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(U, V)-LUCAS POLYNOMIAL COEFFICIENT RELATIONS OF THE BI-UNIVALENT FUNCTION CLASS

Arzu AKGÜL[1] and Timilehin Gideon SHABA[2]

[1]Department of Mathematics, Faculty of Arts and Science, Kocaeli University, Kocaeli, TURKEY
[2]Department of Mathematics, University of Ilorin, P. M. B. 1515, Ilorin, NIGERIA

Abstract. In geometric function theory, Lucas polynomials and other special polynomials have recently gained importance. In this study, we develop a new family of bi-univalent functions. Also we examined coefficient inequalities and Fekete-Szegö problem for this new family via these polynomials.

1. Introduction

Let $\mathfrak{A}$ denote the family of all functions $\theta(\xi)$ that are analytic in the unit disc $\mathfrak{U} = \{\xi : \xi \in C, |\xi| < 1\}$ normalized by the conditions $\theta(0) = \theta'(0) - 1 = 0$. Such a function $\theta(\xi)$ takes the form

$$\theta(\xi) = \xi + \sum_{r=2}^{\infty} n_r \xi^r \quad (\xi \in \mathfrak{U}). \tag{1}$$

Assume that $\mathcal{S}$ be the subclass of $\mathfrak{A}$ compose of univalent functions.

As a subclass of $\mathfrak{A}$, the class of bi-univalent functions was first presented by Lewin [18]. He indicated that $|n_2| \le 1.15$. After that, a lot of studies have been made about coefficient estimates. See for example [4, 10, 11, 14, 15, 27, 30–41]. According to the Koebe 1/4 theorem (see [12]), the range of every function $\theta \in \mathcal{S}$ contains the disc $d_\omega = \{\omega : |\omega| < 0.25\}$, thus, for all $\theta \in \mathcal{S}$ with its inverse θ^{-1}, such that $\theta^{-1}(\theta(\xi)) = \xi \quad (\xi \in \mathfrak{U})$ and $\theta(\theta^{-1}(\omega)) = \omega, \quad (\omega : |\omega| < r_0(\theta); r_0(\theta) \ge 0.25)$ where $\theta^{-1}(\omega)$ is expressed as

$$\vartheta(\omega) = \omega - n_2\omega^2 + (2n_2^2 - n_3)\omega^3 - (5n_2^3 - 5n_2n_3 + n_4)\omega^4 + \cdots . \tag{2}$$

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akgulcagla@hotmail.com-Corresponding author; 0000-0001-7934-0339
shabatimilehin@gmail.com; 0000-0001-5881-9260.

Thus, a function $\theta \in \mathfrak{A}$ is said that bi-univalent in $\mathfrak{U}$, if both $\theta(\xi)$ and $\theta^{-1}(\omega)$ are univalent in $\mathfrak{U}$. Let we show the class of holomorphic and bi-univalent functions in $\mathfrak{U}$ by $\mathfrak{B}$.

It is known that some similar functions $\theta \in \mathcal{S}$ for instance the Koebe function $\kappa(\xi) = \xi/(1-\xi)^2$, its rotation function $\kappa_\varsigma(\xi) = \xi/(1-e^{i\varsigma}\xi)^2$, $\theta(\xi) = \xi - \xi^2/2$ and $\theta(\xi) = \xi/(1-\xi^2)$ are in $\mathfrak{B}$. Also some functions $\theta \in (\mathcal{S} \cap \mathfrak{B})$ contains $\theta(\xi) = \xi$, $\theta(\xi) = 1/2 \log[(1-\xi)/(1-\xi)]$, $\xi/(1-\xi)$.

