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COMPARISON OF ESTIMATION METHODS FOR THE KUMARASWAMY WEIBULL DISTRIBUTION

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Abstract. In this study, the performances of the different parameter estimation methods are compared for the Kumaraswamy Weibull distribution via Monte Carlo simulation study. Maximum Likelihood (ML), Least Squares (LS), Weighted Least Squares (WLS), Cramer-von Mises (CM) and Anderson Darling (AD) methods are used in the comparisons. The results of the Monte Carlo simulation study demonstrate that ML estimators for the parameters of the Kumaraswamy Weibull distribution are more efficient than the other estimators. It is followed by AD estimator. At the end of the study, a real data set taken from the literature is used to illustrate the applicability of the Kumaraswamy Weibull distribution.

1. Introduction

The Weibull is one of the most popular and widely used distribution in many fields of science such as engineering, reliability, biology, ecology and hydrology (see for example, Calabria and Pulcini [4], Keats et al. [16], Saeed et al. [20], Serban et al. [22]). However, the Weibull distribution does not provide a good fit to data sets with bathtub shaped or upside down bathtub shaped failure rates frequently encountered in engineering and reliability studies (see Cordeiro et al. [6], Akgül et al. [2], Maurya et al.[17]). Therefore, many generalized distributions have been developed for modeling these data sets (see, for example, Mudholkar and Srivastava [18], Sarhan and Zaindin [21], Elbatal et al. [8]). A new family of generalized Kumaraswamy (KwG) distributions obtained by combining the work of Eugene

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et al. [11] and Jones [14] is one of these generalized distributions, (see Cordeiro and Castro [5]). Probability density function (pdf) and the cumulative distribution function (cdf) of the KwG distribution for an arbitrary baseline pdf $g(x)$ and cdf $G(x)$ are given by

$$f(x) = abg(x)G(x)^{(a-1)}\{1 - G(x)^a\}^{(b-1)} \tag{1}$$

and

$$F(x) = 1 - [1 - G(x)^a]^b, \quad a, b > 0, x \in R, \tag{2}$$

respectively. Here, a and b are the shape parameters. KwG is a flexible distribution for modeling many different data sets including censored data therefore it is widely used in engineering and biology (see Gomes et al. [12], Elbatal and Elgarhy [9], Rocha et al.[19]).

The Kumaraswamy Weibull (KwWeibull) distribution is a special case of the KwG distribution obtained by inserting the pdf $g(x) = \frac{p}{\sigma^p}(x-\mu)^{p-1}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}$ and the cdf $G(x) = 1 - exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}$ of the well known Weibull distribution into (1). KwWeibull is a better alternative to Weibull distribution since it contains some well known distributions discussed in the literature as special cases such as the Weibull (see Cordeiro et al. [6]).

In this study, the estimators of the location and scale parameters of the KwWeibull distribution are obtained by using Maximum Likelihood (ML), Least Squares (LS), Weighted Least Squares (WLS), Cramer-von Mises (CM) and Anderson Darling (AD) estimation methods. Shape parameters are assumed to be known throughout the study. The most efficient estimators are identified by using an extensive Monte-Carlo simulation study for the different sample sizes and the parameter settings.

The remainder of this paper is organized as follows: In Section 2, a brief description of the KwWeibull distribution is given. In Section 3, the parameter estimation methods are presented. Results of the Monte-Carlo simulation study are given in Section 4. In Section 5, the KwWeibull distribution is used to model a real data set taken from the literature. Finally, the concluding remarks are given in Section 6.

2. Kumaraswamy Weibull Distribution

The pdf and cdf of KwWeibull distribution are given below:

$$f(x) = ab\frac{p}{\sigma^p}(x-\mu)^{p-1}\exp\left\{-(\tfrac{x-\mu}{\sigma})^p\right\}\left[1-\exp\left\{-(\tfrac{x-\mu}{\sigma})^p\right\}\right]^{a-1}$$
$$\times\left\{1-\left[1-\exp\left\{-(\tfrac{x-\mu}{\sigma})^p\right\}\right]^a\right\}^{b-1} \quad \mu<x<\infty,\ \ \mu,\sigma>0\ \ a,b,p>0 \tag{3}$$

and

$$F(x) = 1-\left\{1-\left[1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^p\right\}\right]^a\right\}^b, \tag{4}$$

respectively. Here, μ and σ represent the location (or threshold) and the scale parameters, respectively and a, b and p are the shape parameters. For different values of the shape parameters a, b and p, the plots of the pdf of KwWeibull distribution are shown in Figure 1.

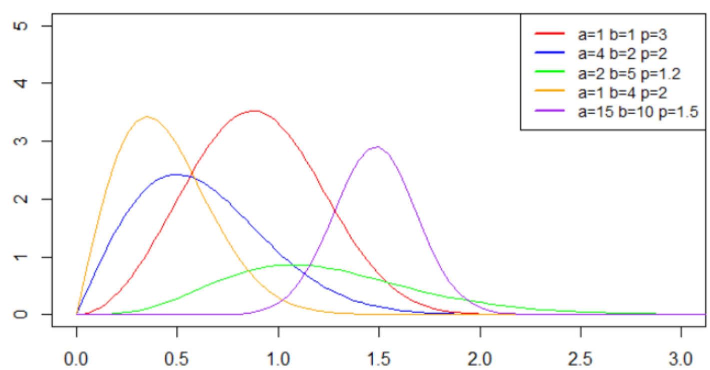

FIGURE 1. The pdf plots of the KwWeibull distribution

For better understanding the shape of the KwWeibull distribution, simulated skewness and kurtosis values of the KwWeibull distribution are given for different values of the shape parameters, see Table 1. It is clear from Table 1 that KwWeibull can be positively or negatively skewed depending on the values of the shape parameters. It can also be seen that kurtosis values can be less than or greater than that of Normal distribution subject to the values of shape parameters.

Table 1. Simulated skewness and kurtosis values for the KwWeibull distribution.

