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Conformal Quasi-Hemi-Slant Riemannian Maps

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Abstract
In this paper, we state some geometric properties of conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We give necessary and sufficient conditions for certain distributions to be integrable and get examples. For such distributions, we examine which conditions define totally geodesic foliations on base manifold. In addition, we apply notion of pluriharmonicity to get some relations between horizontally homothetic maps and conformal quasi-hemi-slant Riemannian maps.

Keywords: Riemannian map, Conformal Riemannian map.
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1. Introduction

The theory of Riemannian submersions between Riemannian manifolds was initially studied by O'Neill [17] and Gray [10]. Particularly, the concept of Riemannian submersions [7] and isometric immersions [6] were studied by Falcitelli and Chen. Then, Riemannian submersions were studied in various types as an anti-invariant, a semi-invariant, a slant, a hemi-slant, etc [13, 25]. Submersions between almost Hermitian manifolds expanded to almost Hermitian submersions [30]. Then, this concept was generalized to the notion of Riemannian map by Fischer [8]. Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions. Riemannian submersions have many application. Let $\Phi:(M_1,g_1)\longrightarrow(M_2,g_2)$ be a smooth map between Riemannian manifolds such that $0<rank\Phi<\min\{\dim(M_1),\dim(M_2)\}$. Then the tangent bundle TM_1 of M_1 has the following decomposition:

$$TM_1 = ker\Phi_* \oplus (ker\Phi_*)^{\perp}.$$

Since $rank\Phi<\min\{\dim(M_1),\dim(M_2)\}$, always we have $(range\Phi_*)^{\perp}$. In this way, tangent bundle TM_2 of M_2 has the following decomposition:

$$TM_2 = (range\Phi_*) \oplus (range\Phi_*)^{\perp}.$$

A smooth map $\Phi:(M_1^m,g_1)\longrightarrow(M_2^m,g_2)$ is called Riemannian map at $p_1\in M_1$ if the horizontal restriction $\Phi^h_{*p_1}:(ker\Phi_{*p_1})^{\perp}\longrightarrow(range\Phi_*)$ is a linear isometry. Hence, a Riemannian map satisfies the equation

$$g_1(Z_1,Z_2) = g_2(\Phi_*(Z_1),\Phi_*(Z_2)) \tag{1.1}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $ker\Phi_*=\{0\}$ and $(range\Phi_*)^{\perp}=\{0\}$ [7]. An important application field of Riemannian maps is the eikonal equation. It acts as a bridge between geometric optics and physical optics. Also, Riemannian maps and their applications studied by Garcia-Rio and Kupeli in semi-Riemannian geometry [9]. Recently, some optimal inequalities for Riemannian maps from Riemannian manifolds onto space forms were established in [12].

Moreover, Şahin introduced any other types of Riemannian maps [20, 21, 22, 23], see also [18, 19]. In further studies, in particular Akyol, Şahin and Yanan searched this type submersions [1, 2, 3, 4, 31] and Riemannian maps [26, 27, 32, 35] under conformality case, see also [11]. All these studies have many applications as texture mapping, remeshing and simulation [14], computer graphics and medical imaging fields [28], brain mapping research [29]. For a comprehensive study in which these issues are introduced and their applications are given, see [25]. We say that $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rank\Phi_{*p} \leq \min\{m,n\}$ and Φ_{*p} maps the horizontal space $(ker(\Phi_{*p})^{\perp})$ conformally onto $range(\Phi_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(\Phi_{*p}(Z_1), \Phi_{*p}(Z_2)) = \lambda^2(p) g_M(Z_1, Z_2) \tag{1.2}$$

for $Z_1, Z_2 \in \Gamma((ker(\Phi_{*p})^{\perp})$. Also Φ is called conformal Riemannian if Φ is conformal Riemannian at each $p \in M$ [24].

An even-dimensional Riemannian manifold (M, g_M, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^2 = -\mathbb{I}$ where $\mathbb{I}$ denotes the identity transformation of TM and

$$g_M(E,F) = g_M(JE, JF), \forall E, F \in \Gamma(TM). \tag{1.3}$$

Let (M, g_M, J) is an almost Hermitian manifold and its Levi-Civita connection is ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_E J)F = 0, \tag{1.4}$$

we say M is a Kaehler manifold [36].

Therefore, in section 2, we present necessary background concepts to be used in this paper. In section 3, we study conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We introduce some properties as integrability conditions and totally geodesic foliation defining of distributions. In section 4, we use the concept of pluriharmonicity to introduce relations between horizontally homothetic maps and conformal quasi-hemi-slant Riemannian maps.

2. Preliminaries

In this section, we give several definitions and results to be used throughout the study for conformal quasi-hemi-slant Riemannian maps. Let $\Phi : (M, g_M) \longrightarrow (N, g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(E,F) = \overset{N}{\nabla}{}^{\Phi}_{E}\Phi_*(F) - \Phi_*(\overset{M}{\nabla}_E F) \tag{2.1}$$

for $E, F \in \Gamma(TM)$. The second fundamental form $\nabla\Phi_*$ is symmetric [15].

Then we define O'Neill's tensor fields $\mathscr{T}$ and $\mathscr{A}$ for Riemannian submersions as

$$\mathscr{A}_E F = h\overset{M}{\nabla}_{hE} vF + v\overset{M}{\nabla}_{hE} hF, \tag{2.2}$$

$$\mathscr{T}_E F = h\overset{M}{\nabla}_{vE} vF + v\overset{M}{\nabla}_{vE} hF \tag{2.3}$$

for $E, F \in \Gamma(TM)$ with the Levi-Civita connection $\overset{M}{\nabla}$ of g_M [17]. For any $E \in \Gamma(TM)$, $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. Also, $\mathscr{T}$ is vertical, $\mathscr{T}_E = \mathscr{T}_{vE}$, and $\mathscr{A}$ is horizontal, $\mathscr{A}_E = \mathscr{A}_{hE}$. Note that the tensor field $\mathscr{T}$ is symmetric on the vertical distribution [17]. Additionally, from (2.2) and (2.3) we have

$$\overset{M}{\nabla}_{\xi_1}\xi_2 = \mathscr{T}_{\xi_1}\xi_2 + \hat{\nabla}_{\xi_1}\xi_2, \tag{2.4}$$

$$\overset{M}{\nabla}_{\xi_1}Z_1 = h\overset{M}{\nabla}_{\xi_1}Z_1 + \mathscr{T}_{\xi_1}Z_1, \tag{2.5}$$

$$\overset{M}{\nabla}_{Z_1}\xi_1 = \mathscr{A}_{Z_1}\xi_1 + v\overset{M}{\nabla}_{Z_1}\xi_1, \tag{2.6}$$

$$\overset{M}{\nabla}_{Z_1}Z_2 = h\overset{M}{\nabla}_{Z_1}Z_2 + \mathscr{A}_{Z_1}Z_2 \tag{2.7}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ and $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$, where $\hat{\nabla}_{\xi_1}\xi_2 = v\overset{M}{\nabla}_{\xi_1}\xi_2$ [7].