For the functions $h, H \in \mathfrak{A}$, The function h is said to be subordinate to H or H is said to be superordinate to h, if there exists a function η, analytic in $\mathfrak{U}$, with $\eta(0) = 0$ and $|\eta(z)| < 1$ and such that $h(\xi) = H(\eta(\xi))$. In such a case we write $h \prec H$ or $h(\xi) \prec H(\xi)$. If h is one-to-one, this $h \prec H$ iff $h(0) = H(0)$ and $h(U) \subset H(U)$. Babalola [9] studied the class $\mathfrak{L}_\sigma(\varphi)$ of σ-pseudo- starlike functions of order $\varphi (0 \le \varphi < 1)$ which is own geometric conditions fulfill

$$\Re\left(\frac{\xi(\theta'(\xi))^\sigma}{\theta(\xi)}\right) > \varphi.$$

He discover that every pseudo-starlike functions are Bazilevič of type $\left(1-\frac{1}{\sigma}\right)$ order $\varphi^{\frac{1}{\sigma}}$ and univalent in $\mathfrak{U}$.

In recent years, theory and applications of Dickson, Fibonacci, Lucas, Chebyshev, Lucas-Lehmer polynomials in modern science have emerged as a very current subject. These polynomials are important in mathematics due to the fact that they can applicable to number theory, numerical analysis, combinatorics, and other fields. Nowadays, these polyinomials have been studied and different generalizations have been made by many authors: see [1–3,5–8].Also see [13,17,19–26,28,29,43–46].

We recall some important properties interested in which we use to construct our new class. Assume that polynomials with real coefficients are written by $U(x)$ and $V(x)$. By using the recurrence relation, the (U,V)-Lucas polynomials $\mathcal{L}_{U,V,t}(x)$ are explained [17] as

$$\mathcal{L}_{U,V,t}(x) = U(x)\mathcal{L}_{U,V,t-1}(x) + V(x)\mathcal{L}_{U,V,t-2}(x) \ (t \ge 2). \tag{3}$$

Also

$$\begin{aligned}
&\mathcal{L}_{U,V,0}(x) = 2,\\
&\mathcal{L}_{U,V,1}(x) = U(x),\\
&\mathcal{L}_{U,V,2}(x) = U^2(x) + 2V(x),\\
&\mathcal{L}_{U,V,3}(x) = U^3(x) + 3U(x)V(x).
\end{aligned} \tag{4}$$

The generating function of the (U,V)-Lucas polynomial sequence $\mathcal{L}_{U,V,t}(x)$ is expressed by [17]

$$\mathcal{K}_{\{\mathcal{L}_t(x)\}}(\xi) = \sum_{t=0}^{\infty} \mathcal{L}_{U,V,t}(x)\xi^t = \frac{2-U(x)\xi}{1-U(x)\xi - V(x)\xi^2}. \tag{5}$$

In the next section, using this polynomials as a tool, we define the family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, \sigma; x)$ as follow:

Definition 1. *For $\beta \geq 0$, $\sigma \geq 1$, $|\gamma| \leq 1$ but $\gamma \neq 1$, a function $\theta \in \mathfrak{B}$ is called in the family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, \sigma; x)$ if the following subordinations are satisfied:*

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^{\sigma}}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{6}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^{\sigma}}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{7}$$

Taking special values for β, γ and σ, the class $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, \sigma; x)$ reduces some exciting new families:

Remark 1. *For $\sigma = 1$, we get the new family $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$. If $\theta \in \mathfrak{B}$, is in $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$ then following condition fulfilles*

$$\frac{((1-\gamma)\xi)^{1-\beta}\theta'(\xi)}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{8}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}\vartheta'(w)}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{9}$$

Remark 2. *For $\beta = 0$, we obtain the new class*

$$\mathfrak{H}^{\mathfrak{B},0}(\gamma, \sigma; x) = \mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x).$$

If $\theta \in \mathfrak{B}$ is in $\mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x)$, then following condition fulfilles

$$\frac{\xi(1-\gamma)(\theta'(\xi))^{\sigma}}{\theta(\xi) - f(\gamma\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{10}$$

and

$$\frac{\omega(1-\gamma)(\vartheta'(w))^{\sigma}}{\vartheta(\omega) - \vartheta(\gamma\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{11}$$

