\alpha_i^2}{(\lambda_i + 1)^2(\lambda_i + k)^2} \quad (42)$$

Differentiating $f(k, d)$ with respect to d gives

$$\frac{\partial s(k,d)}{\partial d} = 2\sigma^2 \sum_i^p \frac{(\lambda_i + d)^2(\lambda_i - k)^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k)^2} + 2\sum_i^p \frac{[k(2\lambda_i) + d + 1) + \lambda_i(1 - d)](k - \lambda_i)}{(\lambda_i + 1)^2(\lambda_i + k)^2}$$

Let $\frac{\partial s(k,d)}{\partial d} = 0$;

$$d = \frac{\lambda_i\left(\alpha_i^2 - \sigma^2\right) + \lambda_i k\left(2\alpha_i^2\lambda_i + \alpha_i^2 - \sigma^2\right)}{\sigma^2(\lambda_i - k) + \alpha_i^2\lambda_i(\lambda_i - k)} \quad (43)$$

For practical purposes, σ^2 and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (43) becomes

$$d = \frac{\lambda_i(\hat{\alpha}_i^2 - \hat{\sigma}^2) + \lambda_i k(2\hat{\alpha}_i^2\lambda_i + \hat{\alpha}_i^2 - \hat{\sigma}^2)}{\hat{\sigma}^2(\lambda_i - k) + \hat{\alpha}_i^2\lambda_i(\lambda_i - k)} \tag{44}$$

Fort he biasing parameter k fort he proposed L-KL estimator, we adopted the biasing parameter k proposed by Kibria and Lukman (2020). The biasing k is given as:

$$k = \min\left[\frac{\hat{\sigma}^2}{2\hat{\alpha}_{i,OLS}^2 + \frac{\hat{\sigma}^2}{e_i}}\right]$$

5. SIMULATION STUDY AND NUMERICAL EXAMPLE

5. 1. Simulation Study

This section conducted a Monte Carlo simulation using the R studio to support the theoretical comparison and compare the estimators' performance. The explanatory variables in this study were generated following the studies of McDonald and Galarneau (1975), Wichern and Churchill (1978), and Kibria (2003). This is given as:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i,p+1}, \qquad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,p \tag{45}$$

where z_{ij} are independent standard normal pseudorandom numbers, ρ is the correlation between any two explanatory variables, and p is the number of explanatory variables. This study considers the values of ρ to be $0.8, 0.9, 0.95$, and 0.99. Also, explanatory variables (p) will be three (3) and sample sizes 50 and 100 for the simulation study. For $p = 3$, the response variable is defined as:

$$Y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + e_i \tag{46}$$

where e_i is normally distributed with mean 0 and variance σ^2. We choose β such that $\beta'\beta = 1$ (Newhouse and Oman, 1971). The selected values for σ are $1, 3, 5$, and 10, and the replication for the study is 1000 times.

The mean square error is then obtained as:

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\alpha}_{ij} - \alpha_i)'(\hat{\alpha}_{ij} - \alpha_i) \tag{47}$$

Table 1. Estimated MSE when p=3, n=50

K	D	Sigma	Rho	OLS	RIDGE	LIU	K-L	MRT	TSS	MNTP	LKL
0.3	**0.2**	**1**	**0.8**	0.1363	0.1311	0.1234	0.1261	0.1301	0.1301	0.1224	**0.1142**
			0.9	0.2462	0.2288	0.2046	0.212	0.2255	0.2253	0.2016	**0.1764**
			0.95	0.4682	0.4064	0.3333	0.3492	0.3956	0.3946	0.3238	**0.2495**
			0.99	2.2417	1.2173	0.7223	0.5101	1.0998	1.0505	0.6305	**0.1721**
		3	**0.8**	1.227	1.1802	1.1098	1.1344	1.1712	1.1709	1.1011	**1.0265**
			0.9	2.2158	2.0587	1.8412	1.9076	2.0293	2.028	1.8141	**1.587**
			0.95	4.2136	3.6579	2.9999	3.1428	3.5604	3.5516	2.9146	**2.2452**
			0.99	20.175	10.956	6.5009	4.5913	9.8985	9.4548	5.6746	**1.5486**
		5	**0.8**	3.4082	3.2782	3.0825	3.1509	3.2532	3.2525	3.0585	**2.8511**
			0.9	6.1551	5.7185	5.1142	5.2987	5.637	5.6332	5.0389	**4.4079**
			0.95	11.704	10.161	8.333	8.73	9.8901	9.8657	8.096	**6.2368**
			0.99	56.042	30.434	18.058	12.754	27.496	26.263	15.763	**4.3018**
		10	**0.8**	13.633	13.113	12.329	12.603	13.012	13.01	12.234	**11.403**
			0.9	24.62	22.874	20.457	21.194	22.548	22.533	20.155	**17.631**
			0.95	46.818	40.644	33.332	34.92	39.56	39.463	32.384	**24.947**
			0.99	224.17	121.73	72.233	51.015	109.98	105.05	63.052	**17.207**
0.3	**0.5**	**1**	**0.8**	0.1363	0.1311	0.1281	0.1261	0.1287	0.1286	0.1257	**0.1186**
			0.9	0.2462	0.2288	0.2197	0.212	0.2207	0.2203	0.2118	**0.1894**
			0.95	0.4682	0.4064	0.3811	0.3492	0.3802	0.3773	0.3548	**0.2849**
			0.99	2.2417	1.2173	1.1925	0.5101	0.9535	0.8241	0.8603	**0.2773**
		3	**0.8**	1.227	1.1802	1.153	1.1344	1.1578	1.1572	1.1307	**1.0663**
			0.9	2.2158	2.0587	1.9774	1.9076	1.9865	1.9824	1.9056	**1.7036**
			0.95	4.2136	3.6579	3.4302	3.1428	3.4215	3.3953	3.1931	**2.5635**
			0.99	20.175	10.956	10.733	4.5913	8.5816	7.4169	7.7427	**2.4951**
		5	**0.8**	3.4082	3.2782	3.2026	3.1509	3.2161	3.2142	3.1408	**2.9617**
			0.9	6.1551	5.7185	5.4928	5.2987	5.5181	5.5065	5.2933	**4.7319**
			0.95	11.704	10.161	9.5284	8.73	9.5043	9.4314	8.8698	**7.121**
			0.99	56.042	.30.434	29.813	12.754	23.838	20.603	21.508	**6.9309**
		10	**0.8**	13.633	13.113	12.81	12.603	12.864	12.857	12.563	**11.846**
			0.9	24.62	22.874	21.971	21.194	22.072	22.026	21.173	**18.927**
			0.95	46.818	40.644	38.114	34.92	38.017	37.726	35.479	**28.484**
			0.99	224.17	121.73	119.25	51.015	95.351	82.41	86.03	**27.724**
0.3	**0.8**	**1**	**0.8**	0.1363	0.1311	0.133	0.1261	0.1272	0.1271	0.129	**0.123**
			0.9	0.2462	0.2288	0.2354	0.212	0.2162	0.2153	0.222	**0.2028**
			0.95	0.4682	0.4064	0.4323	0.3492	0.3657	0.3603	0.3858	**0.3226**
			0.99	2.2417	1.2173	1.7822	0.5101	0.8349	0.6262	1.0792	**0.4083**
		3	**0.8**	1.227	1.1802	1.1971	1.1344	1.1448	1.1435	1.1604	**1.1069**