If a vector field Z on M is related to a vector field Z' on N, we say Z is a projectable vector field. If Z is both a horizontal and a projectable vector field, we say Z is a basic vector field on M. From now on, when we mention a horizontal vector field, we always consider a basic vector field [5].

On the other hand, let $\Phi : (M^m, g_M) \longrightarrow (N^n, g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$(\nabla\Phi_*)(Z_1, Z_2)\mid_{range\Phi_*} = Z_1(\ln\lambda)\Phi_*(Z_2) + Z_2(\ln\lambda)\Phi_*(Z_1) - g_M(Z_1, Z_2)\Phi_*(grad(\ln\lambda)), \tag{2.8}$$

where $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$. Hence from (2.8), we obtain $\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2)$ as

$$\begin{aligned} \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) &= \Phi_*(h\overset{M}{\nabla}_{Z_1}Z_2) + Z_1(\ln\lambda)\Phi_*(Z_2) + Z_2(\ln\lambda)\Phi_*(Z_1) \\ &- g_M(Z_1, Z_2)\Phi_*(grad(\ln\lambda)) + (\nabla\Phi_*)^\perp(Z_1, Z_2), \end{aligned} \tag{2.9}$$

where $(\nabla\Phi_*)^\perp(Z_1, Z_2)$ is the component of $(\nabla\Phi_*)(Z_1, Z_2)$ on $(range\Phi_*)^\perp$ for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$ [26, 27].

Lastly, a map Φ from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if Φ satisfies the following equation

$$(\nabla\Phi_*)(E, F) + (\nabla\Phi_*)(JE, JF) = 0 \tag{2.10}$$

for $E, F \in \Gamma(TM)$ [16].

3. Conformal quasi-hemi-slant Riemannian map

We give definition of conformal quasi-hemi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In the rest of this paper, we take (M, g_M, J) as a Kaehler manifold.

Definition 3.1. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal Riemannian map such that its vertical distribution $ker\Phi_*$ admits three orthogonal distributions D, D_θ and $D_\perp$ which are invariant, slant and anti-invariant distributions, respectively, i.e.*

$$ker\Phi_* = D \oplus D_\theta \oplus D_\perp. \tag{3.1}$$

Then, we say Φ is a conformal quasi-hemi-slant Riemannian map and the angle θ is called the quasi-hemi-slant angle of the map.

Here, we say that;

i) Φ is a conformal hemi-slant Riemannian map [33] if $D = \{0\}$.

ii) Φ is a conformal semi-invariant Riemannian map [27] if $D_\theta = \{0\}$.

iii) Φ is a conformal semi-slant Riemannian map [34] if $D_\perp = \{0\}$.

Therefore, conformal quasi-hemi-slant Riemannian maps are generalization of conformal hemi-slant Riemannian maps, conformal semi-invariant Riemannian maps and conformal semi-slant Riemannian maps. Hence, all these maps provide examples to conformal quasi-hemi-slant Riemannian maps.

We say that conformal quasi-hemi-slant Riemannian map Φ is a *proper conformal quasi-hemi-slant Riemannian map* if the invariant distribution $D \neq \{0\}$, the anti-invariant distribution $D_\perp \neq \{0\}$ and the slant angle $\theta \neq 0, \frac{\pi}{2}$. Now, we give an explicit example to proper condition.

Example 3.2. *Define a map by $\Phi : \mathbb{R}^8 \longrightarrow \mathbb{R}^5$ by*

$$\Phi(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \longrightarrow (x_1 + x_2, x_3 - x_5, \sqrt{2}x_4, b, c)$$

with $b, c \in \mathbb{R}$. We get the horizontal distribution

$$(ker\Phi_*)^\perp = \{Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Z_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, Z_3 = \sqrt{2}(\frac{\partial}{\partial x_4})\}$$

and the vertical distribution

$$ker\Phi_* = \{\xi_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \xi_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \xi_3 = \frac{\partial}{\partial x_6}, \xi_4 = \frac{\partial}{\partial x_7}, \xi_5 = \frac{\partial}{\partial x_8}\},$$

respectively. By the complex structure J of $\mathbb{R}^8$ such that $J = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$, we have

$$J\xi_1 = Z_1,\ J\xi_2 = \frac{1}{\sqrt{2}}Z_3 + \xi_3,\ J\xi_3 = \frac{1}{2}Z_2 - \frac{1}{2}\xi_2,\ J\xi_4 = \xi_5,\ J\xi_5 = -\xi_4.$$

Hence, we obtain the distributions as $D = sp\{\xi_4, \xi_5\}$, $D_\theta = sp\{\xi_2, \xi_3\}$ and $D_\perp = sp\{\xi_1\}$. Therefore, Φ is a proper conformal quasi-hemi-slant Riemannian map with $\lambda = \sqrt{2}$ and the slant angle $\theta = \frac{\pi}{4}$.

Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then we have

$$TM = ker\Phi_* \oplus (ker\Phi_*)^\perp. \tag{3.2}$$

A vertical vector field ξ can be written as

$$\xi = \tilde{P}\xi + \tilde{Q}\xi + \tilde{R}\xi \tag{3.3}$$

where $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are projections to D, D_θ and $D_\perp$, respectively. We get

$$J\xi = \phi\xi + \psi\xi \tag{3.4}$$

where $\phi\xi \in \Gamma(ker\Phi_*)$ and $\psi\xi \in \Gamma((ker\Phi_*)^\perp)$. We obtain $\psi\tilde{P}\xi = 0$, $\phi\tilde{R}\xi = 0$ and

$$J\xi = \phi\tilde{P}\xi + \phi\tilde{Q}\xi + \psi\tilde{Q}\xi + \psi\tilde{R}\xi \tag{3.5}$$

from (3.3) and (3.4). So, we can write

$$J(ker\Phi_*) = D \oplus \phi D_\theta \oplus \psi D_\theta \oplus J(D_\perp). \tag{3.6}$$

From (3.6), we have

$$(ker\Phi_*)^\perp = \psi D_\theta \oplus J(D_\perp) \oplus \mu \tag{3.7}$$

where μ is the orthogonal complement distributions of $\psi D_\theta \oplus J(D_\perp)$ in $(ker\Phi_*)^\perp$ and μ is the invariant with respect to J. At last, for $Z \in \Gamma((ker\Phi_*)^\perp)$, we have

$$JZ = BZ + CZ \tag{3.8}$$

where $BZ \in \Gamma(\psi D_\theta \oplus J(D_\perp))$ and $CZ \in \Gamma(\mu)$.

Here that easily we obtain from (3.1) - (3.7);

$$\phi D_\theta = D_\theta,\ \phi D_\perp = \{0\},\ B\psi D_\theta = D_\theta,\ B\psi D_\perp = D_\perp,\ \psi D = \{0\}, \tag{3.9}$$

$$\phi^2 + B\psi = -\mathbb{I},\ \ \psi\phi + C\psi = 0,\ \ \psi B + C^2 = \mathbb{I},\ \ \phi B + BC = 0. \tag{3.10}$$

Following theorem has same proof with hemi-slant submanifolds as hemi-slant Riemannian maps; see Theorem 3.6. of [23].

