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Shape Preserving Properties of the Generalized Baskakov Operators

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ABSTRACT

The present paper deals with the shape preserving properties of a new Baskakov type operators. Our results are based on a ρ function such as the ρ-convexity, ρ-star-shaped, and the ρ-monotonicty. These results include the preservation properties of the classical Baskakov operators.

Key words: *Baskakov Operators; Shape Preserving Properties; Convexity; Star-shaped.*

1. INTRODUCTION

In [1] discussed the following positive linear operators on the unbounded interval $[0,\infty)$,

$$V_n(f;x)=\sum_{k=0}^{\infty}\upsilon_{n,k}(x)f\left(\frac{k}{n}\right), n\in\mathbb{N}, x\in[0,\infty), \qquad (1.1)$$

for appropriate functions f defined on $[0,\infty)$ for which the above series is convergent and

$$\upsilon_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

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In 2011, Cardenas-Morales et al. [2] introduced a generalized Bernstein operator fixing e_0 and e_2, given by

$$L_n(f;x)=\sum_{k=0}^{n} x^{2k}\left(1-x^2\right)^{n-k} f\left(\sqrt{\frac{k}{n}}\right),\ f\in C[0,1],\ x\in[0,1].$$

This is a special case of the operator

$$B_n^{\tau} f = B_n\left(f\circ\tau^{-1}\right)\circ\tau$$

for $\tau = e_2$, where B_n is the classical Bernstein operator.

Recently, in [3], the following generalization of Szasz-Mirakyan operators are constructed,

$$\begin{aligned} S_n^{\rho}(f;x) &= \exp\left(-n\rho(x)\right)\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\frac{\left(n\rho(x)\right)^k}{k!},\ n\in\mathbb{N},\ x\in[0,\infty), \\ &= \left(S_n\left(f\circ\rho^{-1}\right)\circ\rho\right)(x), \end{aligned} \tag{1.2}$$

where ρ is a real valued function on $[0,\infty)$ satisfied following two conditions:

(1) ρ is a continuously differentiable function on $[0,\infty)$,

(2) $\rho(0)=0$ and $\inf_{x\in[0,\infty)} \rho'(x)\geq 1.$

Throughout the manuscript, we denote the above two conditions as c_1 and c_2.

Notice that if $\rho = e_1$, then the operators (1.2) reduces to well known Szasz-Mirakyan operators. Aral et. al. [3] gave quantitative type theorems in order to obtain the degree of weighted convergence with the help of a weighted modulus of continuity constructed using the function ρ of the operators (1.2). Very recently, some researchers have discussed approximation properties of the generalized Bernstein [4,5], Szasz-Mirakyan operators [6,7,8] and Baskakov [9,10,11].

2. CONSTRUCTION OF THE OPERATORS V_n^{ρ}

The studies presented in introduction motivated us to generalize the Baskakov operators (1.1) as

$$\begin{aligned} V_n^{\rho}(f;x) &= \sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{\left(\rho(x)\right)^k}{\left(1+\rho(x)\right)^{n+k}} \\ &= \left(V_n\left(f\circ\rho^{-1}\right)\circ\rho\right)(x) \\ &= \sum_{k=0}^{\infty} f\left(\rho^{-1}\left(\frac{k}{n}\right)\right)\upsilon_{\rho,n,k}(x), \end{aligned} \tag{2.1}$$

where $n\in\mathbb{N}$, $x\in[0,\infty)$, ρ is a function defined as in conditions c_1 and c_2. Observe that,

$V_n^{\rho}(f;.)=V_n(f;.)$ if $\rho = e_1$. In fact, direct calculation gives that

$$V_n^{\rho}(e_0;x)=1$$
$$V_n^{\rho}(e_1;x)=\rho(x)$$
$$V_n^{\rho}(e_2;x)=\rho^2(x)+\frac{\rho^2(x)+\rho(x)}{n}.$$

In this manuscript, we are dealing with the shape preserving properties of the operators (2.1). In the next section, we discuss the properties of the generalized Baskakov operators $V_n^{\rho}(f;.)$. The generalizes existing results of the classical Baskakov operators (2.1).

We consider the notion of convexity with respect to ρ as used in [2]. A function f is convex with respect to ρ if and only if $f\circ\rho^{-1}$ is convex in the classical sense. Further, we need following notations to discuss shape preserving properties of the operators:

Let $x_0, x_1, ..., x_n$ be distinct points in the domain of f.

$$f\left[x_0, x_1, ..., x_n\right] = \sum_{r=0}^{n} \frac{f\left(x_r\right)}{\prod_{j \neq r}^{n}\left(x_r - x_j\right)},$$

where r remains fixed and j takes all values from 0 to n, excluding r, which is same as

$$f\left[x_0\right] = f\left(x_0\right);$$

$$f\left[x_0, x_1, ..., x_n\right] = \frac{f\left[x_1, x_2, ..., x_n\right] - f\left[x_0, x_1, ..., x_{n-1}\right]}{x_n - x_0}, \text{ for } n \geq 1.$$

3. SHAPE PRESERVING PROPERTIES

Throughout the theorems we consider the appropriate functions f defined on $\left[0, \infty\right)$ for which the series (2.1) is convergent. Note that, the series on the right side of (2.1) is absolutely convergent because $f \in C_\rho\left[0, \infty\right)$; any continuous function f on $\left[0, \infty\right)$ with $\left|f\left(x\right)\right| \leq M_f\left(1 + \rho^2\left(x\right)\right).$ Furthermore, since $C_\rho\left[0, \infty\right) \supset C_B\left[0, \infty\right);$ the space of all bounded and continuous functions on $\left[0, \infty\right)$, the series (2.1) is convergent for $f \in C_B\left[0, \infty\right)$.

