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Research Article

Epitrochoidal Hypersurfaces in 4-Space

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Abstract

We introduce the epitrochoidal hypersurfaces in four dimensional Euclidean space $\mathbb{E}^4$. We serve notations of a Euclidean geometry $\mathbb{E}^4$. Giving a definition of the rotational hypersurface, we define the epitrochoidal hypersurface, and calculate its differential geometric objects, such as the Gauss map and the curvatures. In the end, we reveal some relations for the curvatures of that type hypersurfaces.

Keywords: 4-space, epitrochoidal hypersurface, Gauss map, Gaussian curvature, mean curvature.

4-Boyutta Epitrokhoidal Hiperyüzeyler

Öz

Dört boyutlu Öklid uzayı $\mathbb{E}^4$'de epitrokhoidal hiperyüzeylere giriş yaptık. Öklid geometrisi $\mathbb{E}^4$'ün notasyonlarını verdik. Dönel hiperyüzeyin bir tanımını vererek, epitrokhoidal hiperyüzeyi tanımladık ve Gauss tasviri ve eğrilikler gibi diferansiyel geometrik nesneleri hesapladık. Son olarak, bu tip hiperyüzeylerin eğrilikleri için bazı bağıntıları ortaya çıkardık.

Anahtar Kelimeler: 3-boyut, epitrochoidal hiperyüzey, Gauss tasviri, Gauss eğriliği, ortalama eğrilik.

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1. Introduction

In the literature, we see the following papers [2,4] and also some books [1,3,5-7] about hyper-surfaces.

We introduce the epitrochoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We show the fundamental notions of a Euclidean space in section II. In section III, we present the rotation hypersurface. Then, we consider the epitrochoidal hypersurface, and obtain its curvatures in section IV.

2. Hypersurfaces

In $\mathbb{E}^{n+1}$, revealing the i-th curvature formulas $\mathfrak{C}_i$, $i = 0,1,\dots,n$, one can take the characteristic equation of the shape operator $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0$$

$$= \det(\mathbf{S} - \lambda\mathfrak{I}_n)$$

$$= \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k},$$

where $\mathfrak{I}_n$ indicates the identity matrix. Hence, we obtain the curvature formulas

$$\binom{n}{i}\mathfrak{C}_i = s_i.$$

Here, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ by definition. Considering the following fundamental form

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y\rangle,$$

we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}\mathfrak{C}_i\, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0.$$

Identifying a vector (m,n,p,q) and its transpose, we assume that $\mathbf{x} = \mathbf{x}(u,v,w)$ be the hypersurface M^3 in $\mathbb{E}^4$. The dot product of the vectors $\vec{x} = (x_1,x_2,x_3,x_4)$ and $\vec{y} = (y_1,y_2,y_3,y_4)$ is given by

$$\langle \vec{x}, \vec{y}\rangle = \sum_{t=1}^{4} x_t y_t.$$

The vectorial product of the vectors $\vec{x}$, $\vec{y}$, and $\vec{z} = (z_1,z_2,z_3,z_4)$ is served by

$$\vec{x}\times\vec{y}\times\vec{z} = \det\begin{pmatrix} e_1 e_2 e_3 e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 z_2 z_3 z_4 \end{pmatrix}.$$

The Gauss map of the hypersurface $\mathbf{x}$ is presented by

$$\boldsymbol{\mathcal{G}} = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|}, \qquad (2.1)$$

where $\mathbf{x}_u = d\mathbf{x}/du$. For the hypersurface $\mathbf{x}$, we consider