Theorem 3.3. *Let Φ be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then Φ is a conformal quasi-hemi-slant Riemannian map if and only if there exists a constant $\lambda \in [0,1]$ and a distribution $\mathscr{D}$ on $ker\Phi_*$ such that*

i) $\mathscr{D} = \{\xi \in \Gamma(ker\Phi_*) | \phi^2\xi = \lambda\xi\}$,

ii) for any $\xi \in \Gamma(ker\Phi_*)$ orthogonal to $\mathscr{D}$, we have $\phi\xi = 0$.

Further, we have $\lambda = -\cos^2\theta$ where θ is the slant angle of Φ.

Hence, we have followings from Theorem 3.3.

$$g_M(\phi V_1,\phi V_2)=\cos^2\theta g_M(V_1,V_2), \tag{3.11}$$

$$g_M(\psi V_1,\psi V_2)=\sin^2\theta g_M(V_1,V_2) \tag{3.12}$$

for any $V_1,V_2\in\Gamma(D_\theta)$.

Recall that always the vertical distribution $ker\Phi_*$ is integrable [25]. Now, we give integrability conditions for certain distributions on total manifolds.

Theorem 3.4. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the invariant distribution D is integrable if and only if*

$$\hat{\nabla}_{U_1}JU_2-\hat{\nabla}_{U_2}JU_1\in\Gamma(D\oplus D_\perp),$$

$$\phi\mathscr{T}_{U_1-U_2}\psi\xi\in\Gamma(D_\theta\oplus D_\perp)$$

for $U_1,U_2\in\Gamma(D)$ and $\xi\in\Gamma(D_\theta\oplus D_\perp)$.

Proof. Since $\mathscr{T}$ is skew-symmetric and from equations (1.4), (2.4), (3.4), we get

$$g_M(\overset{M}{\nabla}_{U_1}U_2,\xi)=g_M(\hat{\nabla}_{U_1}JU_2,\phi\xi)+g_M(\phi\mathscr{T}_{U_1}\psi\xi,U_2) \tag{3.13}$$

for $U_1,U_2\in\Gamma(D)$ and $\xi\in\Gamma(D_\theta\oplus D_\perp)$. Now, changing the roles of U_1 and U_2, we obtain

$$g_M([U_1,U_2],\xi)=g_M(\hat{\nabla}_{U_1}JU_2-\hat{\nabla}_{U_2}JU_1,\phi\xi)+g_M(\phi\mathscr{T}_{U_1}\psi\xi,U_2)-g_M(\phi\mathscr{T}_{U_2}\psi\xi,U_1). \tag{3.14}$$

The proof is complete from equation (3.14). □

Theorem 3.5. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the slant distribution D_θ is integrable if and only if*

$$\begin{aligned}\lambda^2 g_M(h\overset{M}{\nabla}_{V_1}\psi V_2-h\overset{M}{\nabla}_{V_2}\psi V_1,\psi\tilde{R}\xi) &= g_N((\nabla\Phi_*)(V_1-V_2,\tilde{P}\xi),\Phi_*(\psi\phi V_2))+g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))\\ &- g_N((\nabla\Phi_*)(V_2,\phi V_1),\Phi_*(\psi\tilde{R}\xi))+g_N((\nabla\Phi_*)(V_2,J\tilde{P}\xi),\Phi_*(\psi V_1))\\ &- g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))\end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $\xi\in\Gamma(D\oplus D_\perp)$.

Proof. Now, from equations (2.4), (2.5), (3.3) and (3.5), we have

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi) &= g_M(\overset{M}{\nabla}_{V_1}\phi V_2,J\tilde{P}\xi)+g_M(\overset{M}{\nabla}_{V_1}\phi V_2,\psi\tilde{R}\xi)+g_M(\mathscr{T}_{V_1}\psi V_2+h\overset{M}{\nabla}_{V_1}\psi V_2,J\tilde{P}\xi+\psi\tilde{R}\xi)\\ &= -g_M(\overset{M}{\nabla}_{V_1}J\phi V_2,\tilde{P}\xi)+g_M(\mathscr{T}_{V_1}\phi V_2,\psi\tilde{R}\xi)+g_M(\mathscr{T}_{V_1}\psi V_2,J\tilde{P}\xi)+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi)\end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$ and $\xi\in\Gamma(D\oplus D_\perp)$. Since $\mathscr{T}$ is skew-symmetric and from (1.2), (2.1), Theorem 1., we have

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi) &= -g_M(\overset{M}{\nabla}_{V_1}\phi^2V_2,\tilde{P}\xi)-g_M(\overset{M}{\nabla}_{V_1}\psi\phi V_2,\tilde{P}\xi)\\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))-g_M(\mathscr{T}_{V_1}J\tilde{P}\xi,\psi V_2)+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi)\\ &= \cos^2\theta g_M(\overset{M}{\nabla}_{V_1}V_2,\tilde{P}\xi)-g_M(\mathscr{T}_{V_1}\psi\phi V_2,\tilde{P}\xi)\\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}g_M(\overset{M}{\nabla}_{V_1}V_2,\xi)-\cos^2\theta g_M(\overset{M}{\nabla}_{V_1}V_2,\tilde{P}\xi) &= g_M(\mathscr{T}_{V_1}\tilde{P}\xi,\psi\phi V_2)-\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi))\\ &+ \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2))+g_M(h\overset{M}{\nabla}_{V_1}\psi V_2,\psi\tilde{R}\xi).\end{aligned} \tag{3.15}$$

In equation (3.15), if we change the roles of V_1 and V_2 we obtain

$$\begin{aligned} g_M([V_1,V_2],\xi)-\cos^2\theta g_M([V_1,V_2],\tilde{P}\xi) &= \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,\tilde{P}\xi),\Phi_*(\psi\phi V_2))-\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\tilde{P}\xi),\Phi_*(\psi\phi V_2)) \\ &+ g_M(h\overset{M}{\nabla}_{V_1}\psi V_2-h\overset{M}{\nabla}_{V_2}\psi V_1,\psi\tilde{R}\xi)+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,J\tilde{P}\xi),\Phi_*(\psi V_2)) \\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,J\tilde{P}\xi),\Phi_*(\psi V_1))+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_2,\phi V_1),\Phi_*(\psi\tilde{R}\xi)) \\ &- \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(V_1,\phi V_2),\Phi_*(\psi\tilde{R}\xi)). \end{aligned} \tag{3.16}$$

Therefore, the proof is clear from (3.16). □

Here, we know that integrability case of the anti-invariant distribution $D_\perp$ is same with Theorem 3.8. in [23]. We have next theorem for horizontal distribution.

