MANAS Journal of Engineering
ISSN 1694-7398 — e-ISSN 1694-7398

Volume 10 (Issue 1) (2022) Pages 228-240
https://doi.org/10.51354/mjen.1085695

On $(\alpha, \phi)-$weak Pata contractions

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ABSTRACT

In this paper, we give $(\alpha, \phi)-$weak Pata contractive mapping by using the simulation function and multivalued $(\alpha, \phi)-$weak Pata contractions and establish some fixed point results for such contractions. Also, we give an example related to $(\alpha, \phi)-$weak Pata contractive mappings via simulation function. Our results generalize some Pata type contractions and Banach contractions. Consequently, the obtained results encompass several results in the literature.

ARTICLE INFO

Research article

Received: 10.05.2022
Accepted: 01.12.2022

Keywords: fixed point, pata contraction, $\phi-$weak contraction, metric spaces

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1. Introduction and Preliminaries

One of the fundamental results in fixed point theory, which is called Banach's contraction principle was given by Banach [7]. Several researchers have dealt with this result. Recently, Pata [21] extended the Banach contraction principle and proved some interesting fixed point results. Chakraborty et. al. [9] got an extension of Kannan's based on the result of Pata [21]. Later Pata-Chatterjea type cyclic fixed point theorems were proved by Kadelburg et. al. [12] in metric spaces. After that coupled fixed point theorems for Pata type mappings were proved by Eshaghi [10]. This topic was extended the metric space into various different spaces by some researchers. For instance, Paknazar et. al. [20] gave Pata type fixed point theorems in modular metric space and Balasubramanian [6] obtained a fixed point theorem for Pata type mappings in cone metric spaces. Later, Aktay et. al. [2] proved some fixed point results for generalized Pata–Suzuki type contractive mappings.

Firstly, the concept of $\phi-$weak contraction was defined by Alber et. al. [1] and then, Rhoades [24] studied such contractions for single-valued mappings in Banach spaces. After that, Zhang et. al. [26] introduced generalized $\phi-$weak contraction and they obtained a unique common fixed point of such contractions.

Existence of fixed point for multivalued mappings in metric fixed point theory was initiated by Nadler [18]. Some notable generalizations were obtained by Hong [11].

In a recent work, Khojasteh et al. [15] introduced the notion of $Z-$contraction using simulation functions. Later, Karapınar [14] and Argoubi et. al. [5] studied such contractions.

Samet et. al. [25] and Karapınar et al. [13] gave respectively, the definition of $\alpha-$admissible and triangular $\alpha-$admissible mappings. Further, Asl et al. [4], Mohammadi et al. [17], Patel [22] and Aktay et. al. [2] gave some definitions related to $\alpha-$admissibility.

The aim of this paper is to establish some fixed point results for $(\alpha, \phi)-$weak Pata contractive mapping by using the simulation function and multivalued $(\alpha, \phi)-$weak Pata contractions. Our results give existence of fixed point for a

wider class of Pata type contractions. Moreover, we give an example related to $(\alpha, \phi)-$weak Pata contractive mappings. Consequently, the obtained results encompass various well known results in the literature.
$P(W) = 2^W$ all nonempty subset of W. Let $\wp = P(W) - \{\emptyset\}$ for $U, V \in 2^W$,

$$H_\varrho(U,V) = \max\left\{\sup_{u\in U} \varrho(u,V), \sup_{v\in V}(U,v)\right\}$$

where

$$\varrho(u,V) = \inf_{v\in V} \varrho(u,v)$$

H_ϱ is called the Hausdorff-Pompeiu functional induced by ϱ.
A point $u \in W$ is said to be a fixed point of $\top : W \to \wp$ if $u \in \top u$ (for single valued mapping $u = \top u$). The set of all fixed points of $\top$ is denoted by $F_H(\top)$ (for single valued mapping $F(\top)$).
Alber et. al. [1] gave the following definition.
[1] Let (W, ϱ) be a metric space. A mapping $\top : W \to W$ is said to be $\phi-$weak contraction, if there exists a map $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ and $\phi(s) > 0$ for all $s > 0$ such that

$$\varrho(\top w, \top t) \leq \varrho(w,t) - \phi(\varrho(w,t))$$

for all $w, t \in W$.
Along this work, Ψ denotes the class of all increasing function $\psi : [0,1] \to [0,\infty)$, which vanishes with continuity at zero. For an arbitrary $w_0 \in W$, we denote $\|w\| = \varrho(w, w_0), \forall w \in W$.
The existence of fixed point of Pata contraction mappings was proved by Pata [21] as follow.
[21] Let (W, ϱ) be a complete metric space. Let $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$ and $\top : W \to W$ be a function. If for all $w, t \in W$ the inequality

$$\varrho(\top w, \top t) \leq (1-\epsilon)\,\varrho(w,t) + \Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + \|w\| + \|t\|\right]^{\vartheta}$$

is satisfied for all $\epsilon \in [0,1]$, then $\top$ has a unique fixed point, $u = \top u$.
Samet et al. [25] and Karapınar et al. [13] gave respectively, the following definitions.
Let W be a metric space and $\top : W \to W$ be a map and $\alpha : W \times W \to [0, +\infty)$ be a function. Then *for all* $w, t, z \in W$,

(i) [25] $\top$ *is said to be* $\alpha-$*admissible if* $\alpha(w,t) \geq 1$ *implies* $\alpha(\top w, \top t) \geq 1$.

(ii) [13] $\top$ *is said to be triangular* $\alpha-$*admissible if:*

- $\top$ *is* $\alpha-$*admissible,*
- $\alpha(w,z) \geq 1$ *and* $\alpha(z,t) \geq 1$ *imply* $\alpha(w,t) \geq 1$.

Further, Asl et al. [4] gave the concept of an α^*-admissible mapping which is a multivalued version of the $\alpha-$admissible mapping. Later, Mohammadi et al. [17] and Patel [22] gave respectively, the definitions of $\alpha-$admissible and triangular $\alpha-$ admissible as follows.
Let W be a nonempty set, $\top : W \to P(W)$ and $\alpha : W \times W \to [0,\infty)$ be two given mappings. Then

(i) [17] $\top$ *is said to be an* $\alpha-$*admissible if whenever for each* $x \in W$ *and* $y \in Tx$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(y,z) \geq 1$, *for all* $z \in Ty$.

