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Research Article

On the singular values of the incomplete Beta function

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ABSTRACT. A new definition of the incomplete beta function as a distribution-valued meromorphic function is given and the finite parts of it and of its partial derivatives at the singular values are calculated and compared with formulas in the literature.

Keywords: Beta function, distribution theory, finite parts.

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1. INTRODUCTION AND NOTATION

This paper originated when one of the authors (N.O.) came across the article [3]. The explicit formulas in [3] were interesting, but we could not concur with the overall framework in which they had been derived. The calculations in [3] are based on van der Corput's "neutrix calculus", see [1], a way of evaluating divergent integrals, which was inspired by Hadamard's method. This "technique of neglecting appropriately defined infinite quantities", see [12, p. 984] , produces numbers, not distributions. Accordingly, the results in [3] represent the incomplete beta function only on the open interval $(0, 1)$ and do not furnish a distribution on $\mathbf{R}$. So we thought that it might be reasonable to reconsider the calculations in [3] from the nowadays generally adopted viewpoint of distribution theory.

Let us mention that regularizations in Hadamard's sense but employing L. Schwartz' theory of distributions were investigated in [9, pp. 15–19], for three kinds of distributions.

Classically, the incomplete beta function is defined by the integral

$$B_{\lambda,\mu}(x) = \int_0^x t^{\lambda-1}(1-t)^{\mu-1}\,\mathrm{d}t, \quad 0 \le x \le 1,\ \mathrm{Re}\,\lambda > 0,\ \mathrm{Re}\,\mu > 0,$$

see [4, Equ. 8.931]. The goal of the article [3] as well as of this paper consists in defining and evaluating $B_{\lambda,\mu}$ and its partial derivatives with respect to λ and μ at the "singular values", i.e., if $\lambda \in -\mathbf{N}_0$ or $\mu \in -\mathbf{N}_0$.

In Section 2, we define $B_{\lambda,\mu}$ as distributions depending analytically on $(\lambda, \mu) \in \mathbf{C}^2$. At the poles, e.g. if $\lambda = -k \in -\mathbf{N}_0$, we set $B_{-k,\mu} = \mathrm{Pf}_{\lambda=-k}\, B_{\lambda,\mu}$, i.e., $B_{-k,\mu}$ is defined as the finite part of the Laurent series of $B_{\lambda,\mu}$ about $\lambda = -k$. The procedure of embedding a function into a family of distributions which depend analytically on a parameter goes back to M. Riesz, see [14, pp. 31, 32], L. Schwartz, see [15, p. 39], and J. Dieudonné, see [2, pp. 260–262]. With respect to distribution-valued analytic or meromorphic functions, we refer the reader also to [10].

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In Section 3, we collect some algebraic reduction formulas, which show that our task can be reduced to evaluating $B, \partial_\lambda B, \partial_\mu B$ if λ or μ are 1. This is eventually done for B in Section 4 and for $\partial_\lambda B, \partial_\mu B$ in Section 5, respectively.

Let us introduce some notation. As usual, an empty series, as, e.g., in $\sum_{j=1}^{0} c_j$, sums to zero. $\mathbf{N}$ and $\mathbf{N}_0$ denote the sets of positive and of non-negative integers, respectively. We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{E}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{E}$ of "test functions" and of C^∞ functions, respectively, see [15, 6, 11]. For the evaluation of a distribution T on a test function ϕ, we use angle brackets, i.e., $\langle \phi, T \rangle$. In this paper, all distributions are on the real axis $\mathbf{R}$, i.e., they belong to $\mathcal{D}'(\mathbf{R})$, but usually depend meromorphically on the complex variables λ, μ. Differentiation with respect to x is denoted by the apostrophe, differentiation with respect to λ, μ by $\partial_\lambda, \partial_\mu$ or $\partial/\partial\lambda, \partial/\partial\mu$ or ∂_1, ∂_2.

The Heaviside function is denoted by Y, see [15, p. 36]. We write δ for the delta distribution with support in 0, i.e., $\delta = Y'$, and δ_1 for the delta distribution with support in 1, i.e., $\delta_1 = Y(x-1)'$. The letter ψ denotes the logarithmic derivative Γ'/Γ of the gamma function and $\mathcal{L}_2$ denotes the dilogarithm, i.e., $\mathcal{L}_2(0) = 0$ and

$$\mathcal{L}_2(x) = \oint_0^1 \frac{\log t}{t - x^{-1}}\, \mathrm{d}t, \quad x \in \mathbf{R} \setminus \{0\},$$

see [5, Section 323].

2. Definition of the incomplete beta function

Let us first recall some facts concerning the distribution $x_+^\lambda = Y(x)x^\lambda$, see [6, Section 3.2, p. 68], , [11, Exs. 1.3.9, 1.4.8, pp. 32, 49]. If $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > -1$, then x_+^λ is a locally integrable function on $\mathbf{R}$ and hence belongs to $\mathcal{D}'(\mathbf{R})$. The function

$$\{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda > -1\} \longrightarrow \mathcal{D}'(\mathbf{R}) : \lambda \longmapsto x_+^\lambda$$

is analytic and can analytically be extended to $\mathbf{C} \setminus (-\mathbf{N})$. This extension, which is also denoted by x_+^λ, is meromorphic on $\mathbf{C}$ and has simple poles on $-\mathbf{N}$ with the residues

$$\operatorname*{Res}_{\lambda=-k-1} x_+^\lambda = (-1)^k \delta^{(k)}/k!$$

for $k \in \mathbf{N}_0$. For abbreviation, we also set

$$x_+^{-k} = \operatorname*{Pf}_{\lambda=-k} x_+^\lambda \ \text{ if } k \in \mathbf{N}.$$

In [13, pp. 11, 12], the distributions x_+^λ are called *Hadamard kernels*.