$a = b = 1$						
p=	1.5	2	2.5	3	4	6
Skewness	1.062	0.630	0.354	0.168	-0.088	-0.367
Kurtosis	4.368	3.219	2.843	2.722	2.734	2.998
$a = b = 2$						
p=	1.5	2	2.5	3	4	6
Skewness	0.709	0.381	0.178	0.041	-0.141	-0.336
Kurtosis	3.617	3.071	2.916	2.889	2.964	3.175
$a = 10$ and $b = 2$						
p=	1.5	2	2.5	3	4	6
Skewness	0.485	0.308	0.202	0.132	0.042	-0.046
Kurtosis	3.431	3.203	3.111	3.076	3.054	3.066
$a = 1$ and $b = 8$						
p=	1.5	2	2.5	3	4	6
Skewness	1.062	0.624	0.357	0.167	-0.087	-0.370
Kurtosis	4.348	3.226	2.843	2.720	2.739	3.022

3. Parameter Estimation Methods

Parameter estimation methods for estimating the location parameter μ and the scale parameter σ of KwWeibull distribution are described in the following subsections.

3.1. **The Maximum Likelihood Method.** In this subsection, the ML estimators for the location and scale parameters of the KwWeibull distribution are obtained. Let $x_1, x_2, \ldots, x_n$ be a random sample from KwWeibull(a, b, p, μ, σ), then the log-likelihood ($\ln L$) function of the KwWeibull distribution is expressed as follows:

$$\begin{aligned}\ln L = {} & n(\ln a + \ln b + \ln p - \ln\sigma) + (p-1)\sum_{i=1}^{n}\ln\left(\tfrac{x_i-\mu}{\sigma}\right) - \sum_{i=1}^{n}\left(\tfrac{x_i-\mu}{\sigma}\right)^p \\ & +(a-1)\sum_{i=1}^{n}\ln\left(1-\exp\left\{-\left(\tfrac{x_i-\mu}{\sigma}\right)^p\right\}\right) \\ & +(b-1)\sum_{i=1}^{n}\ln\left(1-\left[1-\exp\left\{-\left(\tfrac{x_i-\mu}{\sigma}\right)^p\right\}\right]^a\right).\end{aligned} \tag{5}$$

$\ln L$ function is maximized with respect to the parameters of interest, i.e., μ and σ. By taking the derivatives of $\ln L$ with respect to the parameters μ and σ and equating them to zero, the following likelihood equations are obtained

$$\begin{aligned}\frac{\partial lnL}{\partial\mu} =& -\frac{(p-1)}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{-1}+\frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{p-1} \\ &-\frac{(a-1)p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p-1}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}} \\ &+\frac{a(b-1)p}{\sigma}\sum_{i=1}^{n}\frac{\left(1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right)^{a-1}\left(\frac{x_i-\mu}{\sigma}\right)^{p-1}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{\left(1-\left[1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right)} \\ =&0 \end{aligned} \tag{6}$$

and

$$\begin{aligned}\frac{\partial \ln L}{\partial\sigma} =& -\frac{n}{\sigma}-\frac{n\,(p-1)}{\sigma}+\frac{p}{\sigma}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^{p}-\frac{(a-1)\,p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}}{1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}} \\ &+\frac{a\,(b-1)\,p}{\sigma}\sum_{i=1}^{n}\frac{\left(\frac{x_i-\mu}{\sigma}\right)^{p}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\left(1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right)^{a-1}}{\left(1-\left[1-\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right)} \\ =&0. \end{aligned} \tag{7}$$

Solutions of these likelihood equations are called as the ML estimators of the parameters. When the likelihood equations for the location and scale parameters are examined, it is seen that the functions are not linear with respect to the parameters of interest. Therefore, numerical methods are needed for estimating the location and scale parameters.

3.2. **The Least Squares Method.** The LS estimators of the unknown parameters are obtained by minimizing the following equation

$$S_{LS}=\sum_{i=1}^{n}\left(F\left(x_{(i)}\right)-\frac{i}{n+1}\right)^{2} \tag{8}$$

with respect to the parameters of interest (see Swain [23]). Here and in the other subsections, $x_1, x_2, \ldots, x_n$ is a random sample from the distribution function $F(.)$, $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ denotes the corresponding order statistics and $\frac{i}{n+1}$, $(i = 1, \ldots, n)$ are the expected values of $F\left(x_{(i)}\right)$. From Eq. (8), the LS estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{LS}\left(\mu,\sigma\right)=\sum_{i=1}^{n}\left(1-\left\{1-\left[1-\exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{i}{n+1}\right)^{2}. \quad (9)$$

3.3. **The Weighted Least Squares Method.** The WLS estimators of the unknown parameters are obtained by minimizing the following equation with respect to the parameters of interest (see Swain [23])

$$S_{WLS}=\sum_{i=1}^{n}w_{i}\left(F\left(x_{(i)}\right)-\frac{i}{n+1}\right)^{2} \quad (10)$$

where, $w_i = 1/Var\left(F\left(x_{(i)}\right)\right)=(n+1)^2\left(n+2\right)/i(n-i+1), (i=1,2,...,n)$. From Eq.(10), the WLS estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{WLS}\left(\mu,\sigma\right)=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(1-\left\{1-\left[1-\exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{i}{n+1}\right)^{2}. \quad (11)$$

3.4. **The Cramér–Von Mises Method.** The CM estimators of the unknown parameters are obtained by minimizing the following equation

$$S_{CM}=\frac{1}{12n}+\sum_{i=1}^{n}\left(F(x_{(i)})-\frac{2i-1}{2n}\right)^{2} \quad (12)$$

with respect to the parameters of interest (see Wolfowitz [24]). From Eq. (12), the CM estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{CM}\left(\mu,\sigma\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left(1-\left\{1-\left[1-\ exp\left\{-\left(\frac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}-\frac{2i-1}{2n}\right)^{2}. \quad (13)$$

3.5. **The Anderson-Darling Method.** The AD estimators of the unknown parameters are obtained by minimizing the following equation

$$S_{AD} = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\,log\left\{F\left(x_{(i)}\right)\left(1-F\left(x_{(n-i+1)}\right)\right)\right\} \tag{14}$$

with respect to the parameters of interest, (see Wolfowitz [25]). From Eq. (14), the AD estimators for the parameters of the KwWeibull distribution are obtained by minimizing the following equation with respect to the parameters μ and σ

$$S_{AD}(\mu,\sigma) = -n - \tfrac{1}{n}\sum_{i=1}^{n}(2i-1)log\left\{1-\left\{1-\left[1-\ exp\left\{-\left(\tfrac{x_{(i)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}\right. \\ \left.\times\left\{1-\left[1-exp\left\{-\left(\tfrac{x_{(n-i+1)}-\mu}{\sigma}\right)^{p}\right\}\right]^{a}\right\}^{b}\right\}. \tag{15}$$

Here it should be noted that similar to ML estimates of parameters, LS, WLS, CM and AD estimates are obtained iteratively (see, Ergenç [10]).