(ii) *[22]* $\top$ *is said to be triangular* $\alpha-$ *admissible if* $\top$ *is* $\alpha-$*admissible and* $\alpha(x,y) \geq 1$ *and* $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1, \forall z \in Ty$.

Khojasteh et. al. [15] gave the simulation function and $Z-$contraction in 2015 as follows.
[15] A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *is called a simulation function if it satisfies the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(w,t) < w - t$;

(ζ_3) *if* $\{w_n\}$ *and* $\{t_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to+\infty} w_n = \lim_{n\to+\infty} t_n > 0$ *then* $\limsup_{n\to+\infty} \zeta(w_n,t_n) < 0$.

[15] Let (W,ϱ) be a metric space and $\top : W \to W$ be a mapping. If there exists $\zeta \in Z$ such that

$$\zeta(\varrho(\top w, \top t), \varrho(w,t)) \geq 0, \text{ for all } w,t \in W$$

then $\top$ is called $Z-$contraction with respect to ζ.
(ζ_1) condition was removed in above definition of simulation function by Argoubi et. al. [5] in 2015. Also, $\acute{Z}$ denotes the set of all simulation functions.
Let $(W,\leq)$ be a partially ordered set and $w,t \in W$. Elements w and t are said to be comparable elements of W if either $w \leq t$ or $t \leq w$.
Hong [11] gave following definitions for multivalued mappings.
[11] Let W be a metric space. A subset $V \subset W$ is said to be approximative if the multivalued mapping $F_V(w) = \{v \in V : \varrho(w,v) = \varrho(w,V)\}\ \forall w \in W$, has nonempty values.
[11] Let $\top : W \to 2^W$ be a multivalued mapping. Then

(i) $\top$ *is said to have approximative values (AV), if* $\top w$ *is approximative for each* $w \in W$.

(ii) $\top$ *is said to have comparable approximative values (CAV), if* $\top$ *has approximative values and, foreach* $t \in W$, *there exists* $u \in F_{\top_t}(w)$ *such that* w *is comparable to* u.

(iii) $\top$ *is said to have upper comparable approximative values (UCAV), (resp. lower comparable approximative values (LCAV)), if* $\top$ *has approximative values and, for each* $t \in W$, *there exists* $u \in F_{\top_t}(w)$ *such that* $u \geq t$ *(resp.*$u \leq t$*).*

Nieto et. al. [19] gave the following definition in 2005.
[19] (H_*) : Let $(W,\varrho,\leq)$ be a partial ordered complete metric space. If $\{w_n\}$ is a non-decreasing (resp. non-increasing) sequence in W such that $w_n \to w$, then $w_n \leq w$(resp.$w_n \geq w$) for all $n \in \mathbb{N}$.
The following Lemma 1 is used to prove our results.
[23] Let (W,ϱ) is a metric space and $\{w_n\}$ be a sequence in W such that $\varrho(w_{n+1},w_n) \to 0$ as $n \to \infty$. If $\{w_n\}$ is not a Cauchy sequence, then there exist a $\varsigma > 0$ and sequences of positive integers $\{m_j\}$ and $\{n_j\}$ with $m_j > n_j > j$ such that $\varrho(w_{m_j},w_{n_j}) \geq \varsigma$ and $\varrho(w_{m_j-1},w_{n_j}) \leq \varsigma$ and $\lim_{j\to\infty} \varrho(w_{m_j-1},w_{n_j+1}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j},w_{n_j}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j-1},w_{n_j}) = \varsigma$. From Lemma 1, we obtain

$$\lim_{j\to\infty} \varrho(w_{m_j+1},w_{n_j+1}) = \varsigma \text{ and } \lim_{j\to\infty} \varrho(w_{m_j},w_{n_j-1}) = \varsigma.$$

2. Main Results

In this section, we introduce the concept of $(\alpha,\phi)-$weak Pata contractions via simulation function and multivalued $(\alpha,\phi)-$weak Pata contractions in metric spaces. We establish some fixed point results for such contractions on metric spaces.
Let $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0,\xi]$ be fixed constants, $\psi \in \Psi$ and $\alpha : W \times W \to [0,+\infty)$, $\top : W \to W$ be two functions. We say that $\top$ is an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function if there exists a function $\zeta \in \acute{Z}$ such that for all $w,t \in W$, and $\epsilon \in [0,1]$, $\top$ satisfies the inequality

$$\zeta(\alpha(w,t)\,\varrho(\top w,\top t), (1-\epsilon)(M(w,t) - \phi(M(w,t)) + P(w,t)) \geq 0, \tag{1}$$

where $\phi : [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$, for all $s > 0$, and

$$P(w,t) = \Lambda \epsilon^{\xi} \psi(\epsilon)\,[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|]^{\vartheta}$$

and

$$M(w,t) = \max\left\{\varrho(w,t), \varrho(w,\top w), \varrho(t,\top t), \frac{\varrho(w,\top t) + \varrho(t,\top w)}{2}\right\}.$$

Now, we state a fixed point result for $(\alpha,\phi)-$weak Pata contractive mapping via simulation function.
Let (W,ϱ) be a complete metric space. $\top : W \to W$ be an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function. Assume that

(i) $\top$ is *triangular* $\alpha-$*admissible;*

(ii) *there exists* $w_0 \in W$ *such that* $\alpha(w_0, \top w_0) \geq 1$*;*

(iii) $\top$ *is continuous;*

(iv) *for all* $u, v \in F(\top)$, $\alpha(u,v) \geq 1$.

Then $\top$ *has a unique fixed point that is* $u = \top u$, $u \in W$.