Note that $x \cdot x_+^\lambda = x_+^{\lambda+1}$ holds for each $\lambda \in \mathbf{C}$. In contrast, the differentiation formula $(x_+^\lambda)' = \lambda x_+^{\lambda-1}$ is valid for $\lambda \in \mathbf{C} \setminus (-\mathbf{N}_0)$ by analytic continuation, but at $\lambda = -k$, $k \in \mathbf{N}_0$, we obtain

$$\begin{aligned}
(x_+^{-k})' = \operatorname*{Pf}_{\lambda=-k} (x_+^\lambda)' &= \operatorname*{Pf}_{\lambda=-k} \lambda x_+^{\lambda-1} \\
&= \operatorname*{Pf}_{\lambda=-k} [(\lambda+k)x_+^{\lambda-1} - k x_+^{\lambda-1}] \\
&= \lim_{\lambda\to-k} (\lambda+k)x_+^{\lambda-1} - k x_+^{-k-1} \\
&= \operatorname*{Res}_{\lambda=-k} x_+^{\lambda-1} - k x_+^{-k-1} \\
&= \frac{(-1)^k \delta^{(k)}}{k!} - k x_+^{-k-1},
\end{aligned}$$

see also [15, Equ. (II, 2; 28), p. 42], [7, p. 151, Remark], [6, Equ. (3.2.2)″, p. 69], , [11, p. 50].

By differentiation with respect to λ, we obtain the distribution-valued function $\lambda \mapsto \partial_\lambda(x_+^\lambda) = x_+^\lambda \log x$, which is meromorphic in λ with double poles on $-\mathbf{N}$. As above we define $x_+^{-k}\log x := \mathrm{Pf}_{\lambda=-k}\, x_+^\lambda \log x$ for $k \in -\mathbf{N}$ and similarly for the higher derivatives with respect to λ. Hence the Laurent series of x_+^λ about the pole $\lambda = -k$, $k \in \mathbf{N}$, is given by

$$x_+^\lambda = \frac{(-1)^{k-1}\delta^{(k-1)}}{(k-1)!(\lambda+k)} + \sum_{j=0}^{\infty} \frac{x_+^{-k}\log^j x}{j!}(\lambda+k)^j, \quad 0 < |\lambda+k| < 1. \tag{2.1}$$

(In fact, $\mathrm{Pf}_{\lambda=-k}\,\partial_\lambda^j x_+^\lambda = \mathrm{Pf}_{\lambda=-k}\, x_+^\lambda \log^j x = x_+^{-k}\log^j x$ for $j \in \mathbf{N}_0$.)

Now we are prepared for giving a distributional definition of the incomplete beta function.

Definition 2.1. *For $\lambda, \mu \in \mathbf{C}$, we call $S_{\lambda,\mu} = x_+^{\lambda-1} \cdot (1-x)_+^{\mu-1} \in \mathcal{E}'(\mathbf{R})$ the M. Riesz kernels of the incomplete beta function and $B_{\lambda,\mu} = Y * S_{\lambda,\mu} \in \mathcal{D}'(\mathbf{R})$ the incomplete (Eulerian) beta function.*

Note that the multiplication of the two distributional factors $x_+^{\lambda-1}$ and $(1-x)_+^{\mu-1}$ of $S_{\lambda,\mu}$ is well-defined since their respective singular supports $\{0\}$ and $\{1\}$ are disjoint, see [6, Thm. 8.2.10, p. 267]. We also observe that $B_{\lambda,\mu}$ is uniquely determined by the two conditions

$$(i)\ B'_{\lambda,\mu} = S_{\lambda,\mu} \qquad \text{and} \qquad (ii)\ \operatorname{supp} B_{\lambda,\mu} \subset [0,\infty).$$

According to the above, the function $(\lambda,\mu) \mapsto S_{\lambda,\mu}$ is analytic for $\lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$. Therefore the same holds true for $B_{\lambda,\mu}$ and its derivatives $(\partial_1 B)_{\lambda,\mu} = \partial B_{\lambda,\mu}/\partial\lambda$ and $(\partial_2 B)_{\lambda,\mu} = \partial B_{\lambda,\mu}/\partial\mu$. As before, we abbreviate

$$(\partial_1 B)_{-k,\mu} := \mathop{\mathrm{Pf}}_{\lambda=-k} (\partial_1 B)_{\lambda,\mu}$$

and

$$(\partial_1 B)_{-k,-l} := \mathop{\mathrm{Pf}}_{\lambda=-k} \mathop{\mathrm{Pf}}_{\mu=-l} (\partial_1 B)_{\lambda,\mu} \ \text{ if } k, l \in \mathbf{N}_0, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0),$$

and similarly for $\partial_2 B$. As related in Section 1, we aim at calculating explicitly $B_{k,l}, (\partial_1 B)_{k,l}, (\partial_2 B)_{k,l}$ for the singular values, i.e., if $k, l \in \mathbf{Z}$ and $[k \in -\mathbf{N}_0$ or $l \in -\mathbf{N}_0]$.