Theorem 3.6. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the horizontal distribution $(ker\Phi_*)^\perp$ is integrable if and only if*

$$\begin{aligned} &\lambda^2 g_M(v\overset{M}{\nabla}_{Z_2}BZ_1-v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi) \\ &+\lambda^2\{CZ_2(\ln\lambda)g_M(Z_1,\psi\xi)-CZ_1(\ln\lambda)g_M(Z_2,\psi\xi)+\psi\xi(\ln\lambda)g_M(Z_2,CZ_1)-\psi\xi(\ln\lambda)g_M(Z_1,CZ_2)\} \\ &=g_N((\nabla\Phi_*)(Z_2,BZ_1)-(\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi))+g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)) \\ &-g_N((\nabla\Phi_*)(Z_2,\phi\xi),\Phi_*(CZ_1))+g_N(\overset{N}{\nabla^\Phi}_{Z_1}\Phi_*(CZ_2)-\overset{N}{\nabla^\Phi}_{Z_2}\Phi_*(CZ_1),\Phi_*(\psi\xi)) \end{aligned}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$.

Proof. To show the horizontal distribution $(ker\Phi_*)^\perp$ is integrable, we only search $0=g_M([Z_1,Z_2],\xi)$ for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$. Since $\mathscr{A}$ is skew-symmetric and from definitions (1.2), (1.4), equations (2.6), (3.4), (3.7) we get

$$\begin{aligned} g_M(\overset{M}{\nabla}_{Z_1}Z_2,\xi) &= g_M(\mathscr{A}_{Z_1}BZ_2,\psi\xi)+g_M(h\overset{M}{\nabla}_{Z_1}CZ_2,\psi\xi)+g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)-g_M(\mathscr{A}_{Z_1}\phi\xi,CZ_2) \\ &= \frac{1}{\lambda^2}g_N(\Phi_*(\mathscr{A}_{Z_1}BZ_2),\Phi_*(\psi\xi))+\frac{1}{\lambda^2}g_N(\Phi_*(h\overset{M}{\nabla}_{Z_1}CZ_2),\Phi_*(\psi\xi)) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)-\frac{1}{\lambda^2}g_N(\Phi_*(\mathscr{A}_{Z_1}\phi\xi),\Phi_*(CZ_2)) \end{aligned}$$

for $Z_1,Z_2\in\Gamma((ker\Phi_*)^\perp)$ and $\xi\in\Gamma(ker\Phi_*)$. Using (2.1), (2.8) and (2.9), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_{Z_1}Z_2,\xi) &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi))+\frac{1}{\lambda^2}g_N(\Phi_*(h\overset{M}{\nabla}_{Z_1}CZ_2),\Phi_*(\psi\xi)) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi)+\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)) \\ &= -\frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi)) \\ &+ \frac{1}{\lambda^2}g_N(\overset{N}{\nabla^\Phi}_{Z_1}\Phi_*(CZ_2),\Phi_*(\psi\xi))-CZ_2(\ln\lambda)g_M(Z_1,\psi\xi) \\ &+ \psi\xi(\ln\lambda)g_M(Z_1,CZ_2)+g_M(v\overset{M}{\nabla}_{Z_1}BZ_2,\phi\xi) \\ &+ \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2)). \end{aligned} \tag{3.17}$$

Similarly, if we change the roles of Z_1 and Z_2 in (3.17) we finally obtain,

$$\begin{aligned} g_M([Z_1,Z_2],\xi) &= \frac{1}{\lambda^2}\{g_N((\nabla\Phi_*)(Z_2,BZ_1)-(\nabla\Phi_*)(Z_1,BZ_2),\Phi_*(\psi\xi)) \\ &+ g_N((\nabla\Phi_*)(Z_1,\phi\xi),\Phi_*(CZ_2))-g_N((\nabla\Phi_*)(Z_2,\phi\xi),\Phi_*(CZ_1)) \\ &+ g_N(\overset{N}{\nabla^{\Phi}}_{Z_1}\Phi_*(CZ_2)-\overset{N}{\nabla^{\Phi}}_{Z_2}\Phi_*(CZ_1),\Phi_*(\psi\xi))\} \\ &- CZ_2(\ln\lambda)g_M(Z_1,\psi\xi)+\psi\xi(\ln\lambda)g_M(Z_1,CZ_2) \\ &+ CZ_1(\ln\lambda)g_M(Z_2,\psi\xi)-\psi\xi(\ln\lambda)g_M(Z_2,CZ_1) \\ &+ g_M(v\overset{M}{\nabla}_{Z_1}BZ_2-v\overset{M}{\nabla}_{Z_2}BZ_1,\phi\xi). \end{aligned} \tag{3.18}$$

Hence, we get the proof from (3.18). □

Note that if $(\nabla\Phi_*)(E,F)=0$ for all $E,F\in\Gamma(TM)$ the map Φ is said to be totally geodesic map [25]. Using this notion we have followings.

Theorem 3.7. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the invariant distribution D defines a totally geodesic foliation on M if and only if*

$$C\mathscr{T}_{U_1}\phi U_2+\psi\hat{\nabla}_{U_1}\phi U_2=0$$

for $U_1,U_2\in\Gamma(D)$.

Proof. Since D is an invariant distribution we have $\psi U_2=0$. From (2.1), (2.4), (3.4) and (3.8) we get

$$\begin{aligned} (\nabla\Phi_*)(U_1,U_2) &= -\Phi_*(\overset{M}{\nabla}_{U_1}U_2) \\ &= \Phi_*(C\mathscr{T}_{U_1}\phi U_2+\psi\hat{\nabla}_{U_1}\phi U_2) \end{aligned} \tag{3.19}$$

for $U_1,U_2\in\Gamma(D)$. The proof can be seen from (3.19). □

Theorem 3.8. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the slant distribution D_θ defines totally geodesic foliation on M if and only if*

$$\mathscr{T}_{V_1}B\psi V_2=0$$

for $V_1,V_2\in\Gamma(D_\theta)$.

Proof. From definition of second fundamental form, (3.4) and (3.8), we have

$$\begin{aligned} (\nabla\Phi_*)(V_1,V_2) &= \Phi_*(J\overset{M}{\nabla}_{V_1}JV_2) \\ &= \Phi_*(\overset{M}{\nabla}_{V_1}J\phi V_2)+\Phi_*(\overset{M}{\nabla}_{V_1}J\psi V_2) \\ &= \Phi_*(\overset{M}{\nabla}_{V_1}\phi^2V_2+\overset{M}{\nabla}_{V_1}\psi\phi V_2)+\Phi_*(\overset{M}{\nabla}_{V_1}B\psi V_2+\overset{M}{\nabla}_{V_1}C\psi V_2) \end{aligned}$$

for $V_1,V_2\in\Gamma(D_\theta)$. From (3.9), (3.10) and Theorem 3.3., we obtain

$$\begin{aligned} &= \Phi_*(-\cos^2\theta\overset{M}{\nabla}_{V_1}V_2)+\Phi_*(h\overset{M}{\nabla}_{V_1}\psi\phi V_2)+\Phi_*(\mathscr{T}_{V_1}B\psi V_2+h\overset{M}{\nabla}_{V_1}C\psi V_2) \\ \cos^2\theta\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= \Phi_*(\mathscr{T}_{V_1}B\psi V_2). \end{aligned} \tag{3.20}$$

The proof is clear from (3.20). □

In a similar way, we have easily the next theorems.