Proof The hypothesis (ii) of the Theorem 2, there exists $w_0 \in W$ such that $\alpha(w_0, \top w_0) \geq 1$. Starting at the point $w_0 \in W$, the iterative sequence $\{w_n\}$ is constructed by $w_n = \top w_{n-1} = \top^n w_0$, $n \geq 1$. If $w_{n_0} = w_{n_0+1}$ for any $n_0 \in \mathbb{N}$, then $w_{n_0} = \top w_{n_0}$. Consequently, we assume that succesive terms are distinct ie. $w_{n_0+1} \neq w_{n_0}$ for all $n_0 \in \mathbb{N}$. First of all, we show that $\alpha(w_n, w_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since $\top$ is an $\alpha-$admissible mapping, we have

$$\alpha(w_0, w_1) \geq 1 = \alpha(w_0, \top w_0) \geq 1 \text{ implies } \alpha(w_1, w_2) \geq 1$$

and

$$\alpha(w_1, w_2) \geq 1 \text{ implies } \alpha(w_2, w_3) \geq 1.$$

By induction, we obtain

$$\alpha(w_n, w_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{2}$$

Since $\top$ is triangular $\alpha-$admissible, we have

$$\alpha(w_n, w_{n+1}) \geq 1 \text{ and } \alpha(w_{n+1}, w_{n+2}) \geq 1 \text{ imply } \alpha(w_n, w_{n+2}) \geq 1.$$

Thus, by induction, we get

$$\alpha(w_n, w_m) \geq 1 \text{ for all } m > n \geq 0. \tag{3}$$

Now, we will show that $\{\varrho(w_{n+1}, w_n)\}$ is a decreasing sequence. Since $\top$ is an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function, we have

$$\zeta(\alpha(w_{n-1}, w_n)\varrho(w_n, w_{n+1}), (1-\epsilon)(M(w_{n-1}, w_n) - \phi(M(w_{n-1}, w_n))) + P(w_{n-1}, w_n)) \geq 0.$$

From ζ_2 and together with (2.2), we obtain

$$\begin{aligned}
\varrho(w_n, w_{n+1}) &\leq \alpha(w_{n-1}, w_n)\varrho(w_n, w_{n+1}) \\
&\leq (1-\epsilon)(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \\
&\quad \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_{n-1}), \\
&\quad \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\})) \\
&\quad + \Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + \|w_{n-1}\| + \|w_n\| + \|w_n\| + \|w_{n+1}\|\right]^{\vartheta} \\
&\leq (1-\epsilon)(\max\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\} \\
&\quad - \phi(\max\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\})) \\
&\quad + K\epsilon^{\xi}\psi(\epsilon),
\end{aligned}$$

for some $K > 0$. If $\varrho(w_n, w_{n-1}) \leq \varrho(w_{n+1}, w_n)$, then we obtain $\varrho(w_{n+1}, w_n) \leq (1-\epsilon)(\varrho(w_{n+1}, w_n) - \phi(\varrho(w_{n+1}, w_n))) + K\epsilon^{\xi}\psi(\epsilon)$. In this way, we obtain $\varrho(w_{n+1}, w_n) = 0$, is a contraction. Therefore we have

$$\varrho(w_{n+1}, w_n) < \varrho(w_n, w_{n-1}) < \cdots < \varrho(w_1, w_0) = \|w_1\|,$$

that is $\{\varrho(w_{n+1}, w_n)\}$ is a decreasing sequence. Since $\{\varrho(w_n, w_{n+1})\}$ is decreasing, and so, it is convergent to $\varrho \geq 0$ and $\lim_{n\to\infty} \varrho(w_n, w_{n+1}) = \varrho$. Now, we will demonstrate that $\{\|w_n\|\}$ is a bounded sequence. By the triangle inequality, we have

$$\|w_n\| = \varrho(w_n, w_0) \leq \varrho(w_n, w_{n+1}) + \varrho(w_{n+1}, w_1) + \varrho(w_1, w_0).$$

Since $\top$ is an $(\alpha, \phi)-$weak Pata contractive mapping via simulation function, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(w_0, w_n)\varrho(\top w_0, \top w_n), (1-\epsilon)(M(w_0, w_n) - \phi(M(w_0, w_n))) + P(w_0, w_n)) \\ &\leq (1-\epsilon)(M(w_0, w_n) - \phi(M(w_0, w_n))) + P(w_0, w_n)) - \alpha(w_0, w_n)\varrho(\top w_0, \top w_n). \end{aligned}$$

Using (2.3), we obtain

$$\begin{aligned} \varrho(w_1, w_{n+1}) &= \alpha(w_0, w_n)\varrho(\top w_0, \top w_n) \\ &\leq (1-\epsilon)(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\ &\quad \frac{\varrho(w_n, w_1) + \varrho(w_0, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \\ &\quad \varrho(w_0, w_1), \frac{\varrho(w_n, w_1) + \varrho(w_0, w_{n+1})}{2}\})) \\ &\quad + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + \|w_n\| + 0 + \|w_{n+1}\| + \|w_1\|]^{\vartheta} \\ &\leq (1-\epsilon)(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\ &\quad \frac{\varrho(w_n, w_0) + \varrho(w_1, w_0) + \varrho(w_{n+1}, w_n) + \varrho(w_n, w_0)}{2}\} \\ &\quad -\phi(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\ &\quad \frac{\varrho(w_n, w_0) + \varrho(w_1, w_0) + \varrho(w_{n+1}, w_n) + \varrho(w_n, w_0)}{2}\})) \\ &\quad + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + 2\|w_n\| + 2\|w_1\|]^{\vartheta} \\ &\leq (1-\epsilon)(\max\{\|w_n\|, \|w_1\|, \|w_n\| + \|w_1\|\} - \phi(\max\{\|w_n\|, \|w_1\|, \\ &\quad \|w_n\| + \|w_1\|\})) + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + 2\|w_n\| + 2\|w_1\|]^{\vartheta} \\ &\leq (1-\epsilon)(\|w_n\| + \|w_1\| - \phi(\|w_n\| + \|w_1\|)) \\ &\quad + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + 2\|w_n\| + 2\|w_1\|]^{\vartheta}. \end{aligned}$$

Since $\vartheta \leq \xi$, we get

$$\|w_n\| \leq (1-\epsilon)(\|w_n\| + \|w_1\| - \phi(\|w_n\| + \|w_1\|)) + 2\|w_1\| + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + 2\|w_n\| + 2\|w_1\|]^{\xi}$$

and

$$\epsilon\|w_n\| \leq k\epsilon^{\xi}\psi(\epsilon)\|w_n\|^{\xi} + l,$$

for some $k, l > 0$. By the same reason as in [21], $\{\|w_n\|\}$ is a bounded sequence. Using (2.2), we have

$$\begin{aligned} \varrho(w_n, w_{n+1}) &\leq \alpha(w_{n-1}, w_n)\varrho(w_n, w_{n+1}) \\ &\quad (1-\epsilon)(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \\ &\quad \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \\ &\quad \varrho(w_n, w_{n-1}), \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\})) \\ &\quad + \Lambda\epsilon^{\xi}\psi(\epsilon)[1 + \|w_n\| + \|w_{n-1}\| + \|w_{n+1}\| + \|w_n\|]^{\vartheta} \\ &\leq (1-\epsilon)(\max\left\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\right\} \\ &\quad -\phi(\max\left\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\right\})) + K\epsilon^{\xi}\psi(\epsilon), \end{aligned}$$

for some $K > 0$. Taking limit as $n \longrightarrow \infty$, we obtain $\varrho \leq K\epsilon^{\xi}\psi(\epsilon)$ and thus $\varrho = 0$.