4. Simulation Study

In this section, we perform an extensive Monte Carlo simulation study to compare the performances of the ML, LS, WLS, CM and AD estimators of the location parameter μ and scale parameter σ of the KwWeibull distribution. Without loss of generality, μ and σ are taken to be 0 and 1, respectively. All the simulations were conducted using R programming language for 10,000 Monte-Carlo runs. We use small ($n = 20$), moderate ($n = 50, 100$) and large ($n = 200, 500$) sample sizes. It is known that the estimation of the shape parameters along with the other parameters yields unreliable results when the sample size is not large enough (see, Bowman and Shenton [3], Kantar and Şenoğlu [15]). Therefore, it is assumed that the shape parameters a, b and p are known throughout the study. The performances of the estimators are compared with respect to the Bias, mean squares error (MSE) and Deficiency (Def) criteria, see the mathematical expressions given below

$$\begin{aligned} Bias &= \frac{1}{10,000}\sum_{i=1}^{10,000}\left(\hat{\theta}_i-\theta\right), \\ MSE &= \frac{1}{10,000}\sum_{i=1}^{10,000}\left(\hat{\theta}_i-\theta\right)^2 \end{aligned} \tag{16}$$

and

$$Def(\hat{\mu},\hat{\sigma}) = MSE(\hat{\mu}) + MSE(\hat{\sigma}). \tag{17}$$

Here, $\hat{\theta}_i$ is the ith simulated estimate of the parameter of interest (i.e. μ or σ) and θ is the true value of the parameter. Also, Def criterion is defined as the joint efficiencies of the estimators $\hat{\mu}$ and $\hat{\sigma}$. Simulated Bias, MSE and Def values for the ML, LS, WLS, CM and AD estimators for the location parameter μ and the scale parameter σ of the KwWeibull distribution are given in Table 2.

TABLE 2. The simulated Bias, MSE and Def values for the ML, LS, WLS, CM and AD estimators of the parameters μ and σ.

	$(a,b,p)=(1,1,1.5)$					$(a,b,p)=(1,1,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,061	0,011	-0,061	0,029	0,040	0,048	0,020	-0,052	0,022	0,043
LS	-0,034	0,017	0,056	0,045	0,062	-0,039	0,035	0,042	0,042	0,077
WLS	-0,019	0,012	0,036	0,037	0,049	-0,024	0,028	0,026	0,033	0,062
CM	0,010	0,013	-0,011	0,037	0,051	0,025	0,031	-0,033	0,037	0,068
AD	-0,005	0,011	0,009	0,033	0,043	-0,007	0,023	0,008	0,026	0,049
					$n=50$					
ML	0,030	0,003	-0,030	0,011	0,014	0,021	0,007	-0,022	0,008	0,016
LS	-0,018	0,006	0,026	0,016	0,022	-0,015	0,013	0,016	0,015	0,028
WLS	-0,007	0,004	0,012	0,013	0,016	-0,006	0,010	0,005	0,011	0,021
CM	-0,002	0,005	0,004	0,015	0,019	0,010	0,012	-0,014	0,014	0,027
AD	-0,006	0,003	0,011	0,012	0,016	-0,004	0,009	0,004	0,010	0,019
					$n=100$					
ML	0,017	0,001	-0,017	0,005	0,006	0,012	0,003	-0,013	0,004	0,007
LS	-0,012	0,003	0,017	0,008	0,010	-0,007	0,006	0,007	0,007	0,013
WLS	-0,004	0,001	0,006	0,006	0,008	-0,001	0,005	0,000	0,005	0,010
CM	-0,005	0,002	0,007	0,007	0,009	0,006	0,006	-0,008	0,007	0,013
AD	-0,006	0,002	0,008	0,006	0,008	-0,002	0,004	0,002	0,005	0,009
					$n=200$					
ML	0,010	0,000	-0,010	0,002	0,003	0,006	0,002	-0,006	0,002	0,004
LS	-0,009	0,001	0,012	0,004	0,005	-0,005	0,003	0,004	0,004	0,007
WLS	-0,003	0,001	0,005	0,003	0,003	-0,001	0,002	0,000	0,003	0,005
CM	-0,005	0,001	0,007	0,004	0,005	0,002	0,003	-0,003	0,004	0,007
AD	-0,004	0,001	0,007	0,003	0,004	-0,002	0,002	0,002	0,002	0,005
					$n=500$					
ML	0,005	0,000	-0,005	0,001	0,001	0,003	0,001	-0,003	0,001	0,001
LS	-0,006	0,000	0,007	0,001	0,002	-0,002	0,001	0,002	0,001	0,003
WLS	-0,002	0,000	0,002	0,001	0,001	0,000	0,001	0,000	0,001	0,002
CM	-0,004	0,000	0,005	0,001	0,002	0,000	0,001	-0,001	0,001	0,003
AD	-0,002	0,000	0,004	0,001	0,001	-0,001	0,001	0,001	0,001	0,002

Table 2. (continued)