			0.9	2.2158	2.0587	2.1188	1.9076	1.9451	1.9373	1.9974	**1.8245**
			0.95	4.2136	3.6579	3.8903	3.1428	3.2909	3.2426	3.4721	**2.9038**
			0.99	20.175	10.956	16.04	4.5913	7.5144	5.6359	9.7129	**3.675**
		5	**0.8**	3.4082	3.2782	3.3251	3.1509	3.1798	3.1761	3.2232	**3.0744**
			0.9	6.1551	5.7185	5.8855	5.2987	5.4031	5.3813	5.5483	**5.068**
			0.95	11.704	10.161	10.806	8.73	9.1413	9.0071	9.6447	**8.0661**
			0.99	56.042	30.434	44.554	12.754	20.873	15.655	26.98	**10.208**
		10	**0.8**	13.633	13.113	13.3	12.603	12.719	12.704	12.893	**12.297**
			0.9	24.62	22.874	23.542	21.194	21.612	21.525	22.193	**20.271**
			0.95	46.818	40.644	43.226	34.92	36.565	36.029	38.579	**32.265**
			0.99	224.17	121.73	178.22	51.015	83.494	62.622	107.92	**40.834**
0.6	**0.2**	**1**	**0.8**	0.1363	0.1263	0.1234	0.1167	0.1244	0.1243	0.1215	**0.1058**
			0.9	0.2462	0.2132	0.2046	0.1827	0.2074	0.2069	0.1987	**0.1523**
			0.95	0.4682	0.3564	0.3333	0.2604	0.3391	0.336	0.3148	**0.1869**
			0.99	2.2417	0.7679	0.7223	0.0801	0.6563	0.5675	0.5556	**0.0317**
		3	**0.8**	1.227	1.1362	1.1098	1.0491	1.1193	1.1185	1.0926	**0.9499**
			0.9	2.2158	1.9183	1.8412	1.6431	1.8663	1.8614	1.7876	**1.3688**
			0.95	4.2136	3.208	2.9999	2.3436	3.0516	3.0238	2.833	**1.6811**
			0.99	20.175	6.9114	6.5009	0.721	5.907	5.1075	5.0003	**0.2841**
		5	**0.8**	3.4082	3.1559	3.0825	2.9139	3.109	3.1067	3.0349	**2.638**
			0.9	6.1551	5.3284	5.1142	4.5638	5.1839	5.1706	4.9652	**3.8017**
			0.95	11.704	8.9111	8.333	6.5101	8.4768	8.3995	7.8693	**4.6697**
			0.99	56.042	19.198	18.058	2.0029	16.408	14.188	13.89	**0.7892**
		10	**0.8**	13.633	12.623	12.329	11.655	12.436	12.426	12.139	**10.55**
			0.9	24.62	21.313	20.457	18.255	20.735	20.682	19.861	**15.206**
			0.95	46.818	35.645	33.332	26.041	33.907	33.598	31.478	**18.679**
			0.99	224.17	76.794	72.233	80.117	65.634	56.75	55.559	**3.1568**
0.6	**0.5**	**1**	**0.8**	0.1363	0.1263	0.1281	0.1167	0.1217	0.1214	0.1233	**0.1098**
			0.9	0.2462	0.2132	0.2197	0.1827	0.1992	0.1976	0.2042	**0.1633**
			0.95	0.4682	0.3564	0.3811	0.2604	0.3154	0.3065	0.3312	**0.213**
			0.99	2.2417	0.7679	1.1925	0.0801	0.5301	0.3258	0.6518	**0.0469**
		3	**0.8**	1.227	1.1362	1.153	1.0491	1.0947	1.0922	1.1092	**0.9865**
			0.9	2.2158	1.9183	1.9774	1.6431	1.7923	1.7779	1.8378	**1.4686**
			0.95	4.2136	3.208	3.4302	2.3436	2.8383	2.7582	2.9805	**1.9161**
			0.99	20.175	6.9114	10.733	0.721	4.7705	2.9321	5.8657	**0.4217**
		5	**0.8**	3.4082	3.1559	3.2026	2.9139	3.0407	3.0336	3.0809	**2.7397**
			0.9	6.1551	5.3284	5.4928	4.5638	4.9784	4.9384	5.105	**4.0789**
			0.95	11.704	8.9111	9.5284	6.5101	7.8841	7.6616	8.2793	**5.3226**
			0.99	56.042	19.198	29.813	2.0029	13.251	8.1447	16.294	**1.1712**