Theorem 3.1. For every $n \in \mathbb{N}, x \in \left[0, \infty\right)$ such that $\rho\left(x\right) \neq \frac{k}{n}, k = 0, 1, 2, ...,$ the following identity holds:

$$V_n^\rho\left(f; x\right) - f\left(x\right) = \frac{\rho\left(x\right)\left(1 + \rho\left(x\right)\right)}{n} \sum_{k=0}^{\infty}\left(f \circ \rho^{-1}\right)\left[\rho\left(x\right), \frac{k}{n}, \frac{k+1}{n}\right] \upsilon_{\rho, n+1, k}\left(x\right).$$

Proof Since $V_n^\rho\left(1; x\right) = 1$, we get

$$V_n^\rho\left(f; x\right) - f\left(x\right) = \sum_{k=0}^{\infty}\left(\left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right) - f\left(x\right)\right)\binom{n+k-1}{k} \frac{\left(\rho\left(x\right)\right)^k}{\left(1 + \rho\left(x\right)\right)^{n+k}}$$

$$= \sum_{k=0}^{\infty}\left(\frac{k}{n} - \rho\left(x\right)\right)\left(f \circ \rho^{-1}\right)\left[\rho\left(x\right), \frac{k}{n}\right]\binom{n+k-1}{k} \frac{\left(\rho\left(x\right)\right)^k}{\left(1 + \rho\left(x\right)\right)^{n+k}}. \quad (3.1)$$

By simple computation following identity archived,

$$\left(k - n\rho\left(x\right)\right)\rho'\left(x\right)\upsilon_{\rho, n, k}\left(x\right) = \rho\left(x\right)\left(1 + \rho\left(x\right)\right)\upsilon'_{\rho, n, k}\left(x\right) \quad (3.2)$$

$$\upsilon'_{\rho, n, k}\left(x\right) = n\rho'\left(x\right)\left(\upsilon_{\rho, n+1, k-1}\left(x\right) - \upsilon_{\rho, n+1, k}\left(x\right)\right), \quad (3.3)$$

where $\upsilon_{\rho, n+1, -1}\left(x\right) = 0$. Using (3.2) and (3.3) in (3.1), we get

$$V_n^\rho\left(f; x\right) - f\left(x\right) = \frac{\rho\left(x\right)\left(1 + \rho\left(x\right)\right)}{\rho'\left(x\right)n} \sum_{k=0}^{\infty}\left(f \circ \rho^{-1}\right)\left[\rho\left(x\right), \frac{k}{n}\right]\upsilon'_{\rho, n, k}\left(x\right)$$

$$=\rho(x)(1+\rho(x))\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n}\right]\left(\upsilon_{\rho,n+1,k-1}(x)-\upsilon_{\rho,n+1,k}(x)\right).$$

Since $\upsilon_{\rho,n+1,-1}(x)=0$, we write

$$V_n^{\rho}(f;x)-f(x)=\rho(x)(1+\rho(x))\left(\sum_{k=1}^{\infty}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n}\right]\upsilon_{\rho,n+1,k-1}(x)\right.$$
$$\left.-\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n}\right]\upsilon_{\rho,n+1,k}(x)\right)$$
$$=\rho(x)(1+\rho(x))\left(\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k+1}{n}\right]-\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n}\right]\right)\upsilon_{\rho,n+1,k}(x).$$

From the definition of the divided difference

$$\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k+1}{n}\right]-\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n}\right]=\frac{1}{n}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n},\frac{k+1}{n}\right]$$

and we have that

$$V_n^{\rho}(f;x)-f(x)=\frac{\rho(x)(1+\rho(x))}{n}\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left[\rho(x),\frac{k}{n},\frac{k+1}{n}\right]\upsilon_{\rho,n+1,k}(x).$$

Corollary 3.1. If f is $\rho-convex$ on $[0,\infty)$, then

$$V_n^{\rho}(f;x)\geq f(x)$$

for $n\geq 0$ and $x\in[0,\infty)$ such that $\rho(x)\neq\frac{k}{n},(k=0,1,2,...)$.

The above corollary is an immediate consequence of Theorem 3.1.

