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Research Article

Some new insights into ideal convergence and subsequences

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Abstract

Some results on the sets of almost convergent, statistically convergent, uniformly statistically convergent, I-convergent subsequences of (s_n) have been obtained by many authors via establishing a one-to-one correspondence between the interval $(0, 1]$ and the collection of all subsequences of a given sequence $s = (s_n)$. However, there are still some gaps in the existing literature. In this paper we plan to fill some of the gaps with new results. Some of them are easily derived from earlier results but they offer some new deeper insights.

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1. Introduction

It is well known that every $x \in (0, 1]$ has a unique binary expansion $x = \sum_{n=1}^{\infty} 2^{-n} d_n(x)$ such that $d_n(x) = 1$ for infinitely many positive integers n, and for every $x \in (0, 1]$ and any sequence $s = (s_n)$ we can generate a subsequence (sx) of s in such a way that: if $d_n(x) = 1$, then $(sx)_n = s_n$. In the existing literature the relationships between a given sequence and its subsequences have been studied in two directions: the first direction is changing the concept of convergence by statistical convergence, A-statistical convergence, uniform statistical convergence, ideal convergence, and the other direction is using measure or category to study the measure and topological largeness of the sets of subsequences (see [2, 4, 15, 18–24]). There are still gaps to examine in this area. Therefore, in this paper we plan to fill some of the gaps with some new results which also offer deeper insights to existing literature.
Now we pause to collect some notation which we need throughout the paper. A family $I \subseteq P(\mathbb{N})$ of subsets of $\mathbb{N}$ is said to be an ideal on $\mathbb{N}$ if I is closed under subsets and finite unions, i.e. for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and $B \subset A$, we have $B \in I$. The ideal I is said to be proper if $\mathbb{N} \notin I$. A proper ideal is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$. It is easy to see that an admissible ideal contains all finite

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subsets of $\mathbb{N}$. In the remainder of the paper, we will assume that the ideal I is admissible. A sequence of real numbers s is said to be I-convergent to l if for every $\varepsilon > 0$ the set $K_\varepsilon = \{n \in \mathbb{N} : |s_n - l| \geq \varepsilon\}$ belongs to I, and we write $I - \lim s = l$ [1, 14]. Note that if $I = I_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then I_d-convergence coincides with statistical convergence where $d(A)$ denotes the natural density of A [10], and if $I = I_u = \{A \subset \mathbb{N} : u(A) = 0\}$, then I_u-convergence reduces to uniform statistical convergence where $u(A)$ denotes the uniform density of A [22, 23]. Here, the term meager will refer to sets of first Baire category, while the term comeager will refer to sets whose complement is of first category. The following result is well known in [2].
Let I be an ideal on $\mathbb{N}$. The following conditions are equivalent:

- I has the Baire Property.
- I is meager.
- There is a sequence $n_1 < n_2 < ...$ of integers in $\mathbb{N}$ such that no member of I contains infinitely many intervals $[n_k, n_{k+1})$.

A Polish space is a separable completely metrizable topological space. That is, a space homeomorphic to a complete metric space that has a countable dense subset. A subset A of a Polish space is called analytic if it is a continuous image of a Polish space. Equivalently, if there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ with range A, where $\mathbb{N}^{\mathbb{N}}$ is the space of irrationals. A subset of a Polish space is coanalytic if its complement is analytic. Ideals on $\mathbb{N}$ can be regarded as subsets of the Polish space $\{0,1\}^{\mathbb{N}}$. Hence they may have the Baire property or be Borel, analytic, coanalytic and so on.

2. Main results

Let us recall some definitions which are needed to obtain our results. These definitions have been studied by Fridy [11] in the case $I = I_d$ and have been formulated in the general case by Kostryko and et al. [14]. They have also been studied by Demirci [8] in detail. Note that some results on I-convergence, I-limit points and I-cluster points can be found in [2, 8, 14, 18].

Definition 2.1. For a given sequence $s = (s_n)$, l is called an I-cluster point of s if $\{n : |s_n - l| < \varepsilon\} \notin I$ holds for every $\varepsilon > 0$.