		10	**0.8**	13.633	12.623	12.81	11.655	12.162	12.134	12.323	**10.958**
			0.9	24.62	21.313	21.971	18.255	19.753	19.753	20.419	**16.315**
			0.95	46.818	35.645	38.114	26.041	31.537	30.646	33.117	**21.291**
			0.99	224.17	76.794	119.25	8.0117	53.006	32.579	65.175	**4.6851**
0.6	**0.8**	**1**	**0.8**	0.1363	0.1263	0.133	0.1167	0.1191	0.1186	0.1251	**0.1139**
			0.9	0.2462	0.2132	0.2354	0.1827	0.1915	0.1886	0.2097	**0.1748**
			0.95	0.4682	0.3564	0.4323	0.2604	0.2941	0.2784	0.3466	**0.2408**
			0.99	2.2417	0.7679	1.7822	0.0801	0.4374	0.1548	0.7267	**0.0657**
		3	**0.8**	1.227	1.1362	1.1971	1.0491	1.071	1.0662	1.1255	**1.0238**
			0.9	2.2158	1.9183	2.1188	1.6431	1.7228	1.6963	1.8866	**1.572**
			0.95	4.2136	3.208	3.8903	2.3436	2.6472	2.5052	3.1196	**2.1673**
			0.99	20.175	6.9114	16.04	0.721	3.9366	1.3933	6.54	**0.5908**
		5	**0.8**	3.4082	3.1559	3.3251	2.9139	2.9747	2.9615	3.1261	**2.8435**
			0.9	6.1551	5.3284	5.8855	4.5638	4.7853	4.7118	5.2404	**4.3665**
			0.95	11.704	8.9111	10.806	6.5101	7.3532	6.9589	8.6657	**6.0202**
			0.99	56.042	19.198	44.554	2.0029	10.935	3.8702	18.167	**1.641**
		10	**0.8**	13.633	12.623	13.3	11.655	11.898	11.845	12.504	**11.373**
			0.9	24.62	21.313	23.542	18.255	19.141	18.847	20.961	**17.465**
			0.95	46.818	35.645	43.226	26.041	29.413	27.836	34.663	**24.081**
			0.99	224.17	76.794	178.22	8.0117	43.741	15.481	72.667	**6.5642**
0.7	**0.2**	**1**	**0.8**	0.1363	0.1247	0.1234	0.1137	0.1226	0.1225	0.1212	**0.1032**
			0.9	0.2462	0.2083	0.2046	0.1738	0.2019	0.2012	0.1977	**0.145**
			0.95	0.4682	0.3419	0.3333	0.2361	0.323	0.3191	0.3118	**0.1697**
			0.99	2.2417	0.6731	0.7223	0.0378	0.5677	0.4714	0.5336	**0.0172**
		3	**0.8**	1.227	1.1221	1.1098	1.0222	1.1028	1.1017	1.0898	**0.9257**
			0.9	2.2158	1.8748	1.8412	1.5634	1.8164	1.8101	1.7789	**1.3031**
			0.95	4.2136	3.0769	2.9999	2.1241	2.9068	2.8716	2.8065	**1.526**
			0.99	20.175	6.0583	6.5009	0.3396	5.1093	4.2427	4.8027	**0.1534**
		5	**0.8**	3.4082	3.1167	3.0825	2.8391	3.0632	3.0601	3.027	**2.5707**
			0.9	6.1551	5.2076	5.1142	4.3425	5.0455	5.028	4.9411	**3.6191**
			0.95	11.704	8.5469	8.333	5.9004	8.0744	7.9767	7.796	**4.2387**
			0.99	56.042	16.829	18.058	0.9434	14.192	11.785	13.341	**0.4259**
		10	**0.8**	13.633	12.467	12.329	11.355	12.252	12.24	12.108	**10.281**
			0.9	24.62	20.83	20.457	17.369	20.182	20.112	19.764	**14.475**
			0.95	46.818	34.188	33.332	23.602	32.298	31.907	31.184	**16.955**
			0.99	224.17	67.315	72.233	3.7737	56.77	47.142	53.364	**1.7036**
0.7	**0.5**	**1**	**0.8**	0.1363	0.1247	0.1281	0.1137	0.1195	0.1191	0.1225	**0.1071**
			0.9	0.2462	0.2083	0.2197	0.1738	0.1927	0.1907	0.2018	**0.1555**
			0.95	0.4682	0.3419	0.3811	0.2361	0.2975	0.2864	0.3238	**0.1932**

			0.99	2.2417	0.6731	1.1925	0.0378	0.4511	0.2388	0.5992	**0.0237**
		3	**0.8**	1.227	1.1221	1.153	1.0222	1.0749	1.0715	1.1021	**0.9613**
			0.9	2.2158	1.8748	1.9774	1.5634	1.7341	1.7154	1.8161	**1.3978**
			0.95	4.2136	3.0769	3.4302	2.1241	2.6776	2.5775	2.9145	**1.7382**
			0.99	20.175	6.0583	10.733	0.3396	4.0594	2.1492	5.3927	**0.2124**
		5	**0.8**	3.4082	3.1167	3.2026	2.8391	2.9856	2.9762	3.0613	**2.6697**
			0.9	6.1551	5.2076	5.4928	4.3425	4.8167	4.7648	5.0445	**3.8822**
			0.95	11.704	8.5469	9.5284	5.9004	7.4378	7.1597	8.0959	**4.8283**
			0.99	56.042	16.829	29.813	0.9434	11.276	5.97	14.98	**0.5899**
		10	**0.8**	13.633	12.467	12.81	11.355	11.942	11.904	12.245	**10.677**
			0.9	24.62	20.83	21.971	17.369	19.266	19.059	20.178	**15.528**
			0.95	46.818	34.188	38.114	23.602	29.751	28.639	32.384	**19.313**
			0.99	224.17	67.315	119.25	3.7737	45.104	23.88	59.919	**2.3598**
0.7	**0.8**	**1**	**0.8**	0.1363	0.1247	0.133	0.1137	0.1166	0.1159	0.1239	**0.111**
			0.9	0.2462	0.2083	0.2354	0.1738	0.1843	0.1805	0.2058	**0.1664**
			0.95	0.4682	0.3419	0.4323	0.2361	0.2751	0.2556	0.3349	**0.2184**
			0.99	2.2417	0.6731	1.7822	0.0378	0.3674	0.0902	0.6475	**0.0317**
		3	**0.8**	1.227	1.1221	1.1971	1.0222	1.0481	1.0418	1.1142	**0.9976**
			0.9	2.2158	1.8748	2.1188	1.5634	1.6574	1.6233	1.8518	**1.496**
			0.95	4.2136	3.0769	3.8903	2.1241	2.4752	2.3	3.0144	**1.9649**
			0.99	20.175	6.0583	16.04	0.3396	3.3064	0.8115	5.8272	**0.2844**
		5	**0.8**	3.4082	3.1167	3.3251	2.8391	2.911	2.8935	3.0948	**2.7706**
			0.9	6.1551	5.2076	5.8855	4.3425	4.6037	4.509	5.1437	**4.1552**
			0.95	11.704	8.5469	10.806	5.9004	6.8757	6.3888	8.3733	**5.4581**
			0.99	56.042	16.829	44.554	0.9434	9.1844	2.2542	16.187	**0.79**
		10	**0.8**	13.633	12.467	13.3	11.355	11.643	11.573	12.379	**11.081**
			0.9	24.62	20.83	23.542	17.369	18.414	18.035	20.574	**16.62**
			0.95	46.818	34.188	43.226	23.602	27.503	25.555	33.493	**21.833**
			0.99	224.17	67.315	178.22	3.7737	36.738	9.017	64.747	**3.1601**
0.9	**0.2**	**1**	**0.8**	0.1363	0.1217	0.1234	0.1081	0.1191	0.1189	0.1205	**0.0981**
			0.9	0.2462	0.1992	0.2046	0.1575	0.1915	0.1904	0.1958	**0.1316**
			0.95	0.4682	0.3154	0.3333	0.1937	0.2941	0.2885	0.3061	**0.1397**
			0.99	2.2417	0.5301	0.7223	0.0227	0.4374	0.3322	0.4936	**0.0107**
		3	**0.8**	1.227	1.0947	1.1098	0.9706	1.071	1.0692	1.0842	**0.8793**
			0.9	2.2158	1.7923	1.8412	1.4156	1.7228	1.7132	1.7617	**1.1811**
			0.95	4.2136	2.8383	2.9999	1.7424	2.6472	2.5967	2.7548	**1.2559**
			0.99	20.175	4.7705	6.5009	0.2033	3.9366	2.9901	4.4422	**0.0948**
		5	**0.8**	3.4082	3.0407	3.0825	2.6954	2.9747	2.9698	3.0115	**2.4415**
			0.9	6.1551	4.9784	5.1142	3.9317	4.7853	4.7587	4.8933	**3.2799**