	$(a,b,p)=(1,1,4)$					$(a,b,p)=(1,1,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,043	0,023	-0,045	0,023	0,046	0,041	0,027	-0,044	0,026	0,054
LS	-0,043	0,041	0,046	0,044	0,086	-0,041	0,047	0,042	0,047	0,093
WLS	-0,028	0,034	0,030	0,036	0,070	-0,027	0,039	0,027	0,038	0,077
CM	0,025	0,036	-0,030	0,039	0,075	0,032	0,042	-0,035	0,042	0,083
AD	-0,010	0,027	0,011	0,027	0,054	-0,008	0,032	0,007	0,031	0,062
					$n=50$					
ML	0,018	0,009	-0,019	0,009	0,018	0,015	0,010	-0,016	0,010	0,020
LS	-0,015	0,015	0,016	0,016	0,031	-0,019	0,017	0,019	0,017	0,033
WLS	-0,006	0,012	0,006	0,012	0,024	-0,008	0,013	0,008	0,013	0,026
CM	0,012	0,015	-0,014	0,015	0,030	0,010	0,016	-0,011	0,016	0,032
AD	-0,004	0,011	0,004	0,011	0,022	-0,006	0,012	0,006	0,012	0,024
					$n=100$					
ML	0,010	0,004	-0,010	0,004	0,009	0,009	0,005	-0,010	0,005	0,010
LS	-0,008	0,007	0,008	0,007	0,015	-0,007	0,008	0,008	0,008	0,016
WLS	-0,002	0,006	0,002	0,006	0,011	-0,001	0,006	0,001	0,006	0,012
CM	0,006	0,007	-0,007	0,007	0,014	0,007	0,008	-0,007	0,008	0,016
AD	-0,002	0,005	0,002	0,005	0,011	-0,001	0,006	0,001	0,006	0,012
					$n=200$					
ML	0,004	0,002	-0,004	0,002	0,004	0,004	0,002	-0,005	0,002	0,005
LS	-0,006	0,004	0,006	0,004	0,007	-0,004	0,004	0,004	0,004	0,008
WLS	-0,002	0,003	0,002	0,003	0,006	0,000	0,003	0,000	0,003	0,006
CM	0,001	0,003	-0,001	0,004	0,007	0,003	0,004	-0,003	0,004	0,008
AD	-0,003	0,003	0,003	0,003	0,005	-0,001	0,003	0,001	0,003	0,006
					$n=500$					
ML	0,002	0,001	-0,002	0,001	0,002	0,002	0,001	-0,002	0,001	0,003
LS	-0,002	0,001	0,002	0,001	0,003	-0,003	0,002	0,003	0,002	0,004
WLS	0,000	0,001	0,000	0,001	0,002	0,000	0,002	0,000	0,002	0,003
CM	0,001	0,001	-0,001	0,001	0,003	0,001	0,002	-0,001	0,002	0,004
AD	-0,001	0,001	0,001	0,001	0,002	-0,001	0,002	0,001	0,002	0,003

TABLE 2. (continued)

	$(a,b,p)=(1,2,1.5)$					$(a,b,p)=(1,2,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,039	0,004	-0,063	0,029	0,033	0,039	0,013	-0,052	0,022	0,035
LS	-0,021	0,007	0,054	0,046	0,053	-0,029	0,023	0,040	0,042	0,064
WLS	-0,012	0,005	0,034	0,038	0,043	-0,018	0,018	0,026	0,033	0,051
CM	0,006	0,005	-0,006	0,038	0,044	0,022	0,020	-0,034	0,037	0,057
AD	-0,003	0,004	0,015	0,033	0,037	-0,004	0,015	0,007	0,025	0,040
					$n=50$					
ML	0,019	0,001	-0,028	0,011	0,012	0,017	0,005	-0,023	0,008	0,013
LS	-0,011	0,002	0,029	0,016	0,018	-0,013	0,008	0,017	0,015	0,023
WLS	-0,004	0,001	0,014	0,013	0,014	-0,004	0,006	0,005	0,012	0,018
CM	-0,001	0,002	0,006	0,014	0,016	0,007	0,008	-0,013	0,015	0,022
AD	-0,004	0,001	0,013	0,012	0,014	-0,003	0,006	0,004	0,010	0,016
					$n=100$					
ML	0,011	0,000	-0,017	0,005	0,005	0,008	0,002	-0,011	0,004	0,006
LS	-0,007	0,001	0,017	0,008	0,008	-0,007	0,004	0,009	0,007	0,011
WLS	-0,002	0,001	0,005	0,006	0,006	-0,002	0,003	0,002	0,005	0,008
CM	-0,003	0,001	0,006	0,007	0,008	0,003	0,004	-0,006	0,007	0,011
AD	-0,003	0,001	0,008	0,006	0,006	-0,003	0,003	0,004	0,005	0,008
					$n=200$					
ML	0,007	0,000	-0,009	0,002	0,003	0,005	0,001	-0,006	0,002	0,003
LS	-0,005	0,000	0,013	0,004	0,004	-0,004	0,002	0,005	0,003	0,005
WLS	-0,002	0,000	0,005	0,003	0,003	-0,001	0,001	0,001	0,003	0,004
CM	-0,003	0,000	0,007	0,004	0,004	0,001	0,002	-0,002	0,003	0,005
AD	-0,003	0,000	0,007	0,003	0,003	-0,001	0,001	0,002	0,002	0,004
					$n=500$					
ML	0,003	0,000	-0,006	0,001	0,001	0,002	0,000	-0,003	0,001	0,001
LS	-0,004	0,000	0,007	0,001	0,002	-0,001	0,001	0,002	0,001	0,002
WLS	-0,001	0,000	0,002	0,001	0,001	0,000	0,001	0,000	0,001	0,002
CM	-0,003	0,000	0,005	0,001	0,002	0,001	0,001	-0,001	0,001	0,002
AD	-0,002	0,000	0,003	0,001	0,001	0,000	0,001	0,001	0,001	0,002

Table 2. (continued)