Definition 2.2. For a given sequence $s = (s_n)$, l is called an I-limit point of $s = (s_n)$ if there exists a sequence $\{n_i : i \in \mathbb{N}\} \notin I$ such that $\lim_{i \to \infty} s_{n_i} = l$.

We remark that it is easy to see that any bounded sequence has at least one I-cluster point. The same does not hold for I-limit points.

Throughout the paper let us denote by $\Gamma_s(I)$ the set of all I-cluster points of s and by $\Lambda_s(I)$ the set of all I- limit points of s. We also introduce the following notation: for s and $x \in (0,1]$, $\Gamma_{(sx)}(I)$ is the set of I-cluster points of (sx), $\Lambda_{(sx)}(I)$ is the set of I-limit points of (sx). It is easy to see that $\Lambda_{(sx)}(I) \subseteq \Gamma_{(sx)}(I)$.

The authors have proved the following (see [18]).

Theorem 2.3. *Suppose s is a bounded sequence, I is an ideal with the Baire property, and L_s is the set of ordinary limit points of s. Then the set $\left\{x \in (0,1] : \Gamma_{(sx)}(I) = \Gamma_s(I)\right\}$ is either meager and comeager. Additionally $\{x \in (0,1] : \Gamma_{(sx)}(I) = L_s\}$ is comeager.*

Likewise the set $\left\{x \in (0,1] : \Lambda_{(sx)}(I) = \Lambda_s(I)\right\}$ is either meager and comeager and the set $\{x \in (0,1] : \Lambda_{(sx)}(I) = L_s\}$ is comeager.

Further work on meager ideals (ideals with Baire property) has been done by Balcerzak, Glab, and Leonetti in [3].

Since analytic and coanalytic ideals have the Baire property, Theorem 2.3 holds for ideals I that are analytic or coanalytic. First we present three corollaries that give some conclusions about non I-convergent sequences. These corollaries are easily derived from some earlier results but offer some new insights.

In [13], A-summability of sequences for certain categories of matrices A is discussed. Using the notation and definitions from [13] we obtain the following corollary. Recall that a matrix A is said to be Schur if it sums every bounded sequence. The next corollary is a simple consequence of the results in [13].

Corollary 2.4. *Let s be a non-I-convergent sequence and A be a non-Shur matrix with convergent columns. Then the set $\{x \in (0,1] : (sx)$ is A-summable$\}$ is meager.*

Proof. Since I is an admissible ideal, every convergent sequence is ideal convergent. So the sequence s is not convergent. On the other hand according to [13] if the set of subsequences of s which is A-summable is of second category, then s is convergent. Hence we have $\{x \in (0,1] : (sx)$ is A-summable$\}$ is of first category. $\square$

It is noteworthy to mention that summability of subsequences is not the primary topic of the paper. However, we have also found Corollary 2.4 suitable to add since it deals with the A-summability of subsequences of non-I-convergent sequences.
The next corollary is a simple consequence of the results in [7].

Corollary 2.5. *Let s be a non-I-convergent sequence. Then the set*

$$\{x \in (0,1] : (sx) \text{ is not convergent}\}$$

has Lebesgue measure 1.

Proof. Since s is not I-convergent, s cannot be convergent. According to [7] almost every subsequence of a given divergent sequence is divergent too. Therefore we obtain that the Lebesgue measure of the set $\{x \in (0,1] : (sx)$ is convergent$\}$ is 1. $\square$

Next we look at the special case when I is an analytic or coanalytic ideal ([9]) with property (G). Initially let us recall some notation from [2] that we need:

Let T denote the set of all $0-1$ sequences with an infinite numbers of ones.

A function $f : \mathbb{N} \to \mathbb{N}$ is called bi-I-invariant if $E \in I \Leftrightarrow f[E] \in I$ for every $E \subset \mathbb{N}$.

For a $0-1$ sequence $x \in T$ let $\{n_1 < n_2 < ...\} := \{k \in \mathbb{N} : \; x_k = 1\}$. Define $f_x : \mathbb{N} \to \mathbb{N}$ by $f_x(k) = n_k$, and let $T_I := \{x \in T : \; f_x$ is bi-I-invariant$\}$.