			0.95	11.704	7.8841	8.333	4.84	7.3532	7.2132	7.6524	**3.4885**
			0.99	56.042	13.251	18.058	0.5647	10.935	8.3058	12.339	**0.263**
		10	**0.8**	13.633	12.162	12.329	10.78	11.898	11.879	22.045	**9.7635**
			0.9	24.62	19.913	20.457	15.726	19.141	19.034	19.573	**13.118**
			0.95	46.818	31.537	33.332	19.36	29.413	28.853	30.61	**13.954**
			0.99	224.17	53.006	72.233	2.2591	43.741	33.223	49.358	**1.0521**
0.9	**0.5**	**1**	**0.8**	0.1363	0.1217	0.1281	0.1081	0.1153	0.1148	0.121	**0.1018**
			0.9	0.2462	0.1992	0.2197	0.1575	0.1808	0.1777	0.1971	**0.141**
			0.95	0.4682	0.3154	0.3811	0.1937	0.2662	0.2506	0.3099	**0.1589**
			0.99	2.2417	0.5301	1.1925	0.0227	0.3386	0.1258	0.5118	**0.0144**
		3	**0.8**	1.227	1.0947	1.153	0.9706	1.0369	1.0316	1.0883	**0.913**
			0.9	2.2158	1.7923	1.9774	1.4156	1.6262	1.5981	1.7737	**1.2664**
			0.95	4.2136	2.8383	3.4302	1.7424	2.3957	2.2554	2.7891	**1.4286**
			0.99	20.175	4.7705	10.733	0.2033	3.0474	1.1322	4.6063	**0.1281**
		5	**0.8**	3.4082	3.0407	3.2026	2.6954	3.2174	2.8653	3.0227	**2.5351**
			0.9	6.1551	4.9784	5.4928	3.9317	4.5169	4.4388	4.9269	**3.517**
			0.95	11.704	7.8841	9.5284	4.84	6.6547	6.265	7.7476	**3.9681**
			0.99	56.042	13.251	29.813	0.5647	8.4651	3.1451	12.795	**0.3558**
		10	**0.8**	13.633	12.162	12.81	10.78	11.519	11.46	12.09	**10.138**
			0.9	24.62	19.913	21.971	15.726	18.067	17.754	19.707	**14.066**
			0.95	46.818	31.537	38.114	19.36	26.619	25.06	30.991	**15.873**
			0.99	224.17	53.006	119.25	2.2591	33.861	12.58	51.182	**1.4231**
0.9	**0.8**	**1**	**0.8**	0.1363	0.1217	0.133	0.1081	0.1118	0.1107	0.1214	**0.1055**
			0.9	0.2462	0.1992	0.2354	0.1575	0.171	0.1654	0.1984	**0.1507**
			0.95	0.4682	0.3154	0.4323	0.1937	0.2422	0.2155	0.3133	**0.1793**
			0.99	2.2417	0.5301	1.7822	0.0227	0.2703	0.0278	0.524	**0.019**
		3	**0.8**	1.227	1.0947	1.1971	0.9706	1.0046	0.9947	1.0922	**0.9473**
			0.9	2.2158	1.7923	2.1188	1.4156	1.5378	1.4872	1.7851	**1.3548**
			0.95	4.2136	2.8383	3.8903	1.7424	2.1794	1.9392	2.8197	**1.6129**
			0.99	20.175	4.7705	16.04	0.2033	2.4324	0.2498	4.7157	**0.1703**
		5	**0.8**	3.4082	3.0407	3.3251	2.6954	2.7899	2.7627	3.0336	**2.6306**
			0.9	6.1551	4.9784	5.8855	3.9317	4.2714	4.1306	4.9584	**3.7629**
			0.95	11.704	7.8841	10.806	4.84	6.0539	5.3867	7.8325	**4.4803**
			0.99	56.042	13.251	44.554	0.5647	6.7567	0.6938	13.099	**0.473**
		10	**0.8**	13.633	12.162	13.3	10.78	11.159	11.049	12.134	**10.521**
			0.9	24.62	19.913	23.542	15.726	17.085	16.521	19.833	**15.05**
			0.95	46.818	31.537	43.226	19.36	24.216	21.547	31.33	**17.922**
			0.99	224.17	53.006	178.22	2.2591	27.027	2.7752	52.397	**1.892**