Next, we demonstrate that $\{w_n\}$ is a Cauchy sequence. We assume that $\{w_n\}$ is not a Cauchy sequence. From Lemma 1, there exist subsequences $\{w_{m_j}\}$ and $\{w_{n_j}\}$ with $n_j > m_j > j$ such that $\lim_{j\to\infty} \varrho(w_{m_j-1}, w_{n_j+1}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j}, w_{nj}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j-1}, w_{n_j}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j+1}, w_{n_j+1}) = \varsigma$ and $\lim_{j\to\infty} \varrho(w_{m_j}, w_{n_j-1}) = \varsigma$. Since $\top$ is an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function, we have

$$\begin{aligned}\varsigma &\leq \varrho(w_{m_j}, w_{n_j}) = \alpha(w_{m_j-1}, w_{n_j-1})\varrho(\top w_{m_j-1}, \top w_{n_j-1}) \\ &\leq (1-\epsilon)\,(\max\{\varrho(w_{m_j-1}, w_{n_j-1}), \varrho(w_{m_j-1}, w_{m_j}), \varrho(w_{n_j-1}, w_{n_j}), \\ &\quad \frac{\varrho(w_{n_{j-1}}, w_{m_j}) + \varrho(w_{m_j-1}, w_{n_j})}{2}\} - \phi(\max\{\varrho(w_{m_j-1}, w_{n_j-1}), \\ &\quad \varrho(w_{m_j-1}, w_{m_j}), \varrho(w_{n_j-1}, w_{n_j}), \frac{\varrho(w_{n_{j-1}}, w_{m_j}) + \varrho(w_{m_j-1}, w_{n_j})}{2}\})) \\ &\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + \|w_{m_j}\| + \|w_{n_j}\| + \|w_{n_j+1}\| + \|w_{m_j+1}\|\right]^{\vartheta}.\end{aligned}$$

Taking the limit as $j \to \infty$, we obtain

$$\varsigma \leq (1-\epsilon)\,(\varsigma - \phi(\varsigma)) + K\epsilon\psi(\epsilon)$$

and

$$\varsigma \leq (1-\epsilon)\,\varsigma + K\epsilon\psi(\epsilon),$$

then

$$\varsigma \leq K\psi(\epsilon),$$

is a contradiction. Hence, $\{w_n\}$ is a Cauchy sequence in (W,ϱ). By the completeness of W, $w_n \to u \in W$ as $n \to +\infty$. Since $\top$ is continuous, $\top w_n \to \top u$ as $n \to +\infty$. By the uniqueness of the limit, we obtain $u = \top u$, that is, $u \in F(\top)$.

Now we demonstrate that fixed point of $\top$ is unique. Assume that u and v are fixed points of $\top$. Since $\top$ is an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function, we have

$$\begin{aligned}0 &\leq \zeta(\alpha(u,v)\varrho(\top u, \top v), (1-\epsilon)\,(M(u,v) - \phi(M(u,v))) + P(u,v)) \\ &\leq (1-\epsilon)\,(M(u,v) - \phi(M(u,v))) + P(u,v)) - \alpha(u,v)\varrho(\top u, \top v).\end{aligned}$$

Since $\top$ satisfies the hypothesis (iv) of Theorem 2, we have

$$\begin{aligned}\varrho(\top u, \top v) &\leq \alpha(u,v)\varrho(\top u, \top v) \\ &\leq (1-\epsilon)(\max\left\{\varrho(u,v), \varrho(u,\top u), \varrho(v,\top v), \frac{\varrho(u,\top v) + \varrho(v,\top u)}{2}\right\} \\ &\quad -\phi(\max\left\{\varrho(u,v), \varrho(u,\top u), \varrho(v,\top v), \frac{\varrho(u,\top v) + \varrho(v,\top u)}{2}\right\})) + K\epsilon\psi(\epsilon).\end{aligned}$$

We obtain that $\varrho(u,v) \leq K\psi(\epsilon)$, and so, $u = v$. Thus, $\top$ has a unique fixed point in W. □

The following theorem does not require the continuity of $\top$.

Let (W,ϱ) be a complete metric space. $\top : W \to W$ be an $(\alpha,\phi)-$weak Pata contractive mapping via simulation function. Assume that

(i) $\top$ is *triangular $\alpha-$admissible;*

(ii) *there exists $w_0 \in W$ such that $\alpha(w_0, \top w_0) \geq 1$;*

(iii) *if $\{w_n\}$ is a sequence in W such that $\alpha(w_n, w_{n+1}) \geq 1$, for all n and $w_n \to u \in W$ as $n \to +\infty$, then $\alpha(w_n, u) \geq 1$ for all n;*

(iv) *for all $u, v \in F(\top)$, $\alpha(u,v) \geq 1$.*

Then $\top$ has a unique fixed point that is $u = \top u$, $u \in W$.