We will say that an ideal I on $\mathbb{N}$ has property (G) if $\mu(T_I) = 1$. For example, it is simple to show that I_d has property (G) while I_u does not. The next corollary is a simple consequence of the results in [2].

Corollary 2.6. *Let I be an analytic or coanalytic ideal with property (G). If s is a non-I-convergent sequence, the set*

$$\{x \in (0,1] : (sx) \text{ is not } I\text{-convergent}\}$$

has Lebesgue measure 1.

Proof. According to Proposition 3.1 in [2], we have that the set $\{x \in (0,1] : (sx)$ is I-convergent$\}$ has Lebesgue measure 0 or 1. Suppose the Lebesgue measure of the set is 1. Then by Theorem 3.4 in [2], s must be I-convergent. Therefore, this is not possible and we conclude that the set $\{x \in (0,1] : (sx)$ is I-convergent$\}$ has measure 0. The conclusion follows. $\square$

Next we focus our attention on the relationship of sequences and their subsequences regarding their respective sets of I-cluster points, from the point of view of measure (category was treated, as already mentioned, in Theorem 2.3, [18]). These type of results have also been studied by Balcerzak and Leonetti [5].

Theorem 2.7. *Suppose s is a bounded sequence, I is an analytic or coanalytic ideal. Then the set $\{x \in (0,1] : \Gamma_{(sx)}(I) = \Gamma_s(I)\}$ has Lebesgue measure 0 or 1.*

Proof. Let L_s denote the set of ordinary limit points of s. Suppose $l \in L_s$ is arbitrarily fixed. We will show that $\{x \in (0,1] : l \in \Gamma_{(sx)}(I)\}$ is measurable. Clearly

$$\{x \in (0,1] : l \in \Gamma_{(sx)}(I)\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} \{x : \{i : |(sx)_i - l| < \varepsilon\} \in I^c\}$$

where $I^c = P(\mathbb{N}) \backslash I$. Observe the characteristic function

$$\chi_\varepsilon : (0,1] \to \{0,1\}^{\mathbb{N}}$$

defined by

$$(\chi_\varepsilon(x))_i = \begin{cases} 1 & , \quad |(sx)_i - l| < \varepsilon \\ 0 & , \quad \textit{otherwise} \end{cases}$$

.

It is easy to check that χ_ε is continuous. It suffices to check that the i-th component of χ_ε, $(\chi_\varepsilon)_i$ is continuous on $(0,1]$. We will show that the set $(\chi_\varepsilon)_i^{-1}(\{1\})$ is open. Suppose that $x \in (\chi_\varepsilon)_i^{-1}(\{1\})$ is arbitrarily fixed. Clearly if $y \in (0,1]$ is such that $(sx)_j = (sy)_j$ for $1 \le j \le i$, then $y \in (\chi_\varepsilon)_i^{-1}(\{1\})$. This in turn means that there is a $k \ge i$ such that: if $y \in (0,1]$ satisfies $x_j = y_j$ for $1 \le j \le k$ (here x_j, y_j are the j-th coordinates of x, y respectively as $0-1$ sequences), then $y \in (\chi_\varepsilon)_i^{-1}(\{1\})$. Therefore $(\chi_\varepsilon)_i^{-1}(\{1\})$ is open . Likewise, $(\chi_\varepsilon)_i^{-1}(\{0\})$ is also open.