	$(a,b,p)=(1,2,4)$					$(a,b,p)=(1,2,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,036	0,016	-0,045	0,023	0,039	0,034	0,021	-0,041	0,025	0,045
LS	-0,035	0,028	0,045	0,043	0,071	-0,040	0,037	0,046	0,047	0,084
WLS	-0,024	0,023	0,030	0,034	0,057	-0,027	0,031	0,032	0,038	0,069
CM	0,022	0,025	-0,031	0,038	0,063	0,025	0,033	-0,031	0,041	0,074
AD	-0,008	0,019	0,010	0,026	0,045	-0,010	0,024	0,011	0,030	0,054
					$n=50$					
ML	0,013	0,006	-0,017	0,009	0,015	0,016	0,008	-0,018	0,010	0,018
LS	-0,015	0,010	0,019	0,015	0,026	-0,015	0,013	0,017	0,017	0,030
WLS	-0,007	0,008	0,009	0,012	0,020	-0,005	0,011	0,006	0,013	0,024
CM	0,008	0,010	-0,011	0,015	0,024	0,011	0,013	-0,013	0,016	0,029
AD	-0,005	0,008	0,006	0,011	0,018	-0,003	0,010	0,003	0,012	0,022
					$n=100$					
ML	0,008	0,003	-0,010	0,004	0,007	0,008	0,004	-0,009	0,005	0,009
LS	-0,007	0,005	0,009	0,008	0,013	-0,007	0,006	0,008	0,008	0,014
WLS	-0,002	0,004	0,002	0,006	0,010	-0,001	0,005	0,002	0,006	0,011
CM	0,005	0,005	-0,006	0,007	0,013	0,005	0,006	-0,007	0,008	0,014
AD	-0,002	0,004	0,002	0,005	0,009	-0,001	0,005	0,002	0,006	0,010
					$n=200$					
ML	0,004	0,001	-0,005	0,002	0,004	0,005	0,003	-0,006	0,004	0,007
LS	-0,003	0,003	0,004	0,004	0,006	-0,006	0,005	0,007	0,006	0,011
WLS	0,000	0,002	0,000	0,003	0,005	-0,001	0,004	0,001	0,005	0,008
CM	0,002	0,002	-0,003	0,004	0,006	0,003	0,005	-0,004	0,006	0,010
AD	-0,001	0,002	0,001	0,003	0,005	-0,002	0,004	0,002	0,004	0,008
					$n=500$					
ML	0,002	0,001	-0,002	0,001	0,001	0,005	0,002	-0,006	0,003	0,005
LS	-0,001	0,001	0,002	0,002	0,003	-0,004	0,004	0,005	0,005	0,009
WLS	0,000	0,001	0,000	0,001	0,002	-0,001	0,003	0,001	0,004	0,007
CM	0,001	0,001	-0,001	0,002	0,003	0,003	0,004	-0,004	0,005	0,008
AD	0,000	0,001	0,000	0,001	0,002	-0,001	0,003	0,001	0,004	0,006

TABLE 2. (continued)

	$(a,b,p)=(6,4.5,1.5)$					$(a,b,p)=(6,4.5,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
$n=20$										
ML	0,050	0,038	-0,042	0,027	0,064	0,046	0,033	-0,042	0,027	0,060
LS	-0,055	0,067	0,047	0,047	0,114	-0,046	0,056	0,043	0,046	0,102
WLS	-0,035	0,054	0,030	0,038	0,093	-0,030	0,046	0,027	0,038	0,084
CM	0,035	0,059	-0,030	0,041	0,100	0,037	0,050	-0,034	0,041	0,090
AD	-0,010	0,043	0,009	0,031	0,074	-0,006	0,037	0,005	0,030	0,068
$n=50$										
ML	0,017	0,014	-0,014	0,010	0,025	0,017	0,013	-0,015	0,010	0,023
LS	-0,024	0,024	0,021	0,017	0,041	-0,020	0,020	0,018	0,016	0,036
WLS	-0,011	0,019	0,010	0,013	0,032	-0,009	0,016	0,008	0,013	0,029
CM	0,011	0,023	-0,009	0,016	0,038	0,013	0,019	-0,012	0,015	0,034
AD	-0,007	0,017	0,006	0,012	0,029	-0,004	0,014	0,004	0,012	0,026
$n=100$										
ML	0,009	0,007	-0,008	0,005	0,012	0,007	0,006	-0,007	0,005	0,012
LS	-0,011	0,011	0,010	0,008	0,019	-0,011	0,010	0,010	0,008	0,018
WLS	-0,003	0,009	0,003	0,006	0,015	-0,003	0,008	0,003	0,006	0,014
CM	0,006	0,011	-0,005	0,008	0,019	0,006	0,010	-0,005	0,008	0,018
AD	-0,003	0,008	0,002	0,006	0,014	-0,003	0,008	0,002	0,006	0,014
$n=200$										
ML	0,005	0,004	-0,004	0,002	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,005	0,006	0,005	0,004	0,010	-0,006	0,005	0,005	0,004	0,009
WLS	0,000	0,004	0,000	0,003	0,007	-0,001	0,004	0,001	0,003	0,007
CM	0,003	0,006	-0,003	0,004	0,010	0,002	0,005	-0,002	0,004	0,009
AD	-0,001	0,004	0,001	0,003	0,007	-0,002	0,004	0,001	0,003	0,007
$n=500$										
ML	0,002	0,001	-0,002	0,001	0,002	0,001	0,001	-0,001	0,001	0,002
LS	-0,002	0,002	0,001	0,002	0,004	-0,003	0,002	0,002	0,002	0,004
WLS	0,000	0,002	0,000	0,001	0,003	0,000	0,002	0,000	0,001	0,003
CM	0,002	0,002	-0,002	0,002	0,004	0,001	0,002	-0,001	0,002	0,004
AD	0,000	0,002	0,000	0,001	0,003	-0,001	0,002	0,001	0,001	0,003

Table 2. (continued)