Proof Following the proof of Theorem 2, we have already shown that $\{w_n\}$ is a Cauchy sequence in W. Since W is complete, we have $w_n \to u \in W$ as $n \to +\infty$. Next, we prove that $u \in F(\top)$, that is, $u = \top u$. From (2.2) and the hypothesis (iii) of Theorem 2, we have $\alpha(w_n, u) \geq 1$ for all n. Also, we have

$$0 \leq \zeta(\alpha(w_n,u)\varrho(\top u, w_{n+1}), (1-\epsilon)(M(w_n,u) - \phi(M(w_n,u)) + P(w_n,u))$$

and

$$\begin{aligned}
\varrho(\top u,u) &= \varrho(\top u,w_{n+1}) + \varrho(w_{n+1},u) \\
&\leq \alpha(w_n,u)\varrho(\top u,w_{n+1}) + \varrho(w_{n+1},u) \\
&\leq (1-\epsilon)(\max\left\{\varrho(u,w_n), \varrho(u,\top u), \varrho(w_n,w_{n+1}), \frac{\varrho(u,w_{n+1})+\varrho(w_n,\top u)}{2}\right\} \\
&\quad -\phi(\max\left\{\varrho(u,w_n), \varrho(u,\top u), \varrho(w_n,w_{n+1}), \frac{\varrho(u,w_{n+1})+\varrho(w_n,\top u)}{2}\right\})) \\
&\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1+\|w_n\|+\|u\|+\|\top u\|+\|w_{n+1}\|\right]^{\vartheta} + \varrho(w_{n+1},u). \\
&\leq (1-\epsilon)(\max\left\{\varrho(u,w_n), \varrho(u,\top u), \varrho(w_n,w_{n+1}), \frac{\varrho(u,w_{n+1})+\varrho(w_n,\top u)}{2}\right\} \\
&\quad -\phi(\max\left\{\varrho(u,w_n), \varrho(u,\top u), \varrho(w_n,w_{n+1}), \frac{\varrho(u,w_{n+1})+\varrho(w_n,\top u)}{2}\right\})) \\
&\quad +K\epsilon^{\xi}\psi(\epsilon),
\end{aligned}$$

for some $K > 0$. We take the limit as $n \to \infty$, we get

$$\varrho(\top u,u) \leq (1-\epsilon)(\varrho(\top u,u) - \phi(\varrho(\top u,u))) + K\epsilon^{\xi}\psi(\epsilon).$$

Thus, we obtain that $\top u = u$ and that is $u \in F(\top)$. Similar to the proof of Theorem 2, the uniqueness of fixed point of $\top$ can be obtained. □

Let $W = [0,1]$ with the usual metric and define the mappings $\top : W \to W$ by $\top(w) = \frac{w^2}{4}$, $w \in [0,1]$ and $\phi : [0,\infty] \to [0,\infty]$, $\phi(s) = \frac{s}{3}$. Let $\alpha : W \times W \to [0,+\infty)$ be defined as $\alpha(w,t) = \begin{cases} 1, & w,t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$. It is clear that $\top$ is triangular $\alpha-$admissible. Our goal is to show that $\top$ satisfies (2.1). For $w,t \in [0,1]$, we have

$$\varrho(w,t) - \phi(\varrho(w,t)) = \varrho(w,t) - \frac{1}{3}\varrho(w,t) = \frac{2}{3}\varrho(w,t)$$

and

$$\begin{aligned}
\varrho(\top w,\top t) &\leq \alpha(w,t)\varrho(\top w,\top t) \\
&= \frac{w^2}{6} - \frac{t^2}{6} \\
&= \frac{1}{6}(|w-t|)(w+t) \\
&\leq \frac{1}{3}(|w-t|) \\
&= \frac{1}{3}\varrho(w,t).
\end{aligned}$$

Since $\varrho(w,t) \leq M(w,t)$, we obtain

$$\varrho(\top w,\top t) \leq \frac{1}{3}M(w,t) = \frac{1}{2}\left(\frac{2}{3}M(w,t)\right) = \frac{1}{2}(M(w,t) - \phi(M(w,t))).$$

For arbitrary $\epsilon \in [0,1]$, we can write the above inequality as follows

$$\begin{aligned} \varrho(\top w, \top t) &\leq (1-\epsilon)\left(M(w,t) - \phi\left(M(w,t)\right)\right) + \left(\frac{1}{3} + \epsilon - 1\right) M(w,t) \\ &\leq (1-\epsilon)\left(M(w,t) - \phi\left(M(w,t)\right)\right) \\ &\quad + \left(\frac{1}{3} + \epsilon - 1\right)\left(1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right). \end{aligned}$$

Our goal is to prove that $\gamma \geq 0$ and $\Lambda \geq 0$ such that

$$\left(\frac{1}{3} + \epsilon - 1\right)\left(1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right) \leq \Lambda \epsilon^{\gamma+1}\left(1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right),$$

satisfies for all $w, t \in [0,1]$ and every $0 \leq \epsilon \leq 1$. We can find $\Lambda \geq 0$ such that

$$\Lambda = \frac{\left(\frac{1}{3} + \epsilon - 1\right)}{\epsilon^{\gamma+1}},$$

satisfies for each $0 \leq \epsilon \leq 1$ and some $\gamma \geq 0$. If we choose γ such that $\dfrac{\gamma}{\gamma+1} > 1 - \dfrac{1}{3}$, then

$$\Lambda = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}\left(1 - \frac{1}{3}\right)^{\gamma}}.$$

Thus, we have that

$$\alpha(w,t)\varrho(\top w, \top t) \leq (1-\epsilon)(M(w,t) - \phi(M(w,t)) + \Lambda \epsilon^{\gamma+1}\left(\begin{array}{c} 1 + \|w\| + \|t\| \\ + \|\top w\| + \|\top t\| \end{array}\right)$$

and

$$\zeta\left(\alpha(w,t)\varrho(\top w, \top t), (1-\epsilon)(M(w,t) - \phi(M(w,t)) + \Lambda \epsilon^{\gamma+1}\left(\begin{array}{c} 1 + \|w\| + \|t\| \\ + \|\top w\| + \|\top t\| \end{array}\right)\right) \leq 0,$$

satisfies for all $w, t \in [0,1]$, $\zeta \in \acute{Z}$ and each $\epsilon > 0$. If $\epsilon = 0$, it can be seen that (2.1) is satisfied. Also, the conditions of Theorem 2 are satisfied with $\psi(\epsilon) = \epsilon^{\gamma}$, $\xi = \vartheta = 1$. Hence, $\top$ has a unique fixed point in $W = [0,1]$. It is seen that, $u = 0$ is the unique fixed point of $\top$ in W.

Now, we state a fixed point result for multivalued $(\alpha, \phi)-$weak Pata contractive mapping.