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}$$

$$\mathrm{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}$$

$$\mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix}.$$

Here, the coefficients are obtained by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle,\ \ F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle,\ \ G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle,$$

$$A = \langle \mathbf{x}_u, \mathbf{x}_w\rangle,\ \ B = \langle \mathbf{x}_v, \mathbf{x}_w\rangle,\ \ C = \langle \mathbf{x}_w, \mathbf{x}_w\rangle,$$

$$L = \langle \mathbf{x}_{uu}, \boldsymbol{\mathcal{G}}\rangle,\ \ M = \langle \mathbf{x}_{uv}, \boldsymbol{\mathcal{G}}\rangle,\ \ N = \langle \mathbf{x}_{vv}, \boldsymbol{\mathcal{G}}\rangle,$$

$$P = \langle \mathbf{x}_{uw}, \boldsymbol{\mathcal{G}}\rangle,\ \ T = \langle \mathbf{x}_{vw}, \boldsymbol{\mathcal{G}}\rangle,\ \ V = \langle \mathbf{x}_{ww}, \boldsymbol{\mathcal{G}}\rangle,$$

$$X = \langle \boldsymbol{\mathcal{G}}_u, \boldsymbol{\mathcal{G}}_u\rangle,\ \ Y = \langle \boldsymbol{\mathcal{G}}_u, \boldsymbol{\mathcal{G}}_v\rangle,\ \ Z = \langle \boldsymbol{\mathcal{G}}_v, \boldsymbol{\mathcal{G}}_v\rangle,$$

$$O = \langle \boldsymbol{\mathcal{G}}_u, \boldsymbol{\mathcal{G}}_w\rangle,\ \ R = \langle \boldsymbol{\mathcal{G}}_v, \boldsymbol{\mathcal{G}}_w\rangle,\ \ S = \langle \boldsymbol{\mathcal{G}}_w, \boldsymbol{\mathcal{G}}_w\rangle,$$

and $\boldsymbol{\mathcal{G}}$ is the Gauss map.

Next, we will obtain the curvatures and IV fundamental form matrix for the hypersurface $\mathbf{x}$. By taking the polynomial

$$P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$

we reveal the curvature formulas: $\mathfrak{C}_0 = 1$ (from definition),

$$\mathfrak{C}_1 = -\frac{b}{3a}, \quad \mathfrak{C}_2 = \frac{c}{3a}, \quad \mathfrak{C}_3 = -\frac{d}{a}. \qquad (2.2)$$

Theorem 2.1. *A hypersurface* M^3 *in* $\mathbb{E}^4$, *has the following relation*

$$\mathfrak{C}_0 \mathrm{IV} - 3\mathfrak{C}_1 \mathrm{III} + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 \mathrm{I} = 0.$$

Proof. See [5] for details.

3. Rotational Hypersurfaces

We introduce a kind of rotational hypersurface which its generating curve has epitrochoid curve in $\mathbb{E}^4$.

Let γ, from interval I to plane Π, be a curve, and ℓ be the line in Π. Rotation hypersurface can be defined by a rotating with generating curve γ around axis ℓ in $\mathbb{E}^4$.

We can suppose ℓ obained by $(0,0,0,1)^t$. The rotating matrix is defined by

$$\boldsymbol{A}(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$.

The matrix $\boldsymbol{A}$ supplying the following equations

$$\boldsymbol{A}\ell = \ell, \quad \boldsymbol{A}^t A = \boldsymbol{A}\boldsymbol{A}^t = \mathfrak{I}_4, \quad det\, \boldsymbol{A} = 1.$$

While the rotation axis be ℓ, we have a Euclidean transformation that ℓ transforms to x_4. Generating curve is indicated by

$$\gamma(u) = \left(\mathscr{f}(u), 0, 0, \mathscr{g}(u)\right),$$

where $\mathscr{f}(u), \mathscr{g}(u)$ are the functions which are differentiable $\forall u \in I$. Hence, the rotational hypersurface is given as

$$\mathfrak{R}(u,v,w) = \boldsymbol{A}.\gamma^t,$$

where $0 \le v, w < 2\pi$.

4. Epitrochoidal Hypersurfaces

Next, by using rotation matrix in $\mathbb{E}^4$, and profile curve γ with vector on x_4, we find rotational hypersurface having epitrochoid curve. We named it the epitrochoidal hypersurface $\mathcal{E}(u,v,w)$.

Epitrochoid described a curve that is traced out by a point **p** fixed relative to a circle of radius b rolling outside a fixed circle of radius. Here h denotes the distance from **p** to the center of the rolling circle.