	$(a,b,p)=(6,4.5,4)$					$(a,b,p)=(6,4.5,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,039	0,031	-0,037	0,027	0,058	0,038	0,031	-0,037	0,028	0,059
LS	-0,051	0,055	0,047	0,047	0,102	-0,051	0,054	0,049	0,049	0,103
WLS	-0,034	0,045	0,031	0,039	0,084	-0,035	0,045	0,033	0,040	0,085
CM	0,032	0,048	-0,030	0,041	0,089	0,029	0,047	-0,028	0,042	0,089
AD	-0,010	0,036	0,009	0,031	0,067	-0,011	0,036	0,010	0,032	0,069
					$n=50$					
ML	0,017	0,012	-0,016	0,010	0,023	0,016	0,012	-0,015	0,011	0,022
LS	-0,019	0,019	0,018	0,017	0,036	-0,018	0,019	0,018	0,017	0,036
WLS	-0,007	0,015	0,007	0,013	0,029	-0,008	0,015	0,008	0,013	0,028
CM	0,013	0,018	-0,013	0,016	0,034	0,013	0,018	-0,013	0,016	0,034
AD	-0,003	0,014	0,003	0,012	0,026	-0,003	0,014	0,003	0,012	0,026
					$n=100$					
ML	0,006	0,006	-0,006	0,005	0,011	0,008	0,006	-0,008	0,005	0,011
LS	-0,012	0,010	0,012	0,008	0,018	-0,009	0,009	0,009	0,008	0,017
WLS	-0,005	0,008	0,005	0,006	0,014	-0,002	0,007	0,002	0,006	0,014
CM	0,004	0,009	-0,004	0,008	0,017	0,007	0,009	-0,006	0,008	0,017
AD	-0,005	0,007	0,004	0,006	0,013	-0,002	0,007	0,001	0,006	0,013
					$n=200$					
ML	0,004	0,003	-0,003	0,003	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,006	0,005	0,006	0,004	0,009	-0,005	0,004	0,005	0,004	0,008
WLS	-0,001	0,004	0,001	0,003	0,007	-0,001	0,003	0,001	0,003	0,007
CM	0,002	0,005	-0,002	0,004	0,009	0,003	0,004	-0,003	0,004	0,008
AD	-0,002	0,004	0,002	0,003	0,007	-0,001	0,003	0,001	0,003	0,006
					$n=500$					
ML	0,001	0,001	-0,001	0,001	0,002	0,002	0,001	-0,002	0,001	0,002
LS	-0,002	0,002	0,002	0,002	0,003	-0,001	0,002	0,001	0,002	0,003
WLS	0,000	0,001	0,000	0,001	0,003	0,001	0,001	-0,001	0,001	0,003
CM	0,001	0,002	-0,001	0,002	0,003	0,002	0,002	-0,002	0,002	0,003
AD	-0,001	0,001	0,001	0,001	0,003	0,000	0,001	0,000	0,001	0,003

TABLE 2. (continued)

	$(a,b,p) = (15,5,1.5)$					$(a,b,p) = (15,5,3)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
$n = 20$										
ML	0,064	0,070	-0,039	0,026	0,097	0,046	0,044	-0,036	0,026	0,070
LS	-0,077	0,123	0,048	0,047	0,170	-0,066	0,078	0,052	0,048	0,126
WLS	-0,051	0,101	0,032	0,038	0,139	-0,045	0,064	0,036	0,039	0,103
CM	0,048	0,108	-0,029	0,041	0,148	0,033	0,067	-0,026	0,041	0,108
AD	-0,014	0,081	0,009	0,031	0,112	-0,015	0,051	0,012	0,031	0,082
$n = 50$										
ML	0,024	0,028	-0,015	0,011	0,039	0,018	0,017	-0,014	0,010	0,028
LS	-0,030	0,045	0,018	0,017	0,063	-0,024	0,028	0,019	0,017	0,045
WLS	-0,013	0,036	0,008	0,014	0,050	-0,011	0,022	0,008	0,013	0,035
CM	0,019	0,043	-0,012	0,016	0,059	0,015	0,026	-0,012	0,016	0,042
AD	-0,007	0,033	0,004	0,012	0,045	-0,005	0,020	0,004	0,012	0,032
$n = 100$										
ML	0,010	0,014	-0,006	0,005	0,019	0,008	0,008	-0,007	0,005	0,014
LS	-0,018	0,022	0,011	0,008	0,030	-0,013	0,013	0,011	0,008	0,021
WLS	-0,007	0,017	0,004	0,006	0,024	-0,005	0,010	0,004	0,006	0,016
CM	0,006	0,021	-0,004	0,008	0,029	0,006	0,013	-0,005	0,008	0,020
AD	-0,006	0,016	0,004	0,006	0,023	-0,004	0,010	0,003	0,006	0,016
$n = 200$										
ML	0,006	0,007	-0,004	0,003	0,009	0,005	0,004	-0,004	0,003	0,007
LS	-0,008	0,010	0,005	0,004	0,014	-0,006	0,006	0,005	0,004	0,010
WLS	-0,001	0,008	0,001	0,003	0,011	-0,001	0,005	0,001	0,003	0,008
CM	0,004	0,010	-0,003	0,004	0,014	0,004	0,006	-0,003	0,004	0,010
AD	-0,002	0,008	0,001	0,003	0,011	-0,001	0,005	0,001	0,003	0,008
$n = 500$										
ML	0,003	0,003	-0,002	0,001	0,004	0,001	0,002	-0,001	0,001	0,003
LS	-0,003	0,004	0,002	0,002	0,006	-0,003	0,003	0,002	0,002	0,004
WLS	0,000	0,003	0,000	0,001	0,004	-0,001	0,002	0,000	0,001	0,003
CM	0,002	0,004	-0,001	0,002	0,006	0,001	0,003	-0,001	0,002	0,004
AD	0,000	0,003	0,000	0,001	0,004	-0,001	0,002	0,001	0,001	0,003

TABLE 2. (continued)