Also,

(1) Choosing $\sigma = 1$ in the class $\mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x)$ we have new family $\mathfrak{H}^{\mathfrak{B}}(\gamma, 1; x) = \mathfrak{H}^{\mathfrak{B}}(\gamma; x)$. The class $\mathfrak{H}^{\mathfrak{B}}(\gamma; x)$ consists of the function $f \in \mathfrak{B}$ fulfilling

$$\frac{\xi(1-\gamma)\theta'(\xi)}{\theta(\xi) - \theta(\gamma\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{12}$$

and

$$\frac{\omega(1-\gamma)\vartheta'(w)}{\vartheta(\omega) - \vartheta(\gamma\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{13}$$

(2) Choosing $\gamma = 0$ in the class $\mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x)$ we have the class $\mathfrak{H}^{\mathfrak{B}}(0, \sigma; x) = \mathfrak{H}^{\mathfrak{B}}(\sigma; x) = \mathcal{L}_{\Sigma}(\mathfrak{U}; x)$. The class $\mathcal{L}_{\Sigma}(\mathfrak{U}; x)$ was studied by Murugusundaramoorthy and Yalçin [20]. This class consists of the function $\theta \in \mathfrak{B}$ satisfying

$$\frac{\xi(\theta'(\xi))^{\sigma}}{\theta(\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{14}$$

and

$$\frac{\omega(\vartheta'(w))^{\sigma}}{\vartheta(\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{15}$$

(3) Choosing $\sigma = 2$ in the class $\mathfrak{H}^{\mathfrak{B}}(\sigma; x)$ we have the class

$$\mathfrak{H}^{\mathfrak{B}}(2; x) = \mathfrak{H}^{\mathfrak{B}}(x).$$

The class consists of the function $f \in \mathfrak{B}$ satisfying

$$\theta'(\xi)\frac{\xi\theta'(\xi)}{\theta(\xi)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{16}$$

and

$$\vartheta'(\omega)\frac{\omega\vartheta'(w)}{\vartheta(\omega)} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{17}$$

Remark 3. *For $\beta = 1$, we have the new class $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$. If $\theta \in \mathfrak{B}$, is in $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$, then following condition fulfilles*

$$(\theta'(\xi))^{\sigma} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{18}$$

and

$$(\vartheta'(\omega))^{\sigma} \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{19}$$

Also,

(1) *Choosing $\sigma = 1$ in the class $\mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$ we have the class*

$$\mathfrak{H}^{\mathfrak{B},1}(1; x).$$

This class consists of the function $\theta \in \mathfrak{B}$ satisfying

$$\theta'(\xi) \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\xi) - 1 \tag{20}$$

and

$$\vartheta'(\omega) \prec \mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\omega) - 1. \tag{21}$$

2. Main Theorems for the class $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$

Theorem 1. *Let $\theta(\xi)\in\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$. Then*

$$|n_2|\leq\frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|\begin{array}{c}U^2(x)\left[(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2}+2\sigma(1+\gamma)+(1+\gamma+\gamma^2)\right]+\sigma(2\sigma+1)\right.\\ \left.-[2\sigma+(\beta-1)(1+\gamma)]^2\right]-2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2\end{array}\right|}} \tag{22}$$

$$|n_3|\leq\frac{U^2(x)}{[2\sigma+(\beta-1)(1+\gamma)]^2}+\frac{|U(x)|}{|3\sigma+(\beta-1)(1+\gamma+\gamma^2)|}, \tag{23}$$

where $\beta\geq 0$, $\sigma\geq 1$ and $|\gamma|\leq 1$ but $\gamma\neq 1$.

Proof. Let $\theta(\xi)\in\mathfrak{H}^{\mathfrak{B},\beta}(\gamma,\sigma;x)$. Then, according to the Definition 1, for some holomorphic functions Φ, Υ such that $\Upsilon(0)=\Phi(0)=0$, $|\Upsilon(\omega)|<1$, $|\Phi(\xi)|<1$, $(\xi,\omega\in\mathfrak{U})$, we can write