Considering the following epitrochoid curve for $a, b, h \in \mathbb{R}$ in $\mathbb{E}^4$

$$\gamma[a,b,h](u) = \left(\mathscr{f}(u), 0, 0, \mathscr{g}(u)\right)$$

$$= \begin{pmatrix} (a+b)\cos u - h\cos\left(\frac{a+b}{b}u\right) \\ 0 \\ 0 \\ (a+b)\sin u - h\sin\left(\frac{a+b}{b}u\right) \end{pmatrix},$$

we calculate the Gauss map, and also find the curvatures $\mathfrak{C}_{i=1,2,3}$ of the epitrochoidal hypersurface.

We find Gauss map of the epitrochoidal hypersurface, also draw its figure.

In $\mathbb{E}^4$, the epitrochoidal hypersurface $\mathcal{E}(u,v,w)$ is defined by

$$\mathcal{E}(u,v,w) = \begin{pmatrix} \mathscr{f}(u) \cos v \cos w \\ \mathscr{f}(u) \sin v \cos w \\ \mathscr{f}(u) \sin w \\ \mathscr{g}(u) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \qquad (4.1)$$

Taking $(a,b,h) = (3,1,2)$, $w = \pi/4$ in (4.1), we have projection surface into 3-space. See Figure 1 for the projection.

Using the Gauss map formula (2.1) on (4.1), we have the following Gauss map of the hypersurface

$$\mathcal{G} = \frac{1}{(W)^{1/2}} \begin{pmatrix} \mathscr{s}(u) \cos v \cos w \\ \mathscr{s}(u) \sin v \cos w \\ \mathscr{s}(u) \sin w \\ \mathscr{t}(u) \end{pmatrix}.$$

where

$$\mathscr{s}(u) = b\cos u - h\cos\left(\frac{a+b}{b}u\right),$$

$$\mathscr{t}(u) = b\sin u - h\sin\left(\frac{a+b}{b}u\right),$$

$$W = b^2 - 2bh\cos\left(\frac{a}{b}u\right) + h^2.$$

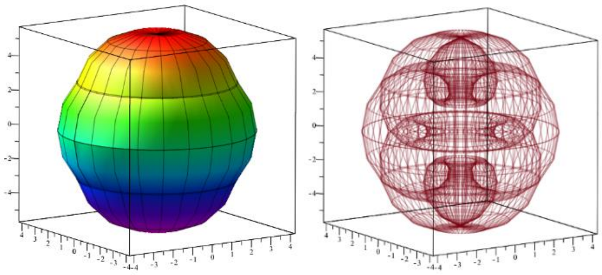

Figure 1. Projection of $\mathcal{E}(u, v, w)$ into $x_1x_2x_4$-space,

Left: outside view, Right: inside view

Theorem 4.1. *The epitrochoidal hypersurface* (4.1) *in* $\mathbb{E}^4$ *has the following curvature formulas*

$\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = -(448\cos^7 u - 784\cos^5 u - 200\cos^4 u + 392\cos^3 u + 188\cos^2 u - 24\cos u - 19)/(4\psi\Omega^{3/2}),$$

$$\mathfrak{C}_2 = -(192\cos^7 u - 336\cos^5 u - 104\cos^4 u + 168\cos^3 u + 96\cos^2 u - 5\cos u - 9).(16\cos^4 u - 16\cos^2 u - \cos u + 2)/(4\psi^2\Omega^2),$$

$$\mathfrak{C}_3 = (16\cos^4 u - 16\cos^2 u - \cos u + 2)^2.(40\cos^3 u - 30\cos u - 17)/\left(16\psi^2\Omega^{5/2}\right),$$

where

$$\Omega(u) = -16\cos^3 u + 12\cos u + 5,$$

$$\psi(u) = 2\cos u - \cos(4u).$$

Proof. Computing eqs. (2.2) on (4.1), we have the curvatures.

5. Conclusion

Epitrochoidal hypersurfaces have never been worked. We introduce the epitrochoidal hypersurfaces, and then reveal some differential geometric results of it.

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