3. Algebraic reduction formulas

The trivial identity

$$S_{\lambda,\mu} = 1 \cdot S_{\lambda,\mu} = (x + 1 - x) \cdot S_{\lambda,\mu} = S_{\lambda+1,\mu} + S_{\lambda,\mu+1}$$

leads to representations of $S_{k,l}$, $k, l \in \mathbf{Z}$, by $S_{j,1}$ and $S_{1,j}$, $j \in \mathbf{Z}$. By convolution with Y and by differentiation with respect to λ and μ, we obtain similar representation formulas for $B_{k,l}, (\partial_1 B)_{k,l}$ and $(\partial_2 B)_{k,l}$, respectively.

Lemma 3.1. *Let $\lambda, \mu \in \mathbf{C}$ and $k, l \in \mathbf{N}_0$. Then the following holds:*

$$S_{\lambda,\mu+l} = \sum_{j=0}^{l} \binom{l}{j} (-1)^j S_{\lambda+j,\mu}; \tag{3.2}$$

$$S_{\lambda-k,\mu-l} = \sum_{j=0}^{k} \binom{k+l-j}{l} S_{\lambda-j,\mu+1} + \sum_{j=0}^{l} \binom{k+l-j}{k} S_{\lambda+1,\mu-j} \tag{3.3}$$

and for $k < l$ we have

$$S_{\lambda-k,\mu+l} = \sum_{j=0}^{k} \binom{l-1}{j} (-1)^j S_{\lambda-k+j,\mu+1} + (-1)^{k+1} \sum_{j=1}^{l-k-1} \binom{l-j-1}{k} S_{\lambda+1,\mu+j}. \tag{3.4}$$

The corresponding formulas hold likewise if S is replaced throughout by $B = Y * S$, by $\partial_1 B$, or by $\partial_2 B$, respectively.

Proof. Equation (3.2) follows directly from the binomial formula:

$$\begin{aligned} S_{\lambda,\mu+l} = x_+^{\lambda-1}(1-x)_+^{\mu+l-1} &= S_{\lambda,\mu}\cdot(1-x)^l \\ &= S_{\lambda,\mu}\cdot\sum_{j=0}^{l}\binom{l}{j}(-1)^j x^j \\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^j S_{\lambda+j,\mu}. \end{aligned}$$

Formula (3.3) follows similarly by using the polynomial identity

$$1 = \sum_{j=0}^{k}\binom{k+l-j}{l}x^{k-j}(1-x)^{l+1} + \sum_{j=0}^{l}\binom{k+l-j}{k}x^{k+1}(1-x)^{l-j}. \tag{3.5}$$

For completeness, let us indicate shortly how the identity (3.5) is derived from a Mittag-Leffler expansion. In fact, in the representation

$$z^{-k-1}(1-z)^{-l-1} = \sum_{j=0}^{k} c_j z^{-j-1} + \sum_{j=0}^{l} d_j(1-z)^{-j-1}, \quad z\in\mathbf{C}\setminus\{0,1\},$$

the coefficients c_j can be determined from the Laurent expansion

$$z^{-k-1}(1-z)^{-l-1} = \sum_{n=0}^{\infty}\binom{-l-1}{n}(-1)^n z^{n-k-1}, \quad 0<|z|<1,$$

i.e.,

$$n = k-j \quad \text{and} \quad c_j = \binom{-l-1}{k-j}(-1)^{k-j} = \binom{k+l-j}{l}, \quad j = 0,\dots,k,$$

and similarly for d_j, $j = 0,\dots,l$.

Equation (3.4) follows in the same way by using the polynomial identity

$$(1-x)^{l-1} = \sum_{j=0}^{k}\binom{l-1}{j}(-1)^j x^j + (-1)^{k+1}\sum_{j=1}^{l-k-1}\binom{l-j-1}{k}x^{k+1}(1-x)^{j-1}.$$

This can be shown by first replacing x by $1-x$ and then employing the Mittag-Leffler expansion of $z^{l-1}(1-z)^{-k-1}$ with respect to the poles 0 and ∞. □

Remark 3.1. *Let us illustrate how the formulas (3.2), (3.3) and (3.4) are applied in order to reduce the singular values $B_{k,l}$ to $B_{j,1}$ and $B_{1,j}$, $j,k,l\in\mathbf{Z}$. E.g., setting $\lambda=\mu=k=l=0$ in formula (3.3) yields the equation $B_{0,0} = B_{0,1} + B_{1,0}$. Instead, if $l\in\mathbf{N}$ and if we set $\lambda = 0$, $\mu = 1$ and replace l by $l-1$, then formula (3.2) implies*

$$B_{0,l} = \sum_{j=0}^{l-1}\binom{l-1}{j}(-1)^j B_{j,1}, \quad l\in\mathbf{N}. \tag{3.6}$$

Note that formula (3.4) leads to a different representation by setting $\lambda = k = \mu = 0$:

$$B_{0,l} = B_{0,1} - \sum_{j=1}^{l-1} B_{1,j}, \quad l\in\mathbf{N}. \tag{3.7}$$

The formulas (3.6) and (3.7) coincide in the cases $l = 1$ and $l = 2$, but yield different representations for $l \geq 3$. E.g.,

$$B_{0,3} = B_{0,1} - 2B_{1,1} + B_{2,1} = B_{0,1} - B_{1,1} - B_{1,2}.$$

(The last equation amounts to $B_{2,1} = B_{1,1} - B_{1,2}$.)