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^{\sigma}}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}}=\mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Phi(\xi))-1$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^{\sigma}}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}}=\mathcal{K}_{\{\mathcal{L}_{U,V,t}(x)\}}(\Upsilon(\omega))-1,$$

by equivalence

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^{\sigma}}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}}=-1+\mathcal{L}_{U,V,0}(x)+\mathcal{L}_{U,V,1}(x)\Phi(\xi)+\mathcal{L}_{U,V,2}(x)\Phi^2(\xi)+\cdots \tag{24}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^{\sigma}}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}}=-1+\mathcal{L}_{U,V,0}(x)+\mathcal{L}_{U,V,1}(x)\Upsilon(\omega)+\mathcal{L}_{U,V,2}(x)\Upsilon^2(\omega)+\cdots \tag{25}$$

From (24) and (25), yields

$$\frac{((1-\gamma)\xi)^{1-\beta}(\theta'(\xi))^{\sigma}}{(\theta(\xi)-\theta(\gamma\xi))^{1-\beta}}=1+\mathcal{L}_{U,V,1}(x)y_1\xi+\left[\mathcal{L}_{U,V,1}(x)y_2+\mathcal{L}_{U,V,2}(x)y_1^2\right]\xi^2+\cdots \tag{26}$$

and

$$\frac{((1-\gamma)\omega)^{1-\beta}(\vartheta'(w))^{\sigma}}{(\vartheta(\omega)-\vartheta(\gamma\omega))^{1-\beta}}=1+\mathcal{L}_{U,V,1}(x)\mu_1\omega+\left[\mathcal{L}_{U,V,1}(x)\mu_2+\mathcal{L}_{U,V,2}(x)\mu_1^2\right]\omega^2+\cdots \tag{27}$$

for $\xi, \omega \in \mathfrak{U}$, it known before that if

$$|\Phi(\xi)| = \left|\sum_{j=1}^{\infty} y_j \xi^j\right| < 1$$

and

$$|\Upsilon(\omega)| = \left|\sum_{j=1}^{\infty} \mu_j \omega^j\right| < 1,$$

thus

$$|y_j| < 1 \tag{28}$$

also

$$|\mu_j| < 1 \tag{29}$$

where $j \in \mathfrak{N} = \{1, 2, 3, \cdots\}$. If we compare corresponding coefficients in (26) and (27), then we have

$$[2\sigma + (\beta - 1)(1 + \gamma)]n_2 = \mathcal{L}_{U,V,1}(x)y_1, \tag{30}$$

$$[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_3+\left[\frac{(\beta-1)(\beta-2)}{2}(1+\gamma)^2+2\sigma(\sigma-1+(\beta-1)(1+\gamma))\right]n_2^2$$
$$= \mathcal{L}_{U,V,1}(x)y_2 + \mathcal{L}_{U,V,2}(x)y_1^2, \tag{31}$$

$$-[2\sigma + (\beta - 1)(1 + \gamma)]n_2 = \mathcal{L}_{U,V,1}(x)\mu_1, \tag{32}$$

$$\left[2(\beta-1)(1+\gamma+\gamma^2) + \frac{(\beta-1)(\beta-2)}{2}(1+\gamma)^2 + 2\sigma(\sigma+2+(\beta-1)(1+\gamma))\right]n_2^2$$
$$-[3\sigma + (\beta-1)(1+\gamma+\gamma^2)]n_3 = \mathcal{L}_{U,V,1}(x)\mu_2 + \mathcal{L}_{U,V,2}(x)\mu_1^2. \tag{33}$$

From (30) and (32)

$$y_1 = -\mu_1, \tag{34}$$

$$2[2\sigma + (\beta - 1)(1 + \gamma)]^2 n_2^2 = \mathcal{L}_{U,V,1}^2(x)(y_1^2 + \mu_1^2). \tag{35}$$

Summation of (31) and (33) gives

$$\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)\right]n_2^2$$
$$= \mathcal{L}_{U,V,1}(x)(y_2 + \mu_2) + \mathcal{L}_{U,V,2}(x)(y_1^2 + \mu_1^2). \tag{36}$$

Applying (35) in (36), yields

$$\left\{\mathcal{L}_{U,V,1}^2(x)\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)\right]\right.$$

$$\left. - 2\mathcal{L}_{U,V,2}(x)[2\sigma + (\beta-1)(1+\gamma)]^2 \right\} n_2^2 = \mathcal{L}_{U,V,1}^3(x)(y_2+\mu_2), \quad (37)$$

$$\Bigg[U^2(x)\Bigg[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + (1+\gamma+\gamma^2)\right] + 2\sigma(2\sigma+1)$$

$$- 2[2\sigma+(\beta-1)(1+\gamma)]^2\Bigg] - 4[2\sigma+(\beta-1)(1+\gamma)]^2 V(x)\Bigg] n_2^2 = \mathcal{L}_{U,V,1}^3(x)(y_2+\mu_2)$$

which gives desired result given by (1).