Let (W, ϱ) be an ordered complete metric space and satisfy (H_*). Let $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$ and $\alpha : W \times W \longrightarrow [0, +\infty)$ be two functions. Assume that $\top : W \longrightarrow 2^{W}$ be a multivalued mapping has UCAV and if for all $w, t \in W$ with w and t comparable, and $\epsilon \in [0,1]$, $\top$ satisfies the inequality

$$\alpha(w,t) H_{\varrho}(\top w, \top t) \leq (1-\epsilon)(M(w,t) - \phi(M(w,t)) + P(w,t), \tag{4}$$

where $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$, for all $s > 0$, and

$$P(w,t) = \Lambda \epsilon^{\xi} \psi(\epsilon)\left[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right]^{\vartheta}$$

and

$$M(w,t) = \max\left\{\varrho(w,t), \varrho(w, \top w), \varrho(t, \top t), \frac{\varrho(w, \top t) + \varrho(t, \top w)}{2}\right\},$$

and also, assume that $\top$ satisfies the following conditions

(i) $\top$ is *triangular* $\alpha-$*admissible;*

(ii) *there exists* $w_0 \in W$, $w_1 \in \top w_0$ *such that* $\alpha(w_0, w_1) \geq 1$*;*

(iii) $\top$ *is continuous;*

(iv) *for all* $u, v \in F_H(\top)$, $\alpha(u, v) \geq 1$.

Then $\top$ *has a unique fixed point that is* $u \in \top u$, $u \in W$.

Proof The hypothesis (ii) of the Theorem 2, there exists $w_0 \in W$ such that $\alpha(w_0, \top w_0) \geq 1$. Starting at the point $w_0 \in W$, if $w_0 \in \top w_0$, proof is clearly completed. Since $\top w_0$ has UCAV, there exists $w_1 \in \top w_0$ with $w_1 \neq w_0$ and $w_1 \geq w_0$ such that

$$\varrho(w_0, w_1) = \inf_{w \in \top w_0} \varrho(w, w_0) = \varrho(\top w_0, w_0).$$

Continuing this process, the iterative sequence $\{w_n\}$ is constructed by $w_{n+1} \in \top w_n$ with $w_{n+1} \neq w_n$ and $w_{n+1} \geq w_n$, for all $n \geq 1$ such that

$$\varrho(w_n, w_{n+1}) = \varrho(\top w_n, w_n).$$

Furthermore

$$\varrho(\top w_n, w_n) \leq \sup_{w \in \top w_{n-1}} \varrho(\top w_n, w) \leq H_\varrho(\top w_n, \top w_{n-1}).$$

Thus, we have

$$\varrho(w_n, w_{n+1}) \leq H_\varrho(\top w_{n-1}, \top w_n), \text{ for } n \geq 2.$$

We denote $\|w_n\| = \varrho(w_n, w_0)$ for $n \geq 1$. If $w_{n_0} = w_{n_0+1}$ for any $n_0 \in \mathbb{N}$, then $w_{n_0} \in \top w_{n_0}$. First of all, we show that $\alpha(w_n, w_{n+1}) \geq 1$, for all $n \in \mathbb{N}$. Since $\top$ is an $\alpha-$admissible mapping, $w_0 \in W$, $w_1 \in \top w_0$ we have

$$\alpha(w_0, w_1) \geq 1 \text{ implies } \alpha(w_1, w_2) \geq 1, \text{ for } w_2 \in \top w_1$$

and

$$\alpha(w_1, w_2) \geq 1 \text{ implies } \alpha(w_2, w_3) \geq 1, \text{ for } w_3 \in \top w_2.$$

By induction, we obtain

$$\alpha(w_n, w_{n+1}) \geq 1, \text{ for } w_{n+1} \in \top w_n, \ n \in \mathbb{N}. \tag{5}$$

Since $\top$ is triangular $\alpha-$admissible, we have

$$\alpha(w_n, w_{n+1}) \geq 1 \text{ and } \alpha(w_{n+1}, w_{n+2}) \geq 1 \text{ imply } \alpha(w_n, w_{n+2}) \geq 1, \ w_{n+2} \in w_{n+1}.$$

Thus, by induction, we get

$$\alpha(w_n, w_m) \geq 1 \text{ for all } m > n \geq 0. \tag{6}$$

Now, we will show that $\{\varrho(w_{n+1}, w_n)\}$ is a decreasing sequence. Using (2.5),

$$\begin{aligned}
\varrho(w_n, w_{n+1}) &\leq H_\varrho(\top w_{n-1}, \top w_n) \\
&\leq \alpha(w_{n-1}, w_n) H_\varrho(\top w_{n-1}, \top w_n) \\
&\leq (1-\epsilon)(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \\
&\quad \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_{n-1}), \\
&\quad \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\})) \\
&\quad + \Lambda \epsilon^\xi \psi(\epsilon) \left[1 + \|w_{n-1}\| + \|w_n\| + \|w_n\| + \|w_{n+1}\|\right]^\vartheta \\
&\leq (1-\epsilon)(\max\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\} \\
&\quad - \phi(\max\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\})) \\
&\quad + K \epsilon^\xi \psi(\epsilon),
\end{aligned}$$

for some $K > 0$. If $\varrho(w_n, w_{n-1}) \leq \varrho(w_{n+1}, w_n)$, then we obtain $\varrho(w_{n+1}, w_n) \leq (1-\epsilon)(\varrho(w_{n+1}, w_n) - \phi(\varrho(w_{n+1}, w_n))) + K\epsilon^\xi \psi(\epsilon)$. In this way, we obtain $\varrho(w_{n+1}, w_n) = 0$, is a contraction. Therefore, we have

$$\varrho(w_{n+1}, w_n) \leq H_\varrho(\top w_{n-1}, \top w_n) < \varrho(w_n, w_{n-1}).$$