Since I is analytic or coanalytic, I^c is also analytic or coanalytic, we conclude that $\chi_\varepsilon^{-1}(I^c)$ (where I^c is regarded as a subset of the Polish space $\{0,1\}^{\mathbb{N}}$) is analytic or coanalytic and hence measurable. Thus
$\{x : \{i : |(sx)_i - l| < \varepsilon\} \in I^c\} = \chi_\varepsilon^{-1}(I^c)$ is measurable for $\varepsilon \in \mathbb{Q}^+$ and hence $\{x \in (0,1] : l \in \Gamma_{(sx)}(I)\}$ is measurable. Now we will show that $\{x \in (0,1] : \Gamma_{(sx)}(I) \supseteq \Gamma_s(I)\}$ is measurable. Since $\Gamma_s(I)$ is closed and separable, then $\Gamma_s(I) = \overline{\{l_j : j \in \mathbb{N}\}}$ for some $l_j \in L$, $j \in \mathbb{N}$. It is easy to see that $\{x \in (0,1] : \Gamma_{(sx)}(I) \supseteq \Gamma_s(I)\} = \bigcap\limits_j \{x \in (0,1] : l_j \in \Gamma_{(sx)}(I)\}$. Hence this set is measurable as a countable intersection of measurable sets it is also measurable. Now we will look at $\{x \in (0,1] : \Gamma_{(sx)}(I) \subseteq \Gamma_s(I)\}$. The set $[\liminf s, \limsup s] \backslash \Gamma_s(I)$ is open in $[\liminf s, \limsup s]$ so it can be written as $[\liminf s, \limsup s] \backslash \Gamma_s(I) = \bigcup\limits_i H_i$ where H_i are open or half-open intervals. Also for i, we can write $H_i = \bigcup\limits_j H_{ij}$ where H_{ij} are closed intervals. Now clearly,

- (A) $\Gamma_{(sx)}(I) \subseteq \Gamma_s(I)$ if and only if $\Gamma_{(sx)}(I) \cap ([\liminf s, \limsup s] \backslash \Gamma_s(I)) = \varnothing$.

Also, it is easy to verify that

- (B) $\Gamma_{(sx)}(I) \cap ([\liminf s, \limsup s] \backslash \Gamma_s(I)) = \varnothing$ if and only if $\{k : (sx)_k \in H_{ij}\} \in I$, for every i,j.

Now for i,j we can define the characteristic function $\chi_{ij} : (0,1] \to \{0,1\}^{\mathbb{N}}$ by

$$(\chi_{ij}(x))_k = \begin{cases} 1 & , \quad (sx)_k \in H_{ij} \\ 0 & , \quad \textit{otherwise} \end{cases} .$$

It is easy to check χ_{ij} is continuous and hence $\chi_{ij}^{-1}(I)$ is analytic or coanalytic and therefore measurable (where again I is regarded as subset of the Polish space $\{0,1\}^{\mathbb{N}}$). Now from (A), (B) we have that $\{x \in (0,1] : \Gamma_{(sx)}(I) \subseteq \Gamma_s(I)\} = \bigcap\limits_i \bigcap\limits_j \chi_{ij}^{-1}(I)$ and is a countable intersection of measurable sets and therefore is measurable. We conclude that $\{x \in (0,1] : \Gamma_{(sx)}(I) = \Gamma_s(I)\} = \{x \in (0,1] : \Gamma_{(sx)}(I) \supseteq \Gamma_s(I)\} \cap \{x \in (0,1] : \Gamma_{(sx)}(I) \subseteq \Gamma_s(I)\}$ is measurable. Since $\{x \in (0,1] : \Gamma_{(sx)}(I) = \Gamma_s(I)\}$ is also a tail set, it must have measure 0 or 1 (see [6] regarding tail sets). $\square$

We remark that the last theorem provides a generalization of the author's earlier result regarding uniform statistical cluster points in [23], and this further indicates that both cases of measure 0 and 1 can occur (see [23]).

Next we look at the special case when I is an analytic or coanalytic ideal with property (G).

Theorem 2.8. *Suppose s is a bounded sequence, I is an analytic or coanalytic ideal with property (G). Then the set $\{x \in (0,1] : \Gamma_{(sx)}(I) = \Gamma_s(I)\}$ has measure 1.*

Proof. First we verify that the set $\{x \in (0,1] : \Gamma_{(sx)}(I) \subseteq \Gamma_s(I)\}$ has measure 1. Suppose $x \in T_I$ is arbitrarily fixed (T_I is as defined earlier). Then $(sx) = \{s_{n_i}\}_i$ for some $n_1 < n_2 < ... < n_i < ....$ Let $l \in \Gamma_{(sx)}(I)$ be fixed. Then, for $\varepsilon > 0$ we have $\{i : |s_{n_i} - l| < \varepsilon\} \in I^c \to f_x(\{i : |s_{n_i} - l| < \varepsilon\}) \in I^c \to \{n_i : |s_{n_i} - l| < \varepsilon\} \in I^c$. Thus $l \in \Gamma_s(I)$. So $\Gamma_{(sx)}(I) \subseteq \Gamma_s(I)$, for all $x \in T_I$ and $m(T_I) = 1$ and hence $\{x \in (0,1] : \Gamma_{(sx)}(I) \subseteq \Gamma_s(I)\}$ has measure 1.