Theorem 3.9. *Let $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the anti-invariant distribution $D_\perp$ defines a totally geodesic foliation on M if and only if*

$$\psi\mathscr{T}_{W_1}\psi W_2+Ch\overset{M}{\nabla}_{W_1}\psi W_2=0$$

for $W_1,W_2\in\Gamma(D_\perp)$.

Proof. From (1.4), (2.5), (3.4) and (3.8), we obtain

$$\begin{aligned}(\nabla\Phi_*)(W_1,W_2) &= \Phi_*(J\overset{M}{\nabla}_{W_1}JW_2)\\ &= \Phi_*(J\mathscr{T}_{W_1}\psi W_2 + Jh\overset{M}{\nabla}_{W_1}\psi W_2)\\ &= \Phi_*(\psi\mathscr{T}_{W_1}\psi W_2 + Ch\overset{M}{\nabla}_{W_1}\psi W_2)\end{aligned} \tag{3.21}$$

for $W_1, W_2 \in \Gamma(D_\perp)$. We have the proof from (3.21). □

Theorem 3.10. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the vertical distribution $ker\Phi_*$ defines a totally geodesic foliation on M if and only if*

$$\psi\{\mathscr{T}_{\xi_1}\psi\xi_2 + \hat{\nabla}_{\xi_1}\phi\xi_2\} + C\{h\overset{M}{\nabla}_{\xi_1}\psi\xi_2 + \mathscr{T}_{\xi_1}\phi\xi_2\} = 0$$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$.

Recall that if $h(grad(\ln\lambda)) = 0$, the map Φ is said to be horizontally homothetic map [5].

Theorem 3.11. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, any two conditions below imply the third condition;*

i) The horizontal distribution $(ker\Phi_*)^\perp$ defines a totally geodesic foliation on M,

ii) The map Φ is a horizontally homothetic map,

iii)

$$\overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_2) = \Phi_*(J[JZ_1, Z_2]) + (\nabla\Phi_*)^\perp(CZ_1, CZ_2) + \Phi_*(\mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2 + \mathscr{T}_{BZ_1}BZ_2)$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$.

Proof. Firstly, from (2.1) and (2.9) we have

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{JZ_1}JZ_2) &= \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1) - (\nabla\Phi_*)(BZ_1, BZ_2) - (\nabla\Phi_*)(CZ_2, BZ_1) - (\nabla\Phi_*)(CZ_1, BZ_2)\\ &- (\nabla\Phi_*)^\perp(CZ_1, CZ_2) - CZ_1(\ln\lambda)\Phi_*(CZ_2) - CZ_2(\ln\lambda)\Phi_*(CZ_1) + g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda))\\ &= \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1) - (\nabla\Phi_*)^\perp(CZ_1, CZ_2) - \Phi_*(\mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2 + \mathscr{T}_{BZ_1}BZ_2)\\ &- CZ_1(\ln\lambda)\Phi_*(CZ_2) - CZ_2(\ln\lambda)\Phi_*(CZ_1)\\ &+ g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda))\end{aligned} \tag{3.22}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^\perp)$. On the other hand, we have

$$\overset{M}{\nabla}_{JZ_1}JZ_2 = J[JZ_1, Z_2] + J\overset{M}{\nabla}_{Z_2}JZ_1. \tag{3.23}$$

Putting equation (3.23) in (3.22), we obtain

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1) &= \Phi_*(J[JZ_1, Z_2]) - \overset{N}{\nabla}{}^{\Phi}_{JZ_1}\Phi_*(CZ_1)\\ &+ \Phi_*(\mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2 + \mathscr{T}_{BZ_1}BZ_2)\\ &+ (\nabla\Phi_*)^\perp(CZ_1, CZ_2) + CZ_1(\ln\lambda)\Phi_*(CZ_2)\\ &+ CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)).\end{aligned} \tag{3.24}$$

Now, suppose that (i) and (ii) are satisfied in (3.24). We have $\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1) = 0$ and

$$0 = CZ_1(\ln\lambda)\Phi_*(CZ_2) + CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)) \tag{3.25}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$, respectively. So, we obtain (iii) clearly. If (ii) and (iii) are provided in (3.24), we have (3.25) and

$$\overset{N}{\nabla^{\Phi}}_{JZ_1}\Phi_*(CZ_2) = \Phi_*(J[JZ_1,Z_2]) + (\nabla\Phi_*)^{\perp}(CZ_1,CZ_2) + \Phi_*(\mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2 + \mathscr{T}_{BZ_1}BZ_2), \tag{3.26}$$

respectively. Easily, we obtain $\Phi_*(\overset{M}{\nabla}_{Z_2}Z_1) = 0$. Hence, we say that the horizontal distribution $(ker\Phi_*)^{\perp}$ defines totally geodesic foliation on M. At last, if (i) and (iii) are provided in (3.24), we obtain (3.25). For $CZ_1 \in \Gamma(\mu)$ in (3.25), we get

$$0 = \lambda^2 CZ_2(\ln\lambda) g_M(CZ_1,CZ_1). \tag{3.27}$$

Hence, we obtain $0 = CZ_2(\ln\lambda)$. It means λ is a constant on μ. Then, for $\xi \in \Gamma(ker\Phi_*)$ and $\psi\xi \in \Gamma(\psi D_\theta \oplus J(D_\perp))$ in (3.25) with $CZ_1 = CZ_2$, we get

$$0 = \lambda^2 \psi\xi(\ln\lambda) g_M(CZ_1,CZ_2). \tag{3.28}$$

Hence, we obtain $0 = \psi\xi(\ln\lambda)$. It means λ is a constant on $\psi D_\theta \oplus J(D_\perp)$. Hence, λ is a constant on horizontal distribution. We obtain (iii) from (3.27) and (3.28). The proof is complete. □

Here, we have conditions for the map Φ which defines a totally geodesic foliations on M.

Theorem 3.12. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. Then, the map Φ defines a totally geodesic foliations on M if and only if*

i) The map Φ is a horizontally homothetic map,

ii)

$$\begin{aligned} \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \overset{N}{\nabla^{\Phi}}_{\bar{E}}\Phi_*(\bar{F}) &= \Phi_*(\mathscr{T}_{\hat{E}}\hat{F} + h\overset{M}{\nabla}_{\hat{E}}\bar{F} + \mathscr{A}_{\bar{E}}\hat{F}) \\ &- (\nabla\Phi_*)^{\perp}(\bar{E},\bar{F}) \end{aligned}$$

holds for $E,F \in \Gamma(TM)$ where $\bar{E},\bar{F}$ and $\hat{E},\hat{F}$ show horizontal and vertical parts of E,F, respectively.