If we continue this process, we get

$$\varrho(w_{n+1}, w_n) < \varrho(w_n, w_{n-1}) < \cdots < \varrho(w_1, w_0) = \|w_1\|,$$

that is $\{\varrho(w_{n+1}, w_n)\}$ is a decreasing sequence and so, this sequence is convergent to $\varrho \geq 0$ and $\lim_{n\to\infty} \varrho(w_n, w_{n+1}) = \varrho$. Now, we will demonstrate that $\{\|w_n\|\}$ is a bounded sequence. By the triangle inequality, we have

$$\|w_n\| = \varrho(w_n, w_0) \leq \varrho(w_n, w_{n+1}) + \varrho(w_{n+1}, w_1) + \varrho(w_1, w_0),$$

is a contradiction. Since $\top$ is a multivalued $(\alpha, \phi)-$weak Pata contractive mapping with (2.6), we obtain

$$\begin{aligned}
\varrho(w_1, w_{n+1}) &\leq H_\varrho(\top w_0, \top w_n) \\
&\leq \alpha(w_0, w_n) H_\varrho(\top w_0, \top w_n) \\
&\leq (1-\epsilon)\,(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\
&\quad \frac{\varrho(w_n, w_1) + \varrho(w_0, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \\
&\quad \varrho(w_0, w_1), \frac{\varrho(w_n, w_1) + \varrho(w_0, w_{n+1})}{2}\})) \\
&\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + \|w_n\| + 0 + \|w_{n+1}\| + \|w_1\|\right]^{\vartheta} \\
&\leq (1-\epsilon)\,(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\
&\quad \frac{\varrho(w_n, w_0) + \varrho(w_1, w_0) + \varrho(w_{n+1}, w_n) + \varrho(w_n, w_0)}{2}\} \\
&\quad -\phi(\max\{\varrho(w_n, w_0), \varrho(w_n, w_{n+1}), \varrho(w_0, w_1), \\
&\quad \frac{\varrho(w_n, w_0) + \varrho(w_1, w_0) + \varrho(w_{n+1}, w_n) + \varrho(w_n, w_0)}{2}\})) \\
&\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + 2\|w_n\| + 2\|w_1\|\right]^{\vartheta} \\
&\leq (1-\epsilon)\,(\max\{\|w_n\|, \|w_1\|, \|w_n\| + \|w_1\|\} - \phi(\max\{\|w_n\|, \|w_1\|, \\
&\quad \|w_n\| + \|w_1\|\})) + \Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + 2\|w_n\| + 2\|w_1\|\right]^{\vartheta} \\
&\leq (1-\epsilon)\,(\|w_n\| + \|w_1\| - \phi(\|w_n\| + \|w_1\|)) \\
&\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + 2\|w_n\| + 2\|w_1\|\right]^{\vartheta}.
\end{aligned}$$

Since $\vartheta \leq \xi$, we get

$$\|w_n\| \leq (1-\epsilon)\,(\|w_n\| + \|w_1\| - \phi(\|w_n\| + \|w_1\|)) + 2\|w_1\| + \Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + 2\|w_n\| + 2\|w_1\|\right]^{\xi}$$

and

$$\epsilon\|w_n\| \leq k\epsilon^{\xi}\psi(\epsilon)\|w_n\|^{\xi} + l,$$

for some $k, l > 0$. By the same reason as in [21], $\{\|w_n\|\}$ is a bounded sequence. Using (2.5), we have

$$\begin{aligned}
\varrho(w_n, w_{n+1}) &\leq \alpha(w_{n-1}, w_n) H_\varrho(\top w_{n-1}, \top w_n) \\
&\quad (1-\epsilon)\,(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \varrho(w_n, w_{n-1}), \\
&\quad \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\} - \phi(\max\{\varrho(w_n, w_{n-1}), \varrho(w_{n+1}, w_n), \\
&\quad \varrho(w_n, w_{n-1}), \frac{\varrho(w_n, w_n) + \varrho(w_{n-1}, w_{n+1})}{2}\})) \\
&\quad +\Lambda\epsilon^{\xi}\psi(\epsilon)\left[1 + \|w_n\| + \|w_{n-1}\| + \|w_{n+1}\| + \|w_n\|\right]^{\vartheta} \\
&\leq (1-\epsilon)\,(\max\left\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\right\} \\
&\quad -\phi(\max\left\{\varrho(w_n, w_{n-1}), \frac{\varrho(w_{n+1}, w_n) + \varrho(w_n, w_{n-1})}{2}\right\})) + K\epsilon^{\xi}\psi(\epsilon),
\end{aligned}$$

for some $K > 0$. Taking limit as $n \to \infty$, we obtain $\varrho \leq K\epsilon^{\xi}\psi(\epsilon)$ and thus $\varrho = 0$.

Next, we demonstrate that $\{w_n\}$ is a Cauchy sequence. We assume that $\{w_n\}$ is not a Cauchy sequence. From Lemma 1, there exist subsequences $\{w_{m_j}\}$ and $\{w_{n_j}\}$ with $n_j > m_j > j$ such that $\lim_{j\to\infty} \varrho(w_{m_j-1}, w_{n_j+1}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j}, w_{nj}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j-1}, w_{n_j}) = \varsigma$, $\lim_{j\to\infty} \varrho(w_{m_j+1}, w_{n_j+1}) = \varsigma$ and $\lim_{j\to\infty} \varrho(w_{m_j}, w_{n_j-1}) = \varsigma$. Since $\top$ is a multivalued $(\alpha, \phi)-$weak Pata contractive mapping with (2.6), we have

$$\begin{aligned} \varsigma &\leq \varrho(w_{m_j}, w_{n_j}) \leq \alpha(w_{m_j-1}, w_{n_j-1}) H_\varrho(\top w_{m_j-1}, \top w_{n_j-1}) \\ &\leq (1-\epsilon)\,(\max\{\varrho(w_{m_j-1}, w_{n_j-1}), \varrho(w_{m_j-1}, w_{m_j}), \varrho(w_{n_j-1}, w_{n_j}), \\ &\quad \frac{\varrho(w_{n_{j-1}}, w_{m_j}) + \varrho(w_{m_j-1}, w_{n_j})}{2}\} - \phi(\max\{\varrho(w_{m_j-1}, w_{n_j-1}), \\ &\quad \varrho(w_{m_j-1}, w_{m_j}), \varrho(w_{n_j-1}, w_{n_j}), \frac{\varrho(w_{n_{j-1}}, w_{m_j}) + \varrho(w_{m_j-1}, w_{n_j})}{2}\})) \\ &\quad + \Lambda\epsilon^\xi \psi(\epsilon) \left[1 + \|w_{m_j}\| + \|w_{n_j}\| + \|w_{n_j+1}\| + \|w_{m_j+1}\|\right]^\vartheta. \end{aligned}$$