	$(a,b,p)=(15,5,4)$					$(a,b,p)=(15,5,6)$				
	$\hat{\mu}$		$\hat{\sigma}$			$\hat{\mu}$		$\hat{\sigma}$		
Methods	Bias	MSE	Bias	MSE	Def	Bias	MSE	Bias	MSE	Def
					$n=20$					
ML	0,046	0,040	-0,038	0,027	0,067	0,044	0,035	-0,039	0,027	0,061
LS	-0,056	0,069	0,047	0,047	0,116	-0,054	0,060	0,048	0,047	0,107
WLS	-0,036	0,057	0,030	0,039	0,096	-0,035	0,050	0,031	0,039	0,089
CM	0,036	0,060	-0,030	0,041	0,101	0,033	0,053	-0,030	0,041	0,093
AD	-0,009	0,046	0,008	0,031	0,077	-0,010	0,040	0,008	0,031	0,070
					$n=50$					
ML	0,016	0,015	-0,014	0,011	0,026	0,017	0,014	-0,015	0,011	0,024
LS	-0,024	0,024	0,020	0,017	0,041	-0,022	0,021	0,019	0,017	0,038
WLS	-0,011	0,020	0,009	0,013	0,033	-0,010	0,017	0,009	0,013	0,031
CM	0,013	0,023	-0,011	0,016	0,039	0,013	0,020	-0,011	0,016	0,036
AD	-0,006	0,018	0,005	0,012	0,030	-0,004	0,016	0,004	0,012	0,028
					$n=100$					
ML	0,011	0,008	-0,009	0,005	0,013	0,009	0,007	-0,008	0,005	0,012
LS	-0,010	0,012	0,008	0,008	0,020	-0,010	0,011	0,008	0,008	0,019
WLS	-0,001	0,009	0,001	0,006	0,016	-0,002	0,008	0,002	0,007	0,015
CM	0,008	0,012	-0,007	0,008	0,019	0,008	0,010	-0,007	0,008	0,019
AD	-0,001	0,009	0,001	0,006	0,015	-0,001	0,008	0,001	0,006	0,014
					$n=200$					
ML	0,005	0,004	-0,004	0,003	0,006	0,004	0,003	-0,004	0,003	0,006
LS	-0,005	0,006	0,004	0,004	0,010	-0,005	0,005	0,004	0,004	0,009
WLS	0,000	0,005	0,000	0,003	0,008	-0,001	0,004	0,000	0,003	0,007
CM	0,004	0,006	-0,003	0,004	0,010	0,004	0,005	-0,003	0,004	0,009
AD	-0,001	0,004	0,001	0,003	0,007	-0,001	0,004	0,001	0,003	0,007
					$n=500$					
ML	0,002	0,002	-0,002	0,001	0,003	0,002	0,001	-0,002	0,001	0,002
LS	-0,002	0,002	0,002	0,002	0,004	-0,002	0,002	0,002	0,002	0,004
WLS	0,000	0,002	0,000	0,001	0,003	0,000	0,002	0,000	0,001	0,003
CM	0,001	0,002	-0,001	0,002	0,004	0,001	0,002	-0,001	0,002	0,004
AD	0,000	0,002	0,000	0,001	0,003	-0,001	0,002	0,000	0,001	0,003

4.1. **Comparisons for the biases.** In this subsection, the biases of the estimators $\hat{\mu}$ and $\hat{\sigma}$ obtained from the ML, LS, WLS, CM and AD methodologies are compared. For the estimators of the location parameter μ and the scale parameter σ, in general, the AD has the smallest bias among the other estimators for all values of the shape parameters and the sample sizes except for the sample size $n = 50$ and shape parameters $a = 1$, $b = 1$, $p = 1.5$ and $a = 1$, $b = 2$, $p = 1.5$ in which case CM provides the smallest bias. When the sample size n=100 and shape parameters $a = 1$, $b = 1$, $p = 1.5$, $a = 1$, $b = 1$, $p = 3$, $a = 1$, $b = 2$, $p = 1.5$ and $a = 1$,

$b = 2$, $p = 3$, WLS provides the smallest biases. AD is followed by the WLS and CM estimators for the small and moderate sample sizes in most of the cases. ML and LS estimators have larger biases than the other estimators for the small and moderate sample sizes. For the large sample sizes, all the estimators have negligible biases.

4.2. **Comparisons for the efficiencies.** Discussions about the efficiencies of the estimators of μ and σ with respect to the MSE criterion are given as follows. For the estimators of the location parameter μ, ML estimator shows the best performance among the others with respect to the MSE criterion in all cases. It is followed by the AD and WLS estimators for the sample sizes $n = 20$ and 50. It should also be pointed out that the LS estimator has shown the worst performance among the others for the sample sizes $n = 20$ and 50. For the sample sizes n $\geq$ 100, ML is the most efficient estimator among the others in general and it is followed by the AD and WLS estimators. For the estimators of the scale parameter σ, the ML is the most efficient among the others for all values of the shape parameters and the sample sizes. It is followed by the AD and WLS estimators for the small and moderate sample sizes. The LS estimator of σ shows the worst performance among the other estimators in almost all cases.

4.3. **Comparisons for the joint efficiencies.** According to the simulation results, the ML estimator shows the highest performance among the others for all values of the shapes parameters and the sample sizes. It is seen that the ML estimator is followed by AD estimator. On the other hand, the LS estimator has the worst performance among the other estimators in almost all cases.

5. Application

In this section, the KwWeibull distribution is used to model the relative humidity data set taken from Cortez and Morais [7]. Table 3 shows the descriptive statistics for the relative humidity data.

Table 3. Descriptive statistics for the relative humidity data.

n	Min	Max	Mean	Variance	Skewness	Kurtosis
517	15.0	100.0	44.29	266.3	0.85	2.59

Before analyzing the relative humidity data, profile likelihood method is used to identify the plausible values of the shape parameters a, b and p of the KwWeibull distribution (see for example, Islam and Tiku [13] and Acıtaş and Şenoğlu [1]). The steps of the profile likelihood procedure are given as follows:

Step 1. Calculate $\hat{\mu}$ and $\hat{\sigma}$ for the given a, b and p values.
Step 2. Calculate $\ln L$value by incorporating $\hat{\mu}$ and $\hat{\sigma}$ into lnL.
Step 3. Repeat *Steps* 1 and 2 for a serious values of a, b and p. Find a, b and p values maximizing the $\ln L$ function among the others and choose them as conceivable values of the shape parameters.