Hence, (31) minus (33) gives us

$$2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_3 + 2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]n_2^2 = \mathcal{L}_{U,V,1}(x)(y_2-\mu_2). \quad (38)$$

Then, by using (34) and (35) in (38), we get

$$n_3 = n_2^2 + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \quad (39)$$

$$n_3 = \frac{\mathcal{L}_{U,V,1}^2(x)(y_1^2+\mu_1^2)}{2[2\sigma+(\beta-1)(1+\gamma)]^2} + \frac{\mathcal{L}_{U,V,1}(x)(y_2-\mu_2)}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}. \quad (40)$$

Applying (4), we have

$$|n_3| \le \frac{U^2(x)}{[2\sigma+(\beta-1)(1+\gamma)]^2} + \frac{|U(x)|}{|3\sigma+(\beta-1)(1+\gamma+\gamma^2)|}.$$

Thus, the proof of our main theorem is completed. □

3. Corollaries

By specializing the parameters γ, β, σ, in Theorem 1, we get the following consequences.

Corollary 1. *Let $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$. Then*

$$|n_2| \le \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left| \begin{array}{c} U^2(x)\left[(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 3(1+\gamma) + \gamma^2\right] + 3 - [2+(\beta-1)(1+\gamma)]^2\right] \\ -2V(x)[2+(\beta-1)(1+\gamma)]^2 \end{array} \right|}} \quad (41)$$

$$|n_3| \le \frac{U^2(x)}{[2+(\beta-1)(1+\gamma)]^2} + \frac{|U(x)|}{|3+(\beta-1)(1+\gamma+\gamma^2)|}. \quad (42)$$

Corollary 2. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, \sigma; x) = \mathfrak{H}^{\mathfrak{B}}(\gamma, \sigma; x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)\left[(2\sigma-1)(\sigma-\gamma)-[2\sigma-(1+\gamma)]^2\right]-2V(x)[2\sigma-(1+\gamma)]^2\right|}}, \tag{43}$$

$$|n_3| \leq \frac{U^2(x)}{[2\sigma-(1+\gamma)]^2} + \frac{|U(x)|}{|3\sigma-(1+\gamma+\gamma^2)|}. \tag{44}$$

Corollary 3. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, 1; x) = \mathfrak{H}^{\mathfrak{B}}(\gamma, 1; x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\left|U^2(x)(\gamma-\gamma^2)-2V(x)(\gamma^2-2\gamma+1)\right|}}, \tag{45}$$

$$|n_3| \leq \frac{U^2(x)}{(1-\gamma)^2} + \frac{|U(x)|}{|2-\gamma(\gamma+1)|}. \tag{46}$$

Corollary 4. *Choosing* $\beta = 0$ *and* $\gamma = 0$ *in Theorem 1, that is if* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(\sigma; x)$, *the results which we obtain reduce to Theorem 2.1 in [20].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)(-2\sigma^2-1+3\sigma)-2V(x)(2\sigma-1)^2|}}, \tag{47}$$

$$|n_3| \leq \frac{U^2(x)}{[2\sigma-1]^2} + \frac{|U(x)|}{|3\sigma-1|}. \tag{48}$$