Table 2. Estimated MSE when p=3, n=100

K	D	Sigma	Rho	OLS	RIDGE	LIU	K-L	MRT	TSS	MNTP	LKL
0.3	**0.2**	**1**	**0.8**	0.0622	0.0611	0.0594	0.06	0.0609	0.0609	0.0592	**0.0574**
			0.9	0.1131	0.1093	0.1036	0.1056	0.1086	0.1086	0.1029	**0.0968**
			0.95	0.2163	0.2023	0.1828	0.1889	0.1997	0.1996	0.1803	**0.1598**
			0.99	1.0445	0.7667	0.5345	0.5329	0.7248	0.7164	0.5009	**0.2761**
		3	**0.8**	0.5596	0.5499	0.5345	0.5403	0.548	0.548	0.5327	**0.5163**
			0.9	1.0178	0.9839	0.9322	0.9506	0.9774	0.9772	0.9259	**0.871**
			0.95	1.9464	1.8211	1.6447	1.7002	1.7976	1.7966	1.6228	**1.4381**
			0.99	9.4002	6.9	4.8107	4.7956	6.5235	6.4475	4.5079	**2.4845**
		5	**0.8**	1.5543	1.5275	1.4848	1.5009	1.5222	1.5222	1.4797	**1.434**
			0.9	2.8273	2.7332	2.5896	2.6407	2.7149	2.7145	2.572	**2.4195**
			0.95	5.4066	5.0586	4.5687	4.7227	4.9932	4.9905	4.5078	**3.9948**
			0.99	26.112	19.167	13.363	13.321	18.121	17.91	12.522	**6.9013**
		10	**0.8**	6.2173	6.11	5.9393	6.0036	6.0889	6.0886	5.9186	**5.7359**
			0.9	11.309	10.933	10.358	10.563	10.86	10.858	10.288	**9.6778**
			0.95	21.626	20.235	18.275	18.891	19.973	19.962	18.031	**15.979**
			0.99	104.45	76.667	53.452	53.285	72.483	71.639	50.087	**27.605**
0.3	**0.5**	**1**	**0.8**	0.0622	0.0611	0.0604	0.06	0.0606	0.0606	0.0599	**0.0584**
			0.9	0.1131	0.1093	0.1071	0.1056	0.1075	0.1075	0.1053	**0.1001**
			0.95	0.2163	0.2023	0.195	0.1889	0.1959	0.1956	0.1886	**0.1704**
			0.99	1.0445	0.7667	0.7055	0.5329	0.6683	0.6443	0.6014	**0.3623**
		3	**0.8**	0.5596	0.5499	0.5438	0.5403	0.5452	0.5451	0.5391	**0.5252**
			0.9	1.0178	0.9839	0.9639	0.9506	0.9677	0.9672	0.9477	**0.9005**
			0.95	1.9464	1.8211	1.7548	1.7002	1.7631	1.7601	1.6971	**1.5337**
			0.99	9.4002	6.9	6.3497	4.7956	6.015	5.7984	5.4129	**3.2607**
		5	**0.8**	1.5543	1.5275	1.5107	1.5009	1.5144	1.5142	1.4976	**1.4589**
			0.9	2.8273	2.7332	2.6774	2.6407	2.6879	2.6867	2.6324	**2.5012**
			0.95	5.4066	5.0586	4.8743	4.7227	4.8976	4.8892	4.7141	**4.2603**
			0.99	26.112	19.167	17.638	13.321	16.708	16.107	15.036	**9.0574**
		10	**0.8**	6.2173	6.11	6.0427	6.0036	6.0574	6.0567	5.9903	**5.8355**
			0.9	11.309	10.933	10.71	10.563	10.752	10.747	10.529	**10.005**
			0.95	21.626	20.235	19.497	18.891	19.59	19.557	18.856	**17.041**
			0.99	104.45	76.667	70.552	53.285	66.834	64.426	60.143	**36.23**
0.3	**0.8**	**1**	**0.8**	0.0622	0.0611	0.0615	0.06	0.0603	0.0603	0.0606	**0.0594**
			0.9	0.1131	0.1093	0.1107	0.1056	0.1065	0.1064	0.1077	**0.1034**
			0.95	0.2163	0.2023	0.2076	0.1889	0.1922	0.1916	0.1968	**0.1814**
			0.99	1.0445	0.7667	0.9008	0.5329	0.6183	0.5761	0.7011	**0.4606**
		3	**0.8**	0.5596	0.5499	0.5532	0.5403	0.5424	0.5422	0.5456	**0.5342**

			0.9	1.0178	0.9839	0.9961	0.9506	0.9581	0.9573	0.9694	**0.9304**
			0.95	1.9464	1.8211	1.8685	1.7002	1.7297	1.724	1.7715	**1.6325**
			0.99	9.4002	6.9	8.1072	4.7956	5.5649	5.1849	6.3102	**4.1455**
		5	**0.8**	1.5543	1.5275	1.5368	1.5009	1.5066	1.5062	1.5155	**1.484**
			0.9	2.8273	2.7332	2.7668	2.6407	2.6614	2.659	2.6928	**2.5844**
			0.95	5.4066	5.0586	5.1902	4.7227	4.8048	4.789	4.9207	**4.5348**
			0.99	26.112	19.167	22.52	13.321	15.458	14.402	17.528	**11.515**
		10	**0.8**	6.2173	6.11	6.1472	6.0036	6.0262	6.0248	60.621	**59.361**
			0.9	11.309	10.933	11.067	10.563	10.645	10.636	10.771	**10.338**
			0.95	21.626	20.235	20.761	18.891	19.219	19.156	19.683	**18.139**
			0.99	104.45	76.667	90.08	53.285	61.832	57.609	70.113	**46.061**
0.6	**0.2**	**1**	**0.8**	0.0622	0.0601	0.0594	0.058	0.0597	0.0596	0.059	**0.0555**
			0.9	0.1131	0.1058	0.1036	0.0987	0.1044	0.1043	0.1022	**0.0905**
			0.95	0.2163	0.1898	0.1828	0.1651	0.1851	0.1847	0.1779	**0.1399**
			0.99	1.0445	0.5881	0.5345	0.2651	0.5341	0.5128	0.4704	**0.1397**
		3	**0.8**	0.5596	0.5405	0.5345	0.5218	0.5368	0.5367	0.5308	**0.4987**
			0.9	1.0178	0.9518	0.9322	0.8881	0.9394	0.9389	0.9197	**0.814**
			0.95	1.9464	1.708	1.6447	1.4859	1.6658	1.6623	1.6014	**1.2582**
			0.99	9.4002	5.2929	4.8107	2.3853	4.8065	4.6154	4.2335	**1.2568**
		5	**0.8**	1.5543	1.5014	1.4848	1.4495	1.4912	1.4909	1.4745	**1.385**
			0.9	2.8273	2.6439	2.5896	2.4669	2.6094	2.608	2.5547	**2.2611**
			0.95	5.4066	4.7444	4.5687	4.1273	4.6272	4.6174	4.4482	**3.4948**
			0.99	26.112	14.702	13.363	6.6258	13.351	12.821	11.76	**3.4911**
		10	**0.8**	6.2173	6.0056	5.9393	5.7978	5.9646	5.9637	5.8981	**5.54**
			0.9	11.309	10.575	10.358	9.8677	10.438	10.432	10.219	**9.0443**
			0.95	21.626	18.978	18.275	16.509	18.509	18.47	17.793	**13.979**
			0.99	104.45	58.81	53.452	26.503	53.405	51.282	47.039	**13.964**
0.6	**0.5**	**1**	**0.8**	0.0622	0.0601	0.0604	0.058	0.0591	0.059	0.0594	**0.0564**
			0.9	0.1131	0.1058	0.1071	0.0987	0.1024	0.1022	0.1036	**0.0935**
			0.95	0.2163	0.1898	0.195	0.1651	0.1784	0.1772	0.1825	**0.1491**
			0.99	1.0445	0.5881	0.7055	0.2651	0.4662	0.4099	0.5193	**0.1819**
		3	**0.8**	0.5596	0.5405	0.5438	0.5218	0.5314	0.5311	0.5345	**0.5073**
			0.9	1.0178	0.9518	0.9639	0.8881	0.9213	0.9197	0.9319	**0.8414**
			0.95	1.9464	1.708	1.7548	1.4859	1.6055	1.5949	1.6423	**1.3412**
			0.99	9.4002	5.2929	6.3497	2.3853	4.1956	3.6891	4.6734	**1.6365**
		5	**0.8**	1.5543	1.5014	1.5107	1.4495	1.476	1.4753	1.4847	**1.409**
			0.9	2.8273	2.6439	2.6774	2.4669	2.5591	2.5546	2.5885	**2.3372**
			0.95	5.4066	4.7444	4.8743	4.1273	4.4597	4.4303	4.5619	**3.7255**
			0.99	26.112	14.702	17.638	6.6258	11.654	10.248	12.982	**4.5459**