Taking the limit as $j \to \infty$, we obtain

$$\varsigma \leq (1-\epsilon)\,(\varsigma - \phi(\varsigma)) + K\epsilon\psi(\epsilon)$$

and $\varsigma \leq (1-\epsilon)\,\varsigma + K\epsilon\psi(\epsilon)$. We obtain that

$$\varsigma \leq K\psi(\epsilon),$$

is a contradiction. Hence, $\{w_n\}$ is a Cauchy sequence in (W, ϱ). Since W is complete metric space, we get $w_n \to u \in W$ as $n \to +\infty$. Since $\top$ is continuous, $\top w_n \to \top u$ as $n \to +\infty$. By the uniqueness of the limit, we obtain $u \in \top u$, that is, $u \in F_H(\top)$.

Now, we demonstrate that fixed point of $\top$ is unique. Assume that u and v are fixed points of $\top$. Since $\top$ satisfies the hypothesis (iv) of Theorem 2 and $\top$ is a multivalued $(\alpha, \phi)-$weak Pata contractive mapping, we have

$$\begin{aligned} \varrho(\top u, fv) &\leq \alpha(u,v) H_\varrho(\top u, fv) \\ &\leq (1-\epsilon)(\max\left\{\varrho(u,v), \varrho(u,\top u), \varrho(v, fv), \frac{\varrho(u, fv) + \varrho(v, \top u)}{2}\right\} \\ &\quad -\phi(\max\left\{\varrho(u,v), \varrho(u,\top u), \varrho(v, fv), \frac{\varrho(u, fv) + \varrho(v, \top u)}{2}\right\})) + K\epsilon\psi(\epsilon). \end{aligned}$$

Thus, we obtain that $\varrho(u, v) \leq K\psi(\epsilon)$, and so, $u = v$. Thus $\top$ has a unique fixed point in W. □

If we take $M(w,t) = \varrho(w,t)$, for all $w, t \in W$ in Theorem 2, then we get the following corollary.

Let (W, ϱ) be a complete metric space, $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$ and $\alpha : W \times W \to [0, +\infty)$, $\top : W \to W$ be two functions. If for all $w, t \in W$, and $\epsilon \in [0,1]$, $\top$ satisfies the inequality

$$\alpha(w,t)\varrho(\top w, \top t) \leq (1-\epsilon)(\varrho(w,t) - \phi(\varrho(w,t)) + \Lambda\epsilon^\xi\psi(\epsilon)\,[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|]^\vartheta,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$, for all $s > 0$, and

(i) $\top$ is *triangular $\alpha-$admissible;*

(ii) *there exists $w_0 \in W$ such that $\alpha(w_0, \top w_0) \geq 1$;*

(iii) $\top$ *is continuous;*

(iv) *for all $u, v \in F(\top)$, $\alpha(u, v) \geq 1$.*

Then $\top$ has a unique fixed point $u = \top u$.

If we take $M(w,t) = \varrho(w,t)$ and $\alpha(w,t) = 1$, for all $w, t \in W$ in Theorem 2, then we get the following corollary.

Let (W, ϱ) be a complete metric space, $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$, $\top : W \to W$ be a function. If for all $w, t \in W$, and $\epsilon \in [0,1]$, $\top$ satisfies the inequality

$$\varrho(\top w, \top t) \leq (1-\epsilon)(\varrho(w,t) - \phi(\varrho(w,t)) + \Lambda\epsilon^\xi\psi(\epsilon)\,[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|]^\vartheta,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$, for all $s > 0$, and $\top$ *is continuous.* Then $\top$ has a unique fixed point $u = \top u$.

Corollary 2 generalizes the results of Pata [21] and Banach [7]. For $\epsilon = 0$, we get the results of [26] and in addition to this, if $\epsilon = 0$ and we take $M(w,t) = \varrho(w,t)$, for all $w, t \in W$ in Theorem 2, then we get the results of [1].

If we take $\alpha(w,t) = 1$, for all $w, t \in W$ in Theorem 2, then we get the following corollary.

Let (W, ϱ) be an ordered complete metric space and satisfy (H_*). Let $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$ be a function. Assume that $\top : W \to 2^W$ be a multivalued mapping has UCAV and if for all $w, t \in W$ with w and t comparable, and $\epsilon \in [0, 1]$, $\top$ satisfies the inequality

$$H_\varrho(\top w, \top t) \leq (1-\epsilon)(M(w,t) - \phi(M(w,t)) + P(w,t),$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$ for all $s > 0$, and

$$P(w,t) = \Lambda \epsilon^\xi \psi(\epsilon) \left[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right]^\vartheta$$

and

$$M(w,t) = \max\left\{\varrho(w,t), \varrho(w,\top w), \varrho(t,\top t), \frac{\varrho(w,\top t) + \varrho(t,\top w)}{2}\right\},$$

and also, $\top$ *is continuous, then* $\top$ *has a unique fixed point, that is,* $u \in \top u$, $u \in W$.

If we take $M(w,t) = \varrho(w,t)$ and $\alpha(w,t) = 1$, for all $w, t \in W$ in Theorem 2, then we get the following corollary.

Let (W, ϱ) be an ordered complete metric space and satisfy (H_*). Let $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants, $\psi \in \Psi$ be a function. Assume that $\top : W \to 2^W$ be a multivalued mapping has UCAV and if for all $w, t \in W$ with w and t comparable, and $\epsilon \in [0, 1]$, $\top$ satisfies the inequality

$$H_\varrho(\top w, \top t) \leq (1-\epsilon)(\varrho(w,t) - \phi(\varrho(w,t)) + P(w,t),$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(s) > 0$, for all $s > 0$, and

$$P(w,t) = \Lambda \epsilon^\xi \psi(\epsilon) \left[1 + \|w\| + \|t\| + \|\top w\| + \|\top t\|\right]^\vartheta,$$

and also $\top$ *is continuous, then* $\top$ *has a unique fixed point, that is,* $u \in \top u$, $u \in W$.

Corollary 2 generalizes the results of Kolagar [16] and Nadler [18].

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