Let us finally investigate how $B_{\lambda,\mu}$ and $B_{\mu,\lambda}$ are connected. For this we extend the definition of the *complete beta function* or, as it is also called, the *Eulerian integral of the first kind*

$$B(\lambda, \mu) = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\,\mathrm{d}x, \quad \lambda, \mu \in \mathbf{C},\ \mathrm{Re}\,\lambda > 0,\ \mathrm{Re}\,\mu > 0,$$

first, as usual, to $[\mathbf{C} \setminus (-\mathbf{N}_0)]^2$ by analytic continuation, i.e.,

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}, \quad \lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0),$$

and then to the singular values in $-\mathbf{N}_0$ by taking the finite part with respect to λ and μ. This implies that $B(\lambda, \mu) = \langle 1, S_{\lambda,\mu}\rangle$ and $B_{\lambda,\mu}(x) = B(\lambda, \mu)$ hold for $x > 1$ and for each $(\lambda, \mu) \in \mathbf{C}^2$.

Lemma 3.2. *For $\lambda, \mu \in \mathbf{C}$, we have $B_{\mu,\lambda}(x) = B(\lambda, \mu) - B_{\lambda,\mu}(1-x)$.*

Proof. If $f, g \in \mathcal{D}(\mathbf{R})$, then

$$\begin{aligned} f(-x) * g(1-x) &= \int f(-t)g(1-(x-t))\,\mathrm{d}t \\ &= \int f(s)g(1-x-s)\,\mathrm{d}s \\ &= (f * g)(1-x) \end{aligned}$$

and this formula holds by density whenever two distributions are convolvable. Hence

$$\begin{aligned} B_{\mu,\lambda} = Y * S_{\mu,\lambda} &= (1 - Y(-x)) * S_{\mu,\lambda} \\ &= \langle 1, S_{\mu,\lambda}\rangle - Y(-x) * S_{\lambda,\mu}(1-x) \\ &= B(\mu, \lambda) - (Y * S_{\lambda,\mu})(1-x) \\ &= B(\lambda, \mu) - B_{\lambda,\mu}(1-x). \end{aligned}$$

□

Let us yet give formulas for the finite parts of the complete beta function $B(\lambda, \mu)$ at the singular points.

Lemma 3.3. *For $k, l \in \mathbf{N}_0$ and $\mu \in \mathbf{C} \setminus \mathbf{Z}$, we have*

$$B(-k, \mu) = (-1)^k \binom{\mu-1}{k} \big[\psi(k+1) - \psi(\mu-k)\big]; \tag{3.8}$$

$$B(-k, l) = \begin{cases} (-1)^k \dbinom{l-1}{k} \Big[\displaystyle\sum_{j=1}^{k} \frac{1}{j} - \sum_{j=1}^{l-k-1} \frac{1}{j}\Big] & : l > k, \\ \dfrac{(-1)^l}{l} \cdot \dbinom{k}{l}^{-1} & : 1 \leq l \leq k; \end{cases} \tag{3.9}$$

$$B(-k, -l) = -\binom{k+l}{k} \Big[\sum_{j=k+1}^{k+l} \frac{1}{j} + \sum_{j=l+1}^{k+l} \frac{1}{j}\Big]. \tag{3.10}$$

Proof. We first calculate

$$\operatorname*{Res}_{\lambda=-k}\Gamma(\lambda)=\operatorname*{Res}_{\lambda=-k}\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k)}=\frac{(-1)^k}{k!}, \tag{3.11}$$

see [8, Section 13.1.4, p. 156], and

$$\begin{aligned}\operatorname*{Pf}_{\lambda=-k}\Gamma(\lambda)&=\operatorname*{Pf}_{\lambda=-k}\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k)}\\&=\partial_\lambda\Big(\frac{\Gamma(\lambda+k+1)}{\lambda(\lambda+1)\cdots(\lambda+k-1)}\Big)\Big|_{\lambda=-k}\\&=\frac{(-1)^k}{k!}\Big(\psi(1)+\sum_{j=1}^k\frac{1}{j}\Big)\\&=\frac{(-1)^k\psi(k+1)}{k!},\end{aligned}\tag{3.12}$$

see [10, p. 65]. This furnishes

$$\begin{aligned}B(-k,\mu)&=\operatorname*{Pf}_{\lambda=-k}\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}\\&=\operatorname*{Pf}_{\lambda=-k}\Gamma(\lambda)\cdot\frac{\Gamma(\mu)}{\Gamma(\mu-k)}+\operatorname*{Res}_{\lambda=-k}\Gamma(\lambda)\cdot\partial_\lambda\Big(\frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)}\Big)\Big|_{\lambda=-k}\\&=(-1)^k\binom{\mu-1}{k}\big[\psi(k+1)-\psi(\mu-k)\big]\end{aligned}$$

and hence formula (3.8).