Following the steps of profile likelihood procedure, the values of shape parameters a, b and p are obtained as 5.637, 6.133 and 0.681, respectively. We also use QQ plot which is a well known and widely used graphical technique to identify the distribution of the relative humidity data set, see Figure 2. It can be seen from Figure 2 that KwWeibull distribution provides a good fit for the relative humidity data.

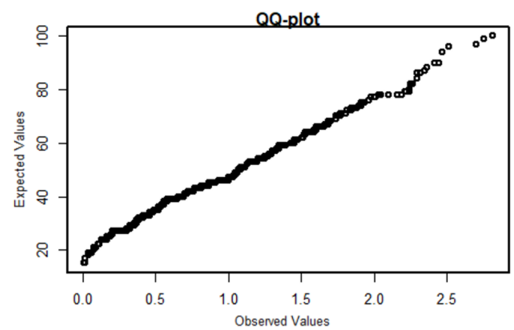

FIGURE 2. KwWeibull QQ plot for the relative humidity data

Based on the estimate values of the shape parameters, the ML estimates of location parameter μ and scale parameter σ are obtained as given in the Table 4. Estimates of the parameters μ, σ and p of Weibull distribution are also given for the relative humidity data to make the comparisons complete in Table 4. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and Corrected AIC (AICc) values along with the Kolmogorov-Smirnov (KS) test statistic and associated $p - values$ are also given in Table 4.

The equalities for the AIC, BIC, AICc and KS are given by

$$\begin{aligned} AIC &= -2\ln L + 2k, \\ BIC &= -2\ln L + kln(n), \\ AICc &= AIC + (2k^2 + k)/(n - k - 1) \end{aligned} \tag{18}$$

and

$$KS = \max \left| \hat{F}(X_{(i)}) - \tfrac{i}{n+1} \right|, \tag{19}$$

respectively. Here, $\hat{F}$ is the estimated cdf, $X_{(i)}$ is the $i-th$ order statistics, k is the number of the unknown parameters and n is the sample size.

Table 4. The estimates of the parameters of the KwWeibull and Weibull distributions for the relative humidity data

	$\hat{a}$	$\hat{b}$	$\hat{p}$	$\hat{\mu}$	$\hat{\sigma}$	KS	p-value	AIC	BIC	AICc
KwWeibull	5.637	6.133	0.681	25.466	11.763	0.043	0.273	4273.80	4295.05	4273.88
Weibull	-	-	1.924	33.662	14.485	0.097	0.063	4337.76	4346.27	4337.81

The smaller AIC, BIC and AICc values imply the better fitting performance. It is clear from Table 4 that the KwWeibull distribution is more preferable than the Weibull distribution in terms of these criteria. See also Figure 3 in which the histogram and the fitted densities based on the KwWeibull and the Weibull distributions are plotted. Here, it should be noted that the ML estimates of the parameters are used in obtaining the fitted densities.

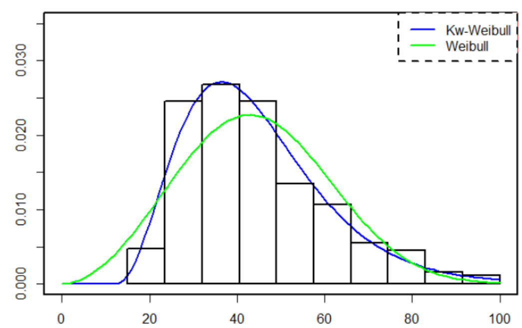

Figure 3. The histogram and the fitted densities based on the KwWeibull and Weibull distributions for the relative humidity data

It is seen from Figure 3 that KwWeibull distribution shows better fitting performance than the Weibull distribution. Then, we obtain the estimates of location parameter μ and scale parameter σ of the KwWeibull distribution when $\hat{a}$ =5.637, $\hat{b}$ =6.133 and $\hat{p}$=0.681 by using ML, LS, WLS, CM and AD methods to see the fitting performance of KwWeibull distribution for each estimation methods. Estimates of the location and scale parameters of KwWeibull distribution for each estimation methods are given in Table 5.

Table 5. Estimates of the location and scale parameters of the KwWeibull distribution for relative humidity data

Estimation Methods	$\hat{\mu}$	$\hat{\sigma}$	AIC	BIC	AICc
ML	25.466	11.763	4273.80	4295.05	4273.88
LS	19.712	14.327	4883.75	4904.99	4883.86
WLS	28.322	12.365	4646.50	4649.39	4646.53
CM	24.553	14.680	4675.32	4696.56	4675.43
AD	18.859	13.140	4622.33	4643.57	4622.44

The histogram and fitted densities based on different estimation methods are given in Figure 4 for the KwWeibull distribution.

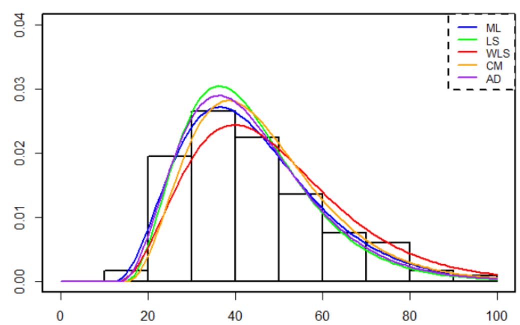

Figure 4. The histogram and the fitted densities based on ML, LS, WLS, CM and AD estimates for the KwWeibull distribution

It can easily be seen from both Table 5 and the Figure 4 that ML method shows the best performance among the others with respect to the fitting performance for the relative humidity data.

6. Conclusions

In this study, we obtain the estimators of location and scale parameters of KwWeibull distribution using the ML, LS, WLS, CM and AD methods. We perform an extensive Monte Carlo simulation study to compare the efficiencies of these estimators. It is concluded that ML estimator shows the best performance among the others and it is followed by AD estimator. The LS estimator demonstrates the worst performance in almost all cases. At the end of the study, we use relative humidity data taken from the literature. Modelling performances of the KwWeibull distribution and the well known and widely used Weibull distribution are compared for this data. It is concluded that KwWeibull distribution shows better fitting performance than the Weibull distribution for modeling the relative humidity data.

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