Next let $l \in \Gamma_s(I)$ be fixed arbitrarily. Let $X = T_I \cap (1 - T_I)$ where $1 - T_I = \{x : 1 - x \in T_I\}$. Then $m(X) = 1$ and $x \in X \to 1 - x \in X$. Let $x \in X$ be fixed. Let $\{n_i\}$ denote the set of indices corresponding to x and $\{n_j\}$ the set of indices corresponding to $1 - x$. Then $\{n_i\} \cap \{n_j\} = \varnothing$, $\{n_i\} \cup \{n_j\} = \mathbb{N}$. For $\varepsilon > 0$ we have:$\{n : |s_n - l| < \varepsilon\} \in I^c \to \{n_i : |s_{n_i} - l| < \varepsilon\} \in I^c$ or $\left\{n_j : |s_{n_j} - l| < \varepsilon\right\} \in I^c$, so analogously as earlier since $x, 1 - x \in T_I \to \{i : |s_{n_i} - l| < \varepsilon\} \in I^c$ or $\left\{j : |s_{n_j} - l| < \varepsilon\right\} \in I^c$. We can conclude that: $\{i : |s_{n_i} - l| < \varepsilon\} \in I^c$ for infinitely many $\varepsilon \to 0$ or $\left\{j : |s_{n_j} - l| < \varepsilon\right\} \in I^c$ for infinitely many $\varepsilon \to 0$. Hence

$$\left(l \in \Gamma_{(sx)}(I) \text{ or } l \in \Gamma_{(s(1-x))}(I)\right), \quad x \in X. \tag{2.1}$$

From the proof of Theorem 2.7, we know that $\left\{x : \ l \in \Gamma_{(sx)}(I)\right\}$ is measurable and it is also a tail set so it has measure 0 or 1. If we assume that $m\left(\left\{x : \ l \in \Gamma_{(sx)}(I)\right\}\right) = 0$, then from (2.1):

$$X = \left(X \cap \left\{x : \ l \in \Gamma_{(sx)}(I)\right\}\right) \cup \left(1 - \left(X \cap \left\{x : \ l \in \Gamma_{(sx)}(I)\right\}\right)\right)$$

would be a union two sets of measure 0, impossible since $m(X) = 1$. Thus $m\left(\left\{x : \ l \in \Gamma_{(sx)}(I)\right\}\right) = 1$ for $l \in \Gamma_s(I)$. Now since $\Gamma_s(I)$ is closed, $\Gamma_s(I) = \overline{\{l_1, l_2, ..., l_i, ...\}}$ for some $\{l_i\} \in \Gamma_s(I)$. Since $\Gamma_{(sx)}(I)$ is also closed for $x \in (0,1]$, $\{x \in (0,1] : \Gamma_s(I) \subseteq \Gamma_{(sx)}(I)\} = \bigcap_i \{x \in (0,1] : l_i \in \Gamma_{(sx)}(I)\}$ is a countable intersection of sets of measure 1. Hence $\{x \in (0,1] : \Gamma_s(I) \subseteq \Gamma_{(sx)}(I)\}$ also has measure 1. This completes the proof. □

We remark that Theorem 2.8 is similar to a result by Leonetti [16]. He has obtained the same conclusion as in Theorem 2.8 only regarding another class of ideals, thinnable ideals. This class is a more restrictive class than ideals with property (G), but his proof uses similar arguments.

Theorems 2.7 and 2.8 state a stability property of I-cluster points of subsequences of a given bounded real sequence s for certain classes of ideals. We conclude the paper by observing that some related results regarding I-limit points have been obtained by Leonetti [17] and additional insights on the relation of I-cluster and I-limit points have been studied in [12].

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