Proof. Using definition of second fundamental form of a map, (2.4), (2.5) and (2.6) we have

$$\begin{aligned} (\nabla\Phi_*)(E,F) &= \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \Phi_*(\overset{M}{\nabla}_{E}F) \\ &= \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \Phi_*(\overset{M}{\nabla}_{\hat{E}}\hat{F} + \overset{M}{\nabla}_{\hat{E}}\bar{F} + \overset{M}{\nabla}_{\bar{E}}\hat{F} + \overset{M}{\nabla}_{\bar{E}}\bar{F}) \\ &= \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \Phi_*(\mathscr{T}_{\hat{E}}\hat{F} + h\overset{M}{\nabla}_{\hat{E}}\bar{F} + \mathscr{A}_{\bar{E}}\hat{F}) - \Phi_*(\overset{M}{\nabla}_{\bar{E}}\bar{F}) \end{aligned} \tag{3.29}$$

for $E,F \in \Gamma(TM)$. Here, from equation (2.9) in (3.29), we obtain

$$\begin{aligned} (\nabla\Phi_*)(E,F) &= \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \Phi_*(\mathscr{T}_{\hat{E}}\hat{F} + h\overset{M}{\nabla}_{\hat{E}}\bar{F} + \mathscr{A}_{\bar{E}}\hat{F}) \\ &- \overset{N}{\nabla^{\Phi}}_{\bar{E}}\Phi_*(\bar{F}) + (\nabla\Phi_*)^{\perp}(\bar{E},\bar{F}) + \bar{E}(\ln\lambda)\Phi_*(\bar{F}) \\ &+ \bar{F}(\ln\lambda)\Phi_*(\bar{E}) - g_M(\bar{E},\bar{F})\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{3.30}$$

Because of Φ defines a totally geodesic foliations on M, we have (3.30). Suppose that Φ is a horizontally homothetic map, we have from (3.30)

$$0 = \bar{E}(\ln\lambda)\Phi_*(\bar{F})\bar{F}(\ln\lambda)\Phi_*(\bar{E}) - g_M(\bar{E},\bar{F})\Phi_*(grad(\ln\lambda)). \tag{3.31}$$

We obtain from (3.31)

$$0 = \lambda^2 \bar{F}(\ln\lambda) g_M(\bar{E},\bar{E}) \tag{3.32}$$

for $\bar{E} \in \Gamma((ker\Phi_*)^{\perp})$. We have $0 = \bar{F}(\ln\lambda)$ from (3.32). It means λ is a constant on horizontal distribution. So, Φ is a horizontally homothetic map and (i) is satisfied. If (i) satisfies in (3.30), we obtain

$$0 = \overset{N}{\nabla^{\Phi}}_{E}\Phi_*(\bar{F}) - \Phi_*(\mathscr{T}_{\hat{E}}\hat{F} + h\overset{M}{\nabla}_{\hat{E}}\bar{F} + \mathscr{A}_{\bar{E}}\hat{F}) - \overset{N}{\nabla^{\Phi}}_{\bar{E}}\Phi_*(\bar{F}) + (\nabla\Phi_*)^{\perp}(\bar{E},\bar{F}). \tag{3.33}$$

From (3.33), we obtain (ii). The proof is complete. □

4. Pluriharmonic conformal quasi-hemi-slant Riemannian map

In this section, we examine some geometric properties of certain distributions with respect to notion of pluriharmonicity, see equation (2.10) or [16]. We present D-pluriharmonicity in the following.

Theorem 4.1. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D-$pluriharmonic map, then one of the below assertions imply the second assertion,*

i) D defines a totally geodesic foliation on M,

ii) $C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2 = 0$

for $U_1, U_2 \in \Gamma(D)$.

Proof. Initially, using definition of pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(U_1, U_2) + (\nabla\Phi_*)(JU_1, JU_2) \tag{4.1}$$

for $U_1, U_2 \in \Gamma(D)$. By some calculations, we obtain from (4.1)

$$\begin{aligned}\Phi_*(\overset{M}{\nabla}_{U_1}U_2) &= -\Phi_*(J(\mathscr{T}_{JU_1}U_2 + \hat{\nabla}_{JU_1}U_2))\\ \Phi_*(\overset{M}{\nabla}_{U_1}U_2) &= -\Phi_*(C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2).\end{aligned} \tag{4.2}$$

If (i) is satisfied in (4.2) we have $\Phi_*(\overset{M}{\nabla}_{U_1}U_2) = 0$. So, we obtain

$$C\mathscr{T}_{JU_1}U_2 + \psi\hat{\nabla}_{JU_1}U_2 = 0.$$

(ii) is provided. In a similar way, if (ii) is satisfied in (4.2), easily one can get (i). □

Theorem 4.2. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D_\theta-$pluriharmonic map, then two of the below assertions imply the third assertion,*

i) D_θ defines a totally geodesic foliation on M,

ii) λ is a constant on ψD_θ and $(\nabla\Phi_*)^\perp(\psi V_1, \psi V_2) = 0$,

iii)

$$\cos^2\theta(C\mathscr{T}_{\phi V_1}V_2 + \psi\hat{\nabla}_{\phi V_1}V_2) = \psi\mathscr{T}_{\phi V_1}\psi\phi V_2 + Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2 - \mathscr{A}_{\psi V_2}\phi V_1 - \mathscr{A}_{\psi V_1}\phi V_2.$$

for $V_1, V_2 \in \Gamma(D_\theta)$.

Proof. If Φ is a $D_\theta-$pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(V_1, V_2) + (\nabla\Phi_*)(JV_1, JV_2) \tag{4.3}$$

for $V_1, V_2 \in \Gamma(D_\theta)$. Using symmetry property of second fundamental form and from equations (2.4), (2.5), (2.6) and (2.9), we get