Corollary 5. *Choosing* $\beta = 0$, $\gamma = 0$ *and* $\sigma = 2$ *in Theorem 1,* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(2; x)$, *our corollary coincides with the corollary 2.3 of Theorem 2.1 in [20].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{3|U^2(x)+6V(x)|}}, \tag{49}$$

$$|n_3| \leq \frac{U^2(x)}{9} + \frac{|U(x)|}{5}. \tag{50}$$

Corollary 6. *Choosing* $\beta = 0$, $\gamma = 0$ *and* $\sigma = 1$ *in Theorem 1,* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B}}(1; x)$, *our corollary coincides with the corollary 2.2 of Theorem 2.1 in [44].*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x)|}} = \sqrt{|U(x)|}, \tag{51}$$

$$|n_3| \leq U^2(x) + \frac{|U(x)|}{2}. \tag{52}$$

Corollary 7. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{\sigma|U^2(x)(1-2\sigma) - 8\sigma V(x)|}}, \tag{53}$$

$$|n_3| \leq \frac{U^2(x)}{4\sigma^2} + \frac{|U(x)|}{3\sigma}. \tag{54}$$

Corollary 8. *Let* $\theta(\xi) \in \mathfrak{H}^{\mathfrak{B},1}(1; x)$. *Then*

$$|n_2| \leq \frac{|U(x)|\sqrt{|U(x)|}}{\sqrt{|U^2(x) + 8V(x)|}}, \tag{55}$$

$$|n_3| \leq \frac{U^2(x)}{4} + \frac{|U(x)|}{3}. \tag{56}$$

Theorem 2. *For* $\beta \geq 0$, $\sigma \geq 1$, $|\gamma| \leq 1$ *but* $\gamma \neq 1$, *let* $\theta \in \mathfrak{A}$ *be in the class* $\mathfrak{H}^{\mathfrak{B},\beta}(\gamma, \sigma; x)$. *Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi - 1| \leqq K \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta - 2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K. \end{cases}$$

Where

$$K = \frac{1}{|3\sigma + (\beta-1)(1+\gamma+\gamma^2)|}\left|\Delta - 2[2\sigma + (\beta-1)(1+\gamma)]^2 \frac{V(x)}{U^2(x)}\right|$$

$$\Delta = (\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma(1+\gamma) + 1 + \gamma + \gamma^2\right] + \sigma(2\sigma+1) - [2\sigma+(\beta-1)(1+\gamma)]^2.$$

Proof. From (37) and (38), we get

$$n_3 - \chi n_2^2 = \mathcal{L}_{U,V,1}(x)\left[\left(\zeta(\chi;x) + \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}\right) y_2 + \left(\zeta(\chi;x) - \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]}\right)\mu_2\right]$$

where

$$\zeta(\chi;x) = \frac{\mathcal{L}^2_{U,V,1}(x)(1-\chi)}{\mathcal{L}^2_{U,V,1}(x)\left[2(\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2} + 2\sigma\right.\right.}$$

$(1+\gamma) + (1+\gamma+\gamma^2) + 2\sigma(2\sigma+1) - 2\mathcal{L}_{U,V,2}(x)[2\sigma + (\beta-1)(1+\gamma)]^2$. Thus, according to (4), we have

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & 0 \leqq |\zeta(\chi;x)| \leqq \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \\ 2|\zeta(\chi;x)| \cdot |U(x)|, & |\zeta(\chi;x)| \geqq \frac{1}{2[3\sigma+(\beta-1)(1+\gamma+\gamma^2)]} \end{cases}$$

hence, after some calculations, gives

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta-2V(x)[2\sigma+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K. \end{cases}$$

□

By choosing special values for the parameters γ, β, σ, in Theorem 2, we get the following corollaries:

Corollary 9. *For $\sigma = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},\beta}(\gamma, 1; x)$. Then*

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3+(\beta-1)(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K_1 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_1-2V(x)[2+(\beta-1)(1+\gamma)]^2|}, & |\chi-1| \geqq K_1. \end{cases}$$

Where

$$K_1 = \frac{1}{|3+(\beta-1)(1+\gamma+\gamma^2)|}\left|\Delta_1 - 2[2+(\beta-1)(1+\gamma)]^2\frac{V(x)}{U^2(x)}\right|$$

$$\Delta_1 = (\beta-1)\left[\frac{(\beta-2)(1+\gamma)^2}{2}+\gamma^2+3\gamma+3\right]+3-[2+(\beta-1)(1+\gamma)]^2.$$