If $l>k$ and if we set $\mu=l$ in formula (3.8), then we immediately obtain the first equation in (3.9) due to $\psi(n+1)=\psi(1)+\sum_{j=1}^n j^{-1}$ for $n\in\mathbf{N}_0$, see [4, Equ. 8.365.3]. On the other hand, if $1\le l\le k$, then

$$\psi(\mu-k)=\psi(\mu-l+1)-\sum_{j=l}^k\frac{1}{\mu-j},$$

see [4, Equ. 8.365.3], and this implies

$$\begin{aligned}B(-k,l)&=\lim_{\mu\to l}(-1)^k\binom{\mu-1}{k}\big[\psi(k+1)-\psi(\mu-k)\big]\\&=(-1)^l\frac{(l-1)!(k-l)!}{k!}\\&=\frac{(-1)^l}{l}\binom{k}{l}^{-1},\end{aligned}$$

i.e., the second equation in formula (3.9).

Finally, we obtain

$$\begin{aligned}B(-k,-l) &= (-1)^k \operatorname*{Pf}_{\mu=-l} \binom{\mu-1}{k}\Big[\psi(k+1)-\psi(\mu+l+1)+\sum_{j=0}^{k+l}\frac{1}{\mu-k+j}\Big]\\ &= (-1)^k\binom{-l-1}{k}\Big[\psi(k+1)-\psi(1)+\sum_{j=0}^{k+l-1}\frac{1}{-k-l+j}\Big]\\ &\quad+(-1)^k\partial_\mu\binom{\mu-1}{k}\Big|_{\mu=-l}\\ &= -\binom{k+l}{k}\Big[\sum_{j=k+1}^{k+l}\frac{1}{j}+\sum_{j=l+1}^{k+l}\frac{1}{j}\Big].\end{aligned}$$

□

4. The singular values of the incomplete Beta function

As explained in Section 3, we can reduce the general case of calculating $B_{k,l}$, $k,l\in\mathbf{Z}$, to the particular cases of $B_{j,1}$ and $B_{1,j}$, $j\in\mathbf{Z}$.

Proposition 4.1. *For $\lambda,\mu\in\mathbf{C}\setminus(-\mathbf{N}_0)$ and $j\in\mathbf{N}$, the following holds:*

(4.13)
$$B_{\lambda,1}=\frac{1}{\lambda}\big[Y(1-x)x_+^\lambda+Y(x-1)\big],\quad B_{1,\mu}=\frac{Y(x)}{\mu}\big[1-(1-x)_+^\mu\big];$$

(4.14)
$$B_{0,1}=Y(x)Y(1-x)\log x,\quad B_{1,0}=-Y(x)Y(1-x)\log(1-x);$$

(4.15)
$$B_{-j,1}=-\frac{1}{j}\big[Y(1-x)x_+^{-j}+Y(x-1)\big]+\frac{(-1)^j\delta^{(j-1)}}{j\cdot j!};$$

(4.16)
$$B_{1,-j}=\frac{Y(x)}{j}\big[(1-x)_+^{-j}-1\big]+\frac{\delta_1^{(j-1)}}{j\cdot j!}.$$

Proof. For $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda>0$, we have

$$B_{\lambda,1}(x)=Y(x)\int_0^x Y(1-t)t^{\lambda-1}\,\mathrm{d}t=\frac{1}{\lambda}\big[Y(1-x)x_+^\lambda+Y(x-1)\big].$$

By analytic continuation, the last expression represents $B_{\lambda,1}$ for all $\lambda\in\mathbf{C}\setminus(-\mathbf{N}_0)$.

For the remaining cases, we use the following formula, which is familiar in the context of complex analysis:

(4.17)
$$\operatorname*{Pf}_{\lambda=\lambda_0}(f_\lambda\cdot T_\lambda)=\operatorname*{Res}_{\lambda=\lambda_0}f_\lambda\cdot\operatorname*{Pf}_{\lambda=\lambda_0}\partial_\lambda T_\lambda+\operatorname*{Pf}_{\lambda=\lambda_0}f_\lambda\cdot\operatorname*{Pf}_{\lambda=\lambda_0}T_\lambda+\operatorname*{Pf}_{\lambda=\lambda_0}\partial_\lambda f_\lambda\cdot\operatorname*{Res}_{\lambda=\lambda_0}T_\lambda.$$

Here f_λ is an analytic $\mathcal{C}^\infty(\mathbf{R})$-valued function for $0<|\lambda-\lambda_0|<\epsilon$ and T_λ is an analytic $\mathcal{D}'(\mathbf{R})$-valued function for $0<|\lambda-\lambda_0|<\epsilon$, $\epsilon>0$, and both f_λ and T_λ have at most a simple pole in λ_0, see [10, Prop. 1.6.3, p. 28].

Hence

$$\begin{aligned}B_{0,1}&=\operatorname*{Pf}_{\lambda=0}\frac{1}{\lambda}\big[Y(1-x)x_+^\lambda+Y(x-1)\big]\\ &=\frac{\partial}{\partial\lambda}\big[Y(1-x)x_+^\lambda+Y(x-1)\big]\big|_{\lambda=0}\\ &=Y(x)Y(1-x)\log x\end{aligned}$$

and

$$B_{-j,1} = -\frac{1}{j} \operatorname*{Pf}_{\lambda=-j} \big[Y(1-x)x_+^\lambda + Y(x-1)\big] - \frac{1}{j^2} Y(1-x) \operatorname*{Res}_{\lambda=-j} x_+^\lambda$$
$$= -\frac{1}{j}\big[Y(1-x)x_+^{-j} + Y(x-1)\big] + \frac{(-1)^j \delta^{(j-1)}}{j \cdot j!}.$$

The formulas for $B_{1,\mu}, B_{1,0}$ and $B_{1,-j}$ then follow from Lemma 3.2. □