$$
\begin{aligned}
0 &= -\Phi_*(\overset{M}{\nabla}_{V_1}V_2) - \Phi_*(\overset{M}{\nabla}_{\phi V_1}\phi V_2) - \Phi_*(\overset{M}{\nabla}_{\psi V_2}\phi V_1) \\
&- \Phi_*(\overset{M}{\nabla}_{\psi V_1}\phi V_2) + (\nabla\Phi_*)^{\perp}(\psi V_1, \psi V_2) + \psi V_1(\ln\lambda)\Phi_*(\psi V_2) \\
&+ \psi V_2(\ln\lambda)\Phi_*(\psi V_1) - g_M(\psi V_1, \psi V_2)\Phi_*(grad(\ln\lambda)) \\
\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= \Phi_*(J\overset{M}{\nabla}_{\phi V_1}J\phi V_2) - \Phi_*(\mathscr{A}_{\psi V_2}\phi V_1 + \mathscr{A}_{\psi V_1}\phi V_2) \\
&+ (\nabla\Phi_*)^{\perp}(\psi V_1, \psi V_2) + \psi V_1(\ln\lambda)\Phi_*(\psi V_2) \\
&+ \psi V_2(\ln\lambda)\Phi_*(\psi V_1) - g_M(\psi V_1, \psi V_2)\Phi_*(grad(\ln\lambda)) \\
\Phi_*(\overset{M}{\nabla}_{V_1}V_2) &= -\cos^2\theta\Phi_*(C\mathscr{T}_{\phi V_1}V_2 + \psi\hat{\nabla}_{\phi V_1}V_2) \\
&+ \Phi_*(\psi\mathscr{T}_{\phi V_1}\psi\phi V_2 + Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2) \\
&- \Phi_*(\mathscr{A}_{\psi V_2}\phi V_1 + \mathscr{A}_{\psi V_1}\phi V_2) + (\nabla\Phi_*)^{\perp}(\psi V_1, \psi V_2) \\
&+ \psi V_1(\ln\lambda)\Phi_*(\psi V_2) + \psi V_2(\ln\lambda)\Phi_*(\psi V_1) \\
&- g_M(\psi V_1, \psi V_2)\Phi_*(grad(\ln\lambda)). \qquad (4.4)
\end{aligned}
$$

Now, if (i) and (ii) are satisfied in (4.4), we have $\Phi_*(\overset{M}{\nabla}_{V_1}V_2) = 0$ and

$$
\begin{aligned}
0 &= \psi V_1(\ln\lambda)\Phi_*(\psi V_2) + \psi V_2(\ln\lambda)\Phi_*(\psi V_1) - g_M(\psi V_1, \psi V_2)\Phi_*(grad(\ln\lambda)), \qquad (4.5) \\
0 &= (\nabla\Phi_*)^{\perp}(\psi V_1, \psi V_2), \qquad (4.6)
\end{aligned}
$$

respectively. Then, we get from (4.4)

$$
\begin{aligned}
0 &= -\cos^2\theta\Phi_*(C\mathscr{T}_{\phi V_1}V_2 + \psi\hat{\nabla}_{\phi V_1}V_2) + \Phi_*(\psi\mathscr{T}_{\phi V_1}\psi\phi V_2 + Ch\overset{M}{\nabla}_{\phi V_1}\psi\phi V_2) \\
&- \Phi_*(\mathscr{A}_{\psi V_2}\phi V_1 + \mathscr{A}_{\psi V_1}\phi V_2). \qquad (4.7)
\end{aligned}
$$

So, (iii) is satisfied at (4.7). If (ii) and (iii) are satisfied in (4.4), we clearly have equations (4.5), (4.6) and (4.7) in (4.4). Therefore, we obtain (i). Lastly, suppose that (i) and (iii) are satisfied in (4.4). Then, we get (4.5) and (4.6). In (4.5), we obtain from (1.2)

$$
\begin{aligned}
0 &= \lambda^2\psi V_1(\ln\lambda)g_M(\psi V_2, \psi V_1) + \lambda^2\psi V_2(\ln\lambda)g_M(\psi V_1, \psi V_1) - \lambda^2 g_M(\psi V_1, \psi V_2)\psi V_1(\ln\lambda) \\
0 &= \lambda^2\psi V_2(\ln\lambda)g_M(\psi V_1, \psi V_1) \qquad (4.8)
\end{aligned}
$$

for $\psi V_1 \in \Gamma(D_\theta)$. At (4.8), we have $0 = \psi V_2(\ln\lambda)$. It means $0 = \psi D_\theta(\ln\lambda)$ which implies that λ is a constant on ψD_θ. Hence, (ii) is satisfied. The proof is completed. □

Similarly, we have the following theorem.

Theorem 4.3. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $D_\perp-$pluriharmonic map, then one of the below assertions imply the second assertion,*

i) $D_\perp$ defines a totally geodesic foliation on M,

ii) λ is a constant on $JD_\perp$ and $(\nabla\Phi_*)^{\perp}(JW_1, JW_2) = 0$

for $W_1, W_2 \in \Gamma(D_\perp)$.

Now, we search properties of horizontal and vertical pluriharmonic maps, respectively.

Theorem 4.4. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a $(ker\Phi_*)^{\perp}-$pluriharmonic map, then any two of the below assertions imply the third assertion,*

i) $(ker\Phi_*)^{\perp}$ defines a totally geodesic foliation on M,

ii) λ is a constant on μ,

iii) $\overset{N}{\nabla^{\Phi}}_{Z_1}\Phi_*(Z_2) = \Phi_*(\mathscr{T}_{BZ_1}BZ_2 + \mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2) - (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2)$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$.

Proof. From definition of pluriharmonic map, we have

$$0 = (\nabla\Phi_*)(Z_1, Z_2) + (\nabla\Phi_*)(JZ_1, JZ_2) \tag{4.9}$$

for $Z_1, Z_2 \in \Gamma((ker\Phi_*)^{\perp})$. By some calculations from (2.8) and (2.9) in (4.9), we get

$$\begin{aligned} 0 &= \overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) - \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2) + (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2) \\ &- \Phi_*(\mathscr{T}_{BZ_1}BZ_2 + \mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2) + CZ_1(\ln\lambda)\Phi_*(CZ_2) \\ &+ CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)). \end{aligned} \tag{4.10}$$

If (i) and (ii) are satisfied in (4.10), we have

$$0 = \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2), \tag{4.11}$$

$$0 = CZ_1(\ln\lambda)\Phi_*(CZ_2) + CZ_2(\ln\lambda)\Phi_*(CZ_1) - g_M(CZ_1, CZ_2)\Phi_*(grad(\ln\lambda)). \tag{4.12}$$

So, we get (iii) from (4.10) such that

$$\overset{N}{\nabla}{}^{\Phi}_{Z_1}\Phi_*(Z_2) = \Phi_*(\mathscr{T}_{BZ_1}BZ_2 + \mathscr{A}_{CZ_2}BZ_1 + \mathscr{A}_{CZ_1}BZ_2) - (\nabla\Phi_*)^{\perp}(CZ_1, CZ_2). \tag{4.13}$$

If (ii) and (iii) are satisfied in (4.10), we have equations (4.12) and (4.13). Clearly, we obtain $0 = \Phi_*(\overset{M}{\nabla}_{Z_1}Z_2)$ which implies that $(ker\Phi_*)^{\perp}$ defines a totally geodesic foliation on M. Lastly, if (i) and (iii) are satisfied in (4.10), we obtain (4.12). From (1.2) in (4.12) we get

$$\begin{aligned} 0 &= \lambda^2 CZ_1(\ln\lambda)g_M(CZ_2, CZ_1) + \lambda^2 CZ_2(\ln\lambda)g_M(CZ_1, CZ_1) - \lambda^2 g_M(CZ_1, CZ_2)CZ_1(\ln\lambda) \\ 0 &= \lambda^2 CZ_2(\ln\lambda)g_M(CZ_1, CZ_1) \end{aligned} \tag{4.14}$$

for $CZ_1, CZ_2 \in \Gamma(\mu)$. Here, we have $CZ_2(\ln\lambda) = 0$ which implies that λ is a constant on μ. (ii) is provided. The proof is completed. □

Theorem 4.5. *Let* $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ *be a conformal quasi-hemi-slant Riemannian map. If* Φ *is a* $ker\Phi_*-$*pluriharmonic map, then any two of the below assertions imply the third assertion,*

i) $ker\Phi_*$ *defines a totally geodesic foliation on* M*,*

ii) Φ *is a horizontally homothetic map and* $(\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) = 0$*,*

iii) $\mathscr{T}_{\phi\xi_1}\phi\xi_2 + \mathscr{A}_{\psi\xi_2}\phi\xi_1 + \mathscr{A}_{\psi\xi_1}\phi\xi_2 = 0$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$.