Corollary 10. *For $\beta = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, \sigma; x)$. Then*

$$\left|n_3-\chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma-(1+\gamma+\gamma^2)}, & |\chi-1| \leqq K_2 \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_2-2V(x)[2\sigma-(1+\gamma)]^2|}, & |\chi-1| \geqq K_2. \end{cases}$$

Where

$$K_2 = \frac{1}{|3\sigma-(1+\gamma+\gamma^2)|}\left|\Delta_2 - 2[2\sigma-(1+\gamma)]^2\frac{V(x)}{U^2(x)}\right|$$

$$\Delta_2 = (2\sigma-1)(\sigma-\gamma)-[2\sigma-(1+\gamma)]^2.$$

Corollary 11. *For $\sigma = 1$, $\beta = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(\gamma, 1; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|2-\gamma(\gamma+1)|}, & |\chi - 1| \leqq K_3 \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_3 - 2V(x)[1-\gamma]^2|}, & |\chi - 1| \geqq K_3. \end{cases}$$

Where

$$K_3 = \frac{1}{|2-\gamma(\gamma+1)|}\left|\Delta_3 - 2[1-\gamma]^2 \frac{V(x)}{U^2(x)}\right|$$

$$\Delta_3 = \gamma(1-\gamma).$$

Corollary 12. *For $\beta = 0$, $\gamma = 0$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(0, \sigma; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{|3\sigma-1|}, & |\chi - 1| \leqq K_4 \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_4 - 2V(x)(2\sigma-1)^2|}, & |\chi - 1| \geqq K_4. \end{cases}$$

Where

$$K_4 = \frac{1}{|3\sigma-1|}\left|\Delta_4 - 2[2\sigma-1]^2 \frac{V(x)}{U^2(x)}\right|.$$

$$\Delta_4 = (2\sigma-1)(1-\sigma)$$

Corollary 13. *For $\sigma = 2$, let $\theta \in \mathfrak{H}^{\mathfrak{B},0}(0, 2; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{5}, & \left(|\chi - 1| \leqq \frac{3}{5}\left|1 + 6\frac{V(x)}{U^2(x)}\right|\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{3|U^2(x)+6V(x)|}, & \left(|\chi - 1| \geqq \frac{3}{5}\left|1 + 6\frac{V(x)}{U^2(x)}\right|\right). \end{cases}$$

Corollary 14. *[44] For $\sigma \geq 1$, let $\theta \in \mathfrak{A}$ be in the class $\mathfrak{H}^{\mathfrak{B},0}(0, 1; x) = \mathfrak{H}^{\mathfrak{B}}(x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{2}, & \left(|\chi - 1| \leqq \frac{|V(x)|}{|U^2 x|}\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{2|V(x)|}, & \left(|\chi - 1| \geqq \frac{|V(x)|}{|U^2 x|}\right). \end{cases}$$

Corollary 15. *For $\beta = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},1}(\sigma; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3\sigma}, & \left(|\chi - 1| \leqq K_5\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x)\Delta_5 - 8\sigma^2 V(x)|}, & \left(|\chi - 1| \geqq K_5\right). \end{cases}$$

Where

$$K_5 = \frac{1}{|3\sigma|}\left|\Delta_5 - 8\sigma^2 \frac{V(x)}{U^2(x)}\right|$$

$$\Delta_5 = \sigma(1 - 2\sigma)$$

Corollary 16. *[8] For $\sigma = 1$, $\beta = 1$, let $\theta \in \mathfrak{H}^{\mathfrak{B},1}(1; x)$. Then*

$$\left|n_3 - \chi n_2^2\right| \leq \begin{cases} \frac{|U(x)|}{3}, & \left(|\chi - 1| \leqq \frac{1}{3}\left|1 + 8\frac{V(x)}{U^2(x)}\right|\right) \\ \\ \frac{|1-\chi|\cdot|U^3(x)|}{|U^2(x) + 8V(x)|}, & \left(|\chi - 1| \geqq \frac{1}{3}\left|1 + 8\frac{V(x)}{U^2(x)}\right|\right). \end{cases}$$

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