Example 4.1. *Let us calculate here $B_{0,n}$ for $n \in \mathbf{Z}$. If $n = l \in \mathbf{N}$, then we use formula (3.7) and obtain from Proposition 4.1 that*

$$B_{0,l} = B_{0,1} - \sum_{j=1}^{l-1} B_{1,j} = Y(x)Y(1-x)\log x - \sum_{j=1}^{l-1} \frac{Y(x)}{j}\big[1 - (1-x)_+^j\big].$$

If $n = -l \in -\mathbf{N}_0$, we set $\lambda = k = \mu = 0$ in formula (3.3) and conclude from Equations (4.14) and (4.16) in Proposition 4.1 that

$$B_{0,-l} = B_{0,1} + \sum_{j=0}^{l} B_{1,-j}$$
$$(4.18)\qquad = Y(x)Y(1-x)\log\Big(\frac{x}{1-x}\Big) + \sum_{j=1}^{l}\Big\{\frac{Y(x)}{j}\big[(1-x)_+^{-j} - 1\big] + \frac{\delta_1^{(j-1)}}{j \cdot j!}\Big\}, \qquad l \in \mathbf{N}_0.$$

In the open interval $(0,1)$, Equation (4.18) coincides with the expression given in Thm. 2.1 in [3, p. 5]. *Note that the calculation in this paper is based on van der Corput's neutrix method, which does not produce a distribution but rather represents $B_{0,-l}$ as a function outside its singular support. Similarly, formulas (1), (2), (3) in* [3, pp. 4, 5], *also follow from Lemma 3.1 and Proposition 4.1 or from the above by Lemma 3.2.*

More generally, formula (3.3) yields a representation of $B_{-k,-l}$, $k, l \in \mathbf{N}_0$, which, on the basis of van der Corput's method, is considered in [12, p. 990].

5. On the singular values of the partial derivatives of the incomplete beta function

As indicated above, we denote $\partial B_{\lambda,\mu}/\partial\lambda$ by $\partial_1 B$ and similarly for $\partial_2 B$. Motivated by the calculations in [3], let us derive formulas for $(\partial_1 B)_{1,j}$ and $(\partial_1 B)_{j,1}$, $j \in \mathbf{Z}$. Lemma 3.1 then immediately yields representations of $\partial_1 B$ at the singular values $(k,l) \in \mathbf{Z}^2$, $k \le 0$ or $l \le 0$. Furthermore, we conclude from Lemma 3.2 that

$$(5.19)\qquad (\partial_2 B)_{\lambda,\mu} = \frac{\partial B_{\lambda,\mu}}{\partial\mu} = \frac{\partial B(\lambda,\mu)}{\partial\mu} - \frac{\partial B_{\mu,\lambda}(1-x)}{\partial\mu} = \frac{\partial B(\lambda,\mu)}{\partial\mu} - (\partial_1 B)_{\mu,\lambda}(1-x),$$

and hence the derivative $\partial_2 B$ can be expressed by $\partial_1 B$.

Proposition 5.2. *For $\lambda, \mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$ and $k, l \in \mathbf{N}$, the following holds:*

$$(5.20)\qquad (\partial_1 B)_{\lambda,1} = \lambda^{-1} Y(1-x)x_+^\lambda \log x - \lambda^{-2}\big[Y(1-x)x_+^\lambda + Y(x-1)\big];$$

$$(5.21)\qquad (\partial_1 B)_{0,1} = \tfrac{1}{2} Y(x)Y(1-x)\log^2 x;$$

(5.22) $$(\partial_1 B)_{-k,1} = -\tfrac{Y(1-x)}{k} x_+^{-k} \log x - \tfrac{x_+^{-k} Y(1-x) + Y(x-1)}{k^2} + \tfrac{(-1)^k \delta^{(k-1)}}{k^2 \cdot k!};$$

(5.23) $$(\partial_1 B)_{1,\mu} = -\mu^{-1} Y(x) \log x \cdot (1-x)_+^{\mu} + \mu^{-1} B_{0,\mu+1};$$

(5.24) $$\begin{aligned}(\partial_1 B)_{1,0} &= -Y(x)Y(1-x)\big[\log x \log(1-x) + \mathcal{L}_2(x)\big] - Y(x-1)\frac{\pi^2}{6} \\ &= Y(x)\Big[Y(1-x)\mathcal{L}_2(1-x) - \frac{\pi^2}{6}\Big].\end{aligned}$$

(5.25) $$\begin{aligned} l(\partial_1 B)_{1,-l} &= Y(x)\log x \cdot (1-x)_+^{-l} - Y(x)Y(1-x)\log\Big(\frac{x}{1-x}\Big) \\ &\quad - \frac{1}{l}Y(x-1) - \sum_{j=1}^{l-1} \frac{Y(x)}{j}\Big\{\big[(1-x)_+^{-j} - 1\big] + \frac{l\,\delta_1^{(j-1)}}{(l-j)\cdot j!}\Big\}.\end{aligned}$$

Proof. Formula (5.20) follows immediately from the first equation in formula (4.13) by differentiation with respect to λ.