Proof. From equations (2.5), (2.6), (2.9), (2.10) and (3.4), we get

$$\begin{aligned} 0 &= (\nabla\Phi_*)(\xi_1, \xi_2) + (\nabla\Phi_*)(J\xi_1, J\xi_2) \\ 0 &= -\Phi_*(\overset{M}{\nabla}_{\xi_1}\xi_2) + (\nabla\Phi_*)(\phi\xi_1, \phi\xi_2) + (\nabla\Phi_*)(\psi\xi_2, \phi\xi_1) \\ &+ (\nabla\Phi_*)(\psi\xi_1, \phi\xi_2) + (\nabla\Phi_*)(\psi\xi_1, \psi\xi_2) \\ \Phi_*(\overset{M}{\nabla}_{\xi_1}\xi_2) &= -\Phi_*(\mathscr{T}_{\phi\xi_1}\phi\xi_2 + \mathscr{A}_{\psi\xi_2}\phi\xi_1 + \mathscr{A}_{\psi\xi_1}\phi\xi_2) \\ &+ (\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) + \psi\xi_1(\ln\lambda)\Phi(\psi\xi_2) \\ &+ \psi\xi_2(\ln\lambda)\Phi(\psi\xi_1) - g_M(\psi\xi_1, \psi\xi_2)\Phi_*(grad(\ln\lambda)) \end{aligned} \tag{4.15}$$

for $\xi_1, \xi_2 \in \Gamma(ker\Phi_*)$. The proof of (i) and (iii) are clear to see. So, we only give proof for (ii). Suppose that (i) and (iii) are provided in (4.15). One can see easily that $(\nabla\Phi_*)^{\perp}(\psi\xi_1, \psi\xi_2) = 0$ and we get

$$0 = \psi\xi_1(\ln\lambda)\Phi(\psi\xi_2) + \psi\xi_2(\ln\lambda)\Phi(\psi\xi_1) - g_M(\psi\xi_1, \psi\xi_2)\Phi_*(grad(\ln\lambda)). \tag{4.16}$$

Since Φ is a conformal map, we obtain from (4.16)

$$0 = \lambda^2\psi\xi_2(\ln\lambda)g_M(\psi\xi_1, \psi\xi_1) \tag{4.17}$$

for $\psi\xi_1 \in \Gamma(\psi D_\theta \oplus JD_\perp)$. It means $0 = \psi\xi_2(\ln\lambda)$ which implies that λ is a constant on $\psi D_\theta \oplus JD_\perp$. On the other hand, we obtain from (4.16)

$$0 = -\lambda^2 CX(\ln\lambda) g_M(\psi\xi_1, \psi\xi_1) \tag{4.18}$$

for $CX \in \Gamma(\mu)$ with $\psi\xi_1 = \psi\xi_2$. It means $0 = CX(\ln\lambda)$ which implies that λ is a constant on μ. Hence, equations (4.17) and (4.18) give us that Φ is a horizontally homothetic map. (ii) is provided. The proof is completed. □

Lastly, we examine mixed pluriharmonicity on conformal quasi-hemi-slant Riemannian maps such that

$$0 = (\nabla\Phi_*)(Z,\xi) + (\nabla\Phi_*)(JZ,J\xi)$$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Theorem 4.6. *Let $\Phi : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal quasi-hemi-slant Riemannian map. If Φ is a mixed$-$pluriharmonic map, then any of the below assertions imply the second assertion,*

i- Φ is a horizontally homothetic map and $(\nabla\Phi_*)^\perp(CZ,\psi\xi) = 0$,

ii- $\mathscr{A}_Z\xi = \mathscr{T}_{BZ}\phi\xi + \mathscr{A}_{\psi\xi}BZ + \mathscr{A}_{CZ}\phi\xi$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$.

Proof. From definition of mixed pluriharmonic map, we get

$$\begin{aligned} 0 &= -\Phi_*(\mathscr{A}_Z\xi) + \Phi_*(\mathscr{T}_{BZ}\phi\xi + \mathscr{A}_{\psi\xi}BZ + \mathscr{A}_{CZ}\phi\xi) \\ &+ (\nabla\Phi_*)^\perp(CZ,\psi\xi) + CZ(\ln\lambda)\Phi_*(\psi\xi) + \psi\xi(\ln\lambda)\Phi_*(CZ) \end{aligned} \tag{4.19}$$

for $\xi \in \Gamma(ker\Phi_*)$ and $Z \in \Gamma((ker\Phi_*)^\perp)$. If (i) is satisfied in (4.19) we have $(\nabla\Phi_*)^\perp(CZ,\psi\xi) = 0$ and

$$0 = CZ(\ln\lambda)\Phi_*(\psi\xi) + \psi\xi(\ln\lambda)\Phi_*(CZ). \tag{4.20}$$

So, one can obtain (ii) easily. Now, if (ii) is satisfied in (4.19) we obtain easily $(\nabla\Phi_*)^\perp(CZ,\psi\xi) = 0$. Then, from (4.20) we obtain

$$0 = \lambda^2 \psi\xi(\ln\lambda) g_M(CZ,CZ) \tag{4.21}$$

for $CZ \in \Gamma(\mu)$. It means $0 = \psi\xi(\ln\lambda)$ which implies that λ is a constant on $\psi D_\theta \oplus JD_\perp$. On the other hand, from (4.20) we obtain

$$0 = \lambda^2 CZ(\ln\lambda) g_M(\psi\xi,\psi\xi) \tag{4.22}$$

for $\psi\xi \in \Gamma(\psi D_\theta \oplus JD_\perp)$. It means $0 = CZ(\ln\lambda)$ which implies that λ is a constant on μ. Hence, (4.21) and (4.22) give us that Φ is a horizontally homothetic map. (i) is provided. The proof is completed. □

5. Conclusion

In this paper, integrability conditions and conditions for defining a totally geodesic foliation by certain distributions were found. Then, by applying the notion of pluriharmonicity onto conformal quasi-hemi-slant Riemannian maps we obtained relations among pluriharmonicity, horizontally homotheticness and totally geodesicness.

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