		10	**0.8**	6.2173	6.0056	6.0427	5.7978	5.9041	5.9012	5.9386	**5.6359**
			0.9	11.309	10.575	10.71	9.8677	10.236	10.218	10.354	**9.3486**
			0.95	21.626	18.978	19.497	16.509	17.839	17.721	18.248	**14.902**
			0.99	104.45	58.81	70.552	26.503	46.617	40.99	51.926	**18.183**
0.6	**0.8**	**1**	**0.8**	0.0622	0.0601	0.0615	0.058	0.0585	0.0584	0.0598	**0.0574**
			0.9	0.1131	0.1058	0.1107	0.0987	0.1004	0.1001	0.1049	**0.0966**
			0.95	0.2163	0.1898	0.2076	0.1651	0.1721	0.1699	0.1869	**0.1586**
			0.99	1.0445	0.5881	0.9008	0.2651	0.4107	0.319	0.5623	**0.2299**
		3	**0.8**	0.5596	0.5405	0.5532	0.5218	0.526	0.5255	0.5381	**0.516**
			0.9	1.0178	0.9518	0.9961	0.8881	0.9037	0.9007	0.9439	**0.8693**
			0.95	1.9464	1.708	1.8685	1.4859	1.5485	1.529	1.6821	**1.4271**
			0.99	9.4002	5.2929	8.1072	2.3853	3.6959	2.8709	5.0602	**2.0684**
		5	**0.8**	1.5543	1.5014	1.5368	1.4495	1.4611	1.4598	1.4947	**1.4332**
			0.9	2.8273	2.6439	2.7668	2.4669	2.5103	2.5018	2.6219	**2.4146**
			0.95	5.4066	4.7444	5.1902	4.1273	4.3015	4.2472	4.6724	**3.964**
			0.99	26.112	14.702	22.52	6.6258	10.266	7.9745	14.056	**5.7455**
		10	**0.8**	6.2173	6.0056	6.1472	5.7978	5.8445	5.8391	5.9789	**5.7328**
			0.9	11.309	10.575	11.067	9.8677	10.041	10.007	10.488	**9.6582**
			0.95	21.626	18.978	20.761	16.509	17.206	16.989	18.69	**15.856**
			0.99	104.45	58.81	90.08	26.503	41.065	31.898	56.225	**22.982**
0.7	**0.2**	**1**	**0.8**	0.0622	0.0597	0.0594	0.0573	0.0593	0.0592	0.0589	**0.0549**
			0.9	0.1131	0.1046	0.1036	0.0965	0.103	0.103	0.102	**0.0885**
			0.95	0.2163	0.1859	0.1828	0.1579	0.1806	0.1801	0.1772	**0.1338**
			0.99	1.0445	0.5425	0.5345	0.2074	0.4873	0.4622	0.4609	**0.1102**
		3	**0.8**	0.5596	0.5374	0.5345	0.5158	0.5332	0.5331	0.5302	**0.4929**
			0.9	1.0178	0.9415	0.9322	0.8682	0.9273	0.9266	0.9176	**0.7959**
			0.95	1.9464	1.6727	1.6447	1.4207	1.6252	1.6206	1.5943	**1.2034**
			0.99	9.4002	4.8827	4.8107	1.8662	4.3853	4.1593	4.1478	**0.9908**
		5	**0.8**	1.5543	1.4929	1.4848	1.4327	1.481	1.4807	1.4728	**1.3691**
			0.9	2.8273	2.6151	2.5896	2.4117	2.5757	2.5737	2.5489	**2.2108**
			0.95	5.4066	4.6464	4.5687	3.9463	4.5145	4.5016	4.4287	**3.3427**
			0.99	26.112	13.563	13.363	5.1838	12.181	11.554	11.522	**2.7521**
		10	**0.8**	6.2173	5.9714	5.9393	5.7309	5.9241	5.9229	5.8912	**5.4763**
			0.9	11.309	10.461	10.358	96.467	10.303	10.295	10.196	**8.8429**
			0.95	21.626	18.586	18.275	15.785	18.058	18.006	17.715	**13.37**
			0.99	104.45	54.252	53.452	20.735	48.726	46.214	46.086	**11.008**
0.7	**0.5**	**1**	**0.8**	0.0622	0.0597	0.0604	0.0573	0.0586	0.0585	0.0592	**0.0558**
			0.9	0.1131	0.1046	0.1071	0.0965	0.1008	0.1005	0.103	**0.0915**
			0.95	0.2163	0.1859	0.195	0.1579	0.1731	0.1716	0.1805	**0.1426**