By taking the finite part at $\lambda = 0$, we infer

$$\begin{aligned}(\partial_1 B)_{0,1} &= \operatorname*{Pf}_{\lambda=0} \frac{1}{\lambda} Y(1-x) x_+^{\lambda} \log x - \operatorname*{Pf}_{\lambda=0} \frac{1}{\lambda^2} Y(1-x) x_+^{\lambda} \\ &= \frac{\partial}{\partial\lambda}\big[Y(1-x)x_+^{\lambda}\log x\big]\big|_{\lambda=0} - \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\big[Y(1-x)x_+^{\lambda}\big]\big|_{\lambda=0} \\ &= \frac{1}{2} Y(x)Y(1-x)\log^2 x\end{aligned}$$

and hence we obtain formula (5.21).

In order to calculate the finite part of $(\partial_1 B)_{\lambda,1}$ at $\lambda = -k \in -\mathbf{N}$, let us first derive the Laurent series of $x_+^{\lambda} \log x$ about $\lambda = -k$ from that of x_+^{λ}, i.e. formula (2.1), by differentiation with respect to λ :

$$x_+^{\lambda}\log x = \frac{(-1)^k \delta^{(k-1)}}{(k-1)!(\lambda+k)^2} + \sum_{j=0}^{\infty} \frac{x_+^{-k}\log^{j+1} x}{j!}(\lambda+k)^j, \quad 0 < |\lambda + k| < 1.$$

Hence $\operatorname{Res}_{\lambda=-k} x_+^{\lambda}\log x = 0$ and we conclude that

$$\begin{aligned}(\partial_1 B)_{-k,1} &= \operatorname*{Pf}_{\lambda=-k}\Big\{\frac{1}{\lambda}Y(1-x)x_+^{\lambda}\log x - \frac{1}{\lambda^2}\big[Y(1-x)x_+^{\lambda} + Y(x-1)\big]\Big\} \\ &= -\frac{1}{k}Y(1-x)x_+^{-k}\log x - \frac{1}{k^2}\big[Y(1-x)x_+^{-k} + Y(x-1)\big] \\ &\quad + \frac{1}{2}\frac{\partial^2\lambda^{-1}}{\partial\lambda^2}\Big|_{\lambda=-k} \cdot \frac{(-1)^k\delta^{(k-1)}}{(k-1)!} - \frac{\partial\lambda^{-2}}{\partial\lambda}\Big|_{\lambda=-k} \cdot \operatorname*{Res}_{\lambda=-k} Y(1-x)x_+^{\lambda} \\ &= -\frac{1}{k}Y(1-x)x_+^{-k}\log x - \frac{1}{k^2}\big[Y(1-x)x_+^{-k} + Y(x-1)\big] + \frac{(-1)^k\delta^{(k-1)}}{k^2\cdot k!}.\end{aligned}$$

This furnishes formula (5.22).

Since $\mu \in \mathbf{C} \setminus (-\mathbf{N}_0)$, we have

$$-\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}(1-x)_+^{\mu} = (1-x)_+^{\mu-1}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[-\frac{1}{\mu}Y(x)\log x\cdot(1-x)_+^{\mu}\Big] = Y(x)\log x\cdot(1-x)_+^{\mu-1} - \frac{1}{\mu}x_+^{-1}(1-x)_+^{\mu}.$$

Thus $(\partial_1 S)_{1,\mu} = Y(x)\log x \cdot (1-x)^{\mu-1}$ is the derivative of the distribution $-\mu^{-1}Y(x)\log x \cdot (1-x)_+^\mu + \mu^{-1}B_{0,\mu+1}$ and this distribution has its support in the positive half-axis $[0,\infty)$ and coincides therefore with $(\partial_1 B)_{1,\mu}$. This implies formula (5.23).

Evaluating the finite part of $(\partial_1 B)_{1,\mu}$ at $\mu = 0$ in formula (5.23) yields

$$\begin{aligned}(\partial_1 B)_{1,0} &= \operatorname*{Pf}_{\mu=0}(\partial_1 B)_{1,\mu} = -\frac{\partial}{\partial\mu}\,Y(x)\log x\cdot(1-x)_+^\mu\big|_{\mu=0} + \frac{\partial B_{0,\mu+1}}{\partial\mu}\bigg|_{\mu=0}\\ &= -Y(x)Y(1-x)\log x\log(1-x) + Y(x)\int_0^x Y(1-t)\log(1-t)\frac{\mathrm{d}t}{t}\\ &= -Y(x)Y(1-x)\big[\log x\log(1-x)+\mathcal{L}_2(x)\big] - Y(x-1)\mathcal{L}_2(1),\end{aligned}$$

see [5, Equ. 323.3a]. Due to $\mathcal{L}_2(1) = \frac{\pi^2}{6}$, this gives the first equation in formula (5.24). On the other hand, a direct calculation yields the following:

$$\begin{aligned}(\partial_1 B)_{1,0} &= Y(x)\int_0^x Y(1-t)(1-t)^{-1}\log t\,\mathrm{d}t\\ &= Y(x)\int_{1-x}^1 Y(t)\log(1-t)\frac{\mathrm{d}t}{t}\\ &= Y(x)\big[Y(1-x)\mathcal{L}_2(1-x)-\mathcal{L}_2(1)\big].\end{aligned}$$

Of course, these two representations of $(\partial_1 B)_{1,0}$ must and do coincide as can be seen from [5, Equ. 323.3g].