			0.99	1.0445	0.5425	0.7055	0.2074	0.4192	0.3541	0.4957	**0.1429**
		3	**0.8**	0.5596	0.5374	0.5438	0.5158	0.5269	0.5266	0.5329	**0.5014**
			0.9	1.0178	0.9415	0.9639	0.8682	0.9066	0.9044	0.9267	**0.8227**
			0.95	1.9464	1.6727	1.7548	1.4207	1.5578	1.544	1.6247	**1.2827**
			0.99	9.4002	4.8827	6.3497	1.8662	3.7726	3.1868	4.4611	**1.2856**
		5	**0.8**	1.5543	1.4929	1.5107	1.4327	1.4636	1.4626	1.4804	**1.3928**
			0.9	2.8273	2.6151	2.6774	2.4117	2.5184	2.5123	2.5742	**2.285**
			0.95	5.4066	4.6464	4.8743	3.9463	4.3272	4.2889	4.5129	**3.5629**
			0.99	26.112	13.563	17.638	5.1838	10.48	8.8522	12.392	**3.5709**
		10	**0.8**	6.2173	5.9714	6.0427	5.7309	5.8543	5.8505	5.9216	**5.5711**
			0.9	11.309	10.461	10.71	9.6467	10.073	10.049	10.297	**9.14**
			0.95	21.626	18.586	19.497	15.785	17.309	17.155	18.052	**14.251**
			0.99	104.45	54.252	70.552	20.735	41.918	35.409	49.567	**14.284**
0.7	**0.8**	**1**	**0.8**	0.0622	0.0597	0.0615	0.0573	0.0579	0.0578	0.0595	**0.0567**
			0.9	0.1131	0.1046	0.1107	0.0965	0.0985	0.0981	0.104	**0.0945**
			0.95	0.2163	0.1859	0.2076	0.1579	0.1661	0.1633	0.1838	**0.1517**
			0.99	1.0445	0.5425	0.9008	0.2074	0.3647	0.2611	0.5252	**0.1801**
		3	**0.8**	0.5596	0.5374	0.5532	0.5158	0.5207	0.5201	0.5356	**0.51**
			0.9	1.0178	0.9415	0.9961	0.8682	0.8867	0.8826	0.9356	**0.8498**
			0.95	1.9464	1.6727	1.8685	1.4207	1.4947	1.4694	1.6539	**1.3646**
			0.99	9.4002	4.8827	8.1072	1.8662	3.2818	2.3494	4.7269	**1.6206**
		5	**0.8**	1.5543	1.4929	1.5368	1.4327	1.4465	1.4447	1.4879	**1.4167**
			0.9	2.8273	2.6151	2.7668	2.4117	2.463	2.4517	2.5989	**2.3606**
			0.95	5.4066	4.6464	5.1902	3.9463	4.1518	4.0815	4.594	**3.7905**
			0.99	26.112	13.563	22.52	5.1838	9.1161	6.5261	13.13	**45014**
		10	**0.8**	6.2173	5.9714	6.1472	5.7309	5.7859	5.7786	5.9516	**5.6667**
			0.9	11.309	10.461	11.067	9.6467	9.8519	9.8067	10.396	**9.4423**
			0.95	21.626	18.586	20.761	15.785	16.607	16.326	18.376	**15.162**
			0.99	104.45	54.252	90.08	20.735	36.464	26.104	52.521	**18.006**
0.9	**0.2**	**1**	**0.8**	0.0622	0.0591	0.0594	0.0561	0.0585	0.0584	0.0588	**0.0537**
			0.9	0.1131	0.1024	0.1036	0.0923	0.1004	0.1003	0.1015	**0.0847**
			0.95	0.2163	0.1784	0.1828	0.1444	0.1721	0.1713	0.1756	**0.1224**
			0.99	10.445	0.4662	0.5345	0.1233	0.4107	0.3788	0.4427	**0.0669**
		3	**0.8**	0.5596	0.5314	0.5345	0.504	0.526	0.5258	0.529	**0.4817**
			0.9	1.0178	0.9213	0.9322	0.8299	0.9037	0.9026	0.9135	**0.761**
			0.95	1.9464	1.6055	1.6447	1.2988	1.5485	1.5414	1.5804	**1.101**
			0.99	9.4002	4.1956	4.8107	1.1091	3.6959	3.4088	3.9842	**0.6009**
		5	**0.8**	1.5543	1.476	1.4848	1.3999	1.4611	1.4606	1.4694	**1.3379**
			0.9	2.8273	2.5591	2.5896	2.3051	2.5103	2.5072	2.5375	**2.1136**

			0.95	5.4066	4.4597	4.5687	3.6077	4.3015	4.2817	4.3899	**3.0582**
			0.99	26.112	11.654	13.363	3.0808	10.266	9.4689	11.067	**1.6691**
		10	**0.8**	6.2173	5.9041	5.9393	5.5996	5.8445	5.8425	5.8776	**5.3513**
			0.9	11.309	10.236	10.358	9.2202	10.041	10.029	10.15	**8.454**
			0.95	21.626	17.839	18.275	14.431	17.206	17.127	17.559	**12.232**
			0.99	104.45	46.617	53.452	12.323	41.065	37.876	44.268	**6.6759**
0.9	**0.5**	**1**	**0.8**	0.0622	0.0591	0.0604	0.0561	0.0576	0.0575	0.0589	**0.0545**
			0.9	0.1131	0.1024	0.1071	0.0923	0.0976	0.0972	0.1018	**0.0875**
			0.95	0.2163	0.1784	0.195	0.1444	0.1632	0.1609	0.1767	**0.1304**
			0.99	1.0445	0.4662	0.7055	0.1233	0.3446	0.2653	0.4532	**0.0859**
		3	**0.8**	0.5596	0.5314	0.5438	0.504	0.5181	0.5176	0.5299	**0.49**
			0.9	1.0178	0.9213	0.9639	0.8299	0.8784	0.8749	0.9165	**0.7864**
			0.95	1.9464	1.6055	1.7548	1.2988	1.4688	1.4479	1.5902	**1.1732**
			0.99	9.4002	4.1956	6.3497	1.1091	3.1008	2.3876	4.0787	**0.7724**
		5	**0.8**	1.5543	1.476	1.5107	1.3999	1.4392	1.4377	1.4719	**1.361**
			0.9	2.8273	2.5591	2.6774	2.3051	2.4399	2.4303	2.5458	**2.1844**
			0.95	5.4066	4.4597	4.8743	3.6077	4.08	4.0218	4.4173	**3.2587**
			0.99	26.112	11.654	17.638	3.0808	8.6133	6.6321	11.33	**2.1454**
		10	**0.8**	6.2173	5.9041	6.0427	5.5996	5.7569	5.7507	5.8877	**5.4437**
			0.9	11.309	10.236	10.71	9.2202	9.7593	9.7213	10.183	**8.7372**
			0.95	21.626	17.839	19.497	14.431	16.32	16.087	17.669	**13.034**
			0.99	104.45	46.617	70.552	12.323	34.453	26.528	45.318	**8.5812**
0.9	**0.8**	**1**	**0.8**	0.0622	0.0591	0.0615	0.0561	0.0568	0.0566	0.059	0.0555
			0.9	0.1131	0.1024	0.1107	0.0923	0.0949	0.0942	0.1022	0.0903
			0.95	0.2163	0.1784	0.2076	0.1444	0.1551	0.1509	0.1777	0.1387
			0.99	1.0445	0.4662	0.9008	0.1233	0.2934	0.173	0.4615	0.1075
		3	**0.8**	0.5596	0.5314	0.5532	0.504	0.5105	0.5094	0.5308	0.4984
			0.9	1.0178	0.9213	0.9961	0.8299	0.8541	0.8477	0.9194	0.8123
			0.95	1.9464	1.6055	1.8685	1.2988	1.3954	1.3574	1.5996	1.2478
			0.99	9.4002	4.1956	8.1072	1.1091	2.6407	1.557	4.1536	0.9668
		5	**0.8**	1.5543	1.476	1.5368	1.3999	1.4179	1.415	1.4744	1.3843
			0.9	2.8273	2.5591	2.7668	2.3051	2.3725	2.3548	2.5539	2.2564
			0.95	5.4066	4.4597	5.1902	3.6077	3.8759	3.7705	4.4432	3.4659
			0.99	26.112	11.654	22.52	3.0808	7.3351	4.3248	11.538	2.6854
		10	**0.8**	6.2173	5.9041	6.1472	5.5996	5.6715	5.6598	5.8976	5.5369
			0.9	11.309	10.236	11.067	9.2202	9.4898	9.4189	10.215	9.0253
			0.95	21.626	17.839	20.761	14.431	15.503	15.082	17.773	13.863
			0.99	104.45	46.617	90.08	12.323	29.34	17.299	46.151	10.741

NOTE: The bolded value is the minimum MSE

Discussions of Results

From Table 1and Table 2, it appears that as σ and ρ increase respectively, the estimated MSE values increase. Likewise, as n increases, the estimated MSE values decrease. As expected and from several simulations and empirical research, when the multicollinearity problem exists, the OLS estimator gives the highest MSE values and performs the worst among all estimators.