Let us finally calculate $(\partial_1 B)_{1,-l}$ for $l\in\mathbf{N}$. From formula (5.23), we obtain

$$\begin{aligned}(\partial_1 B)_{1,-l} &= \operatorname*{Pf}_{\mu=-l}(\partial_1 B)_{1,\mu}\\ &= Y(x)l^{-1}\log x\cdot(1-x)_+^{-l} + Y(x)l^{-2}\log x\cdot\operatorname*{Res}_{\mu=-l}(1-x)_+^\mu - l^{-1}B_{0,1-l} - l^{-2}\operatorname*{Res}_{\mu=-l}B_{0,\mu+1}.\end{aligned}$$

Furthermore,

$$\operatorname*{Res}_{\mu=-l}(1-x)_+^\mu = \Big(\operatorname*{Res}_{\mu=-l}x_+^\mu\Big)(1-x) = \frac{(-1)^{l-1}\delta^{(l-1)}(1-x)}{(l-1)!} = \frac{\delta_1^{(l-1)}}{(l-1)!},$$

and, for a function f which is differentiable at 1 and $m\in\mathbf{N}_0$, we have

$$f\cdot\delta_1^{(m)} = \sum_{j=0}^m\binom{m}{j}(-1)^{m-j}f^{(m-j)}(1)\,\delta_1^{(j)}$$

and hence

$$(\log x)\cdot\operatorname*{Res}_{\mu=-l}(1-x)_+^\mu = -\sum_{j=0}^{l-2}\frac{\delta_1^{(j)}}{(l-j-1)\cdot j!}.$$

From formula (4.18), we infer that

$$B_{0,1-l} = Y(x)Y(1-x)\log\Big(\frac{x}{1-x}\Big) + \sum_{j=1}^{l-1}\Big\{\frac{Y(x)}{j}\big[(1-x)_+^{-j}-1\big] + \frac{\delta_1^{(j-1)}}{j\cdot j!}\Big\}.$$

In order to evaluate the residue $\operatorname{Res}_{\mu=-l}B_{0,\mu+1}$, we note that

$$\operatorname*{Res}_{\mu=-l}S_{0,\mu+1} = \operatorname*{Res}_{\mu=-l}x_+^{-1}(1-x)_+^\mu = x^{-1}\cdot\frac{\delta_1^{(l-1)}}{(l-1)!} = \sum_{j=0}^{l-1}\frac{\delta_1^{(j)}}{j!}$$

and thus

$$\operatorname*{Res}_{\mu=-l} B_{0,\mu+1} = Y * \operatorname*{Res}_{\mu=-l} S_{0,\mu+1} = Y(x-1) + \sum_{j=0}^{l-2} \frac{\delta_1^{(j)}}{(j+1)!}.$$

Collecting terms we arrive at formula (5.25). The proof is complete. □

Remark 5.2. *From formula (5.25) in Proposition 5.2, we conclude that*

$$(\partial_1 B)_{1,-l}(x) = -\frac{1}{l^2} + \frac{1}{l}\sum_{j=1}^{l-1}\frac{1}{j}, \quad l \in \mathbf{N},\ x > 1. \tag{5.26}$$

Let us check this equation by replacing $\log x$ *by its Taylor series about* 1*. If* $l \in \mathbf{N}$ *and* $x > 1$*, then*

$$\begin{aligned}(\partial_1 B)_{1,-l}(x) &= \langle 1, (\partial_1 S)_{1,-l}\rangle \\ &= \langle 1, Y(x)\log x \cdot (1-x)_+^{-l-1}\rangle \\ &= \langle 1, -\sum_{j=1}^{\infty} j^{-1} Y(x)(1-x)_+^{j-l-1}\rangle \\ &= -\langle 1, \sum_{j=1}^{\infty} j^{-1} S_{1,j-l}\rangle.\end{aligned} \tag{5.27}$$

(In fact, these series converge in $\mathcal{E}'(\mathbf{R})$*.) For Re* $\mu > 0$*, we have*

$$\langle 1, S_{1,\mu}\rangle = \langle 1, S_{\mu,1}\rangle = \int_0^1 x^{\mu-1}\,dx = \frac{1}{\mu}$$

and hence

$$\langle 1, S_{1,0}\rangle = 0 \textit{ and } \langle 1, S_{1,l}\rangle = l^{-1} \textit{ for } l \in \mathbf{Z}\setminus\{0\}$$

by analytic continuation and taking finite parts. Therefore Equation (5.27) implies

$$\begin{aligned}(\partial_1 B)_{1,-l}(x) &= -\sum_{j=1,\, j\neq l}^{\infty} \frac{1}{j(j-l)} \\ &= -\frac{1}{l}\sum_{j=1,\, j\neq l}^{\infty}\Big(\frac{1}{j-l} - \frac{1}{j}\Big) \\ &= -\frac{1}{l}\Big(\frac{1}{l} - \sum_{j=1}^{l-1}\frac{1}{j}\Big), \quad l \in \mathbf{N},\ x > 1.\end{aligned}$$

in accordance with the result in formula (5.26).

Remark 5.3. *In the open interval* $(0,1)$*, the representation of* $(\partial_1 B)_{1,-l}$ *in formula (5.25) coincides with* [3, Thm. 2.2, p. 6]*. Similarly, the formulas for* $(\partial_2 B)_{-k,1}$ *and for* $(\partial_2 B)_{-k,l}$*,* $k, l \in \mathbf{N}$*, in* [3, Thms. 2.3, 2.4, pp. 6, 7]*, follow from Equation (5.19), Lemma 3.1 and Proposition 5.2.*

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