Seven already existing estimators were compared with the proposed estimator. The results show that the proposed LKL estimator consistently performs better and dominates other existing estimators considered in this study. This was observed across the study's sample sizes, sigma levels, and rho levels. Thus, the findings agree with the theoretical results earlier discussed in this study.

5. 2. Numerical Example

In this section, two real-life data sets were analyzed to compare the performance of the proposed (LKL) and other existing estimators.

Portland cement

In this section, Portland cement data was used to demonstrate the performance of the proposed estimator. The Portland cement data was originally adopted by Woods *et al.* (1932) and was later adopted by Li and Yang (2012), and then Ayinde *et al.* (2018). The data set is widely known as the Portland cement dataset. The regression model for these data is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i \tag{48}$$

where y_i = heat evolved after 180 days of curing measured in calories per gram of cement, X_1= tricalcium aluminate, X_2 = tricalcium silicate, X_3 = tetracalcium aluminoferrite, and X_4 = β-dicalcium silicate. The variance inflation factors are VIF1 = 38.50, VIF2 = 254.42, VIF3 = 46.87, and VIF4 = 282.51. Eigenvalues of $X'X$ matrix are $\lambda_1 = 44676.206$, $\lambda_2 = 5965.422$, $\lambda_3 = 809.952$, and $\lambda_4 = 105.419$, and the condition number of $X'X$ is approximately 424. The VIFs, eigenvalues, and condition numbers indicate that severe multicollinearity exists. The estimated parameters and the MSE values of the estimators are presented in Table 3.

Table 3. Results regression coefficients and corresponding MSE values for Portland cement data

	OLS	RIDGE	LIU	KL	MRT	TSS	MNTP	LKL
$\hat{\alpha}_0$	62.40537	1.021341	2.87726	47.13058	0.9807	-1.69321	-0.26497	2.18401
$\hat{\alpha}_1$	1.551103	2.18148	2.15033	1.70822	2.10085	2.20936	-3.2682	2.15745
$\hat{\alpha}_2$	0.510168	1.14304	1.1267	0.66759	1.1434	1.171032	-14.8054	1.13384
$\hat{\alpha}_3$	0.101909	0.747	0.71886	0.26262	0.74739	0.775531	-2.74802	0.7261
$\hat{\alpha}_4$	-0.14406	0.47616	0.45923	0.01023	0.47657	0.50359	-6.37113	0.046624
k	-	0.07676	-	0.00017	0.07676	0.07676	0.07676	0.00017
d			0.044222	-	0.044222	0.044222	0.044222	0.567
MSE	4912.09	3770.86	3555.261	3033.81	3615.903	4114.263	3798.261	**2172.55**

From Table 3, the proposed estimator (LKL) performs best among other estimators as it gives the smallest MSE value. As observed in the simulation study results, the OLS estimator did not perform well in the presence of multicollinearity as it has the highest MSE.

Longley cement

The Longley data was originally used by Longley (1967) and then by Lukman and Ayinde (2017), and then Dawoud and Kibria (2020). The regression model for this dataset is:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_6 X_6 + \varepsilon \tag{49}$$

The variance inflation factors are VIF1 = 135.53, VIF2 = 1788.51, VIF3 = 33.62, VIF4 = 3.59, VIF5 = 399.15 and VIF6 = 758.98. Eigenvalues of $X'X$ matrix are $\lambda_1 = 2.76779 \times 10^{12}$, $\lambda_2 = 7{,}039{,}139{,}179$, $\lambda_3 = 11{,}608{,}993.96$, $\lambda_4 = 2{,}504{,}761.021$, $\lambda_5 = 1738.356$, $\lambda_6 = 13.309$ and the condition number of $X'X$ is $456{,}070$. The VIFs, the eigenvalues, and the condition number all indicate that severe multicollinearity exists.

Table 4. Results regression coefficients and corresponding MSE values for Longley data

	OLS	RIDGE	LIU	KL	MRT	TSS	MNTP	LKL
$\hat{\alpha}_1$	-52.9936	-52.8644	-48.982	-52.3929	-52.8644	-52.8645	44.7419	-48.9406
$\hat{\alpha}_2$	0.0711	0.071	0.0702	0.0709	0.071	0.071046	50.89905	0.0701
$\hat{\alpha}_3$	-0.4142	-0.414	-0.4072	-0.4132	-0.414	-0.41398	14.77316	-0.4062
$\hat{\alpha}_4$	-0.4235	-0.4238	-0.4332	-0.4249	-0.4238	-0.42378	0.88509	-0.4345
$\hat{\alpha}_5$	-0.5726	-0.5726	-0.5748	-0.5729	-0.5726	-0.57264	0.38458	-0.5751
$\hat{\alpha}_6$	48.4179	48.4036	47.9725	48.3514	48.4036	48.40359	-5.03442	47.9105
k	-	0.03	-	0.07	0.03	0.03	0.03	0.07
d			0.00003	-	0.00003	0.00003	0.00003	0.005
MSE	17095.15	17018.93	14822.89	16742.29	17018.93	17018.93	14822.89	**14532.72**

From Table 4, the proposed estimator (LKL) performs best among other estimators as it gives the smallest MSE value. The OLS estimator did not perform well in the presence of multicollinearity as it has the highest MSE.

6. CONCLUSION

This paper proposed a new class two-parameter estimator, namely, the Liu–Kibria-Lukman (LKL) estimator, to combat multicollinearity in linear regression models. This study theoretically compares the proposed LKL estimator with some existing estimators like the ordinary least squares (OLS) estimator, the ordinary ridge regression (ORR) estimator, the Liu estimator, Kibria and Lukman (KL) estimator , and the modified ridge-type estimator (Hoerl and Kennard, 1970, Liu, 1993, Kibria and Lukman, 2020, Lukman *et al*., 2019) A simulation study was conducted to compare the performance of these already existing estimators with the proposed LKL estimator. From the simulation study results, the proposed LKL estimator performs better than the existing estimators under some conditions. Two real-life datasets were analyzed to bolster the results of the theoretical comparison and simulation study. We recommended this new estimator for practitioners to combat the problem of multicollinearity in linear regression models.

ETHICAL DECLARATION

In the writing process of the study titled "Combating Multicollinearity: A New Two-Parameter Approach", there were followed the scientific, ethical and the citation rules; was not made any falsification on the collected data and this study was not sent to any other academic media for evaluation.

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