Theorem 3.2. If f is $\rho-convex$ on $[0,\infty)$, then

$$V_{n+1}^{\rho}(f;x)-V_n^{\rho}(f;x)$$
$$=-\frac{1}{n(n+1)^2}\sum_{k=0}^{\infty}\frac{(n+k+1)(\rho(x))^{k+1}}{(1+\rho(x))^{n+k+1}}\binom{n+k}{k}\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1},\frac{k+1}{n+1},\frac{k+1}{n}\right];$$

for $n\geq 0$ and $x\in[0,\infty)$.

Proof We can write

$$V_{n+1}^{\rho}(f;x)=\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)\binom{n+k}{k}\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k}}$$
$$-\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)\binom{n+k}{k}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}$$
$$=\frac{\left(f\circ\rho^{-1}\right)(0)}{\left(1+\rho(x)\right)^{n}}+\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n+1}\right)\binom{n+k+1}{k+1}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}$$
$$-\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)\binom{n+k}{k}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}.$$

On the other hand, we have

$$V_{n}^{\rho}(f;x)=\frac{\left(f\circ\rho^{-1}\right)(0)}{\left(1+\rho(x)\right)^{n}}+\sum_{k=1}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k+1}\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k}}$$
$$=\frac{\left(f\circ\rho^{-1}\right)(0)}{\left(1+\rho(x)\right)^{n}}+\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}.$$

Now, we obtain

$$V_{n+1}^{\rho}(f;x)-V_{n}^{\rho}(f;x)$$
$$=\sum_{k=0}^{\infty}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}\left[\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n+1}\right)\binom{n+k+1}{k+1}\right.$$
$$\left.-\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)\binom{n+k}{k}-\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1}\right].$$

From the equalities $\binom{n+k+1}{k+1}=\frac{n+k+1}{k+1}\binom{n+k}{k}$ and $\binom{n+k}{k+1}=\frac{n}{k+1}\binom{n+k}{k}$, we get

$$V_{n+1}^{\rho}(f;x)-V_{n}^{\rho}(f;x)$$
$$=-\sum_{k=0}^{\infty}\frac{\left(\rho(x)\right)^{k+1}}{\left(1+\rho(x)\right)^{n+k+1}}\binom{n+k}{k}\left[\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)\right. \tag{3.4}$$
$$\left.-\frac{n+k+1}{k+1}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n+1}\right)+\frac{n}{k+1}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\right].$$

Using some simple calculations about divided difference, we have

$$\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1},\frac{k}{n+1},\frac{k+1}{n}\right]$$

$$=\frac{\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]-\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1},\frac{k+1}{n+1}\right]}{\frac{k+1}{n}-\frac{k}{n+1}}$$

$$=\frac{n(n+1)}{n+k+1}\left\{\frac{\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n}\right]-\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n+1}\right]}{\frac{k+1}{n}-\frac{k+1}{n+1}}-\frac{\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n+1}\right]-\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1}\right]}{\frac{k+1}{n+1}-\frac{k}{n+1}}\right\}$$

$$=\frac{n(n+1)}{n+k+1}\left\{\frac{n(n+1)\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n}\right]}{k+1}-\frac{(n+1)(n+k+1)\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n+1}\right]}{k+1}\right.$$

$$\left.+(n+1)\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1}\right]\right\}$$

$$=\frac{n(n+1)^2}{n+k+1}\left\{\frac{n}{k+1}\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n}\right]-\frac{(n+k+1)}{k+1}\left(f\circ\rho^{-1}\right)\left[\frac{k+1}{n+1}\right]\right.$$

$$\left.+\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1}\right]\right\}$$

$$\left(f\circ\rho^{-1}\right)\left[\frac{k}{n+1},\frac{k}{n+1},\frac{k+1}{n}\right]=\frac{n(n+1)^2}{n+k+1}\left[\left(f\circ\rho^{-1}\right)\left(\frac{k}{n+1}\right)-\frac{(n+k+1)}{k+1}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n+1}\right)+\frac{n}{k+1}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\right] \tag{3.5}$$

Combining equations (3.4) and (3.5), we archived our results.

Corollary 3.2. If f is $\rho-convex$ on $[0,\infty)$, then $V_n^{\rho}(f;x)\geq V_{n+1}^{\rho}(f;x)$, for all $n\geq 0$ and $x\in[0,\infty)$. If $\left(f\circ\rho^{-1}\right)$ is linear then $V_n^{\rho}(f;x)=V_{n+1}^{\rho}(f;x)$.

The following corollary is an immediate consequence of Theorem 3.2.

Now, we define the notion of $star-shaped$ with respect to ρ. A function f is $star-shaped$ with respect to ρ if and only if $\left(f\circ\rho^{-1}\right)$ is $star-shaped$ in the classical sense.

Theorem 3.3. Let ρ be $star-shaped$. If f is $\rho-star-shaped$, then $V_n^{\rho}(f;.)$ is $star-shaped$.

Proof By taking derivative of $V_n^\rho\left(f;.\right)$, we get

$$\begin{aligned}\frac{dV_n^\rho\left(f;x\right)}{dx} &= \sum_{k=1}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}k\frac{\left(\rho(x)\right)^{k-1}}{\left(1+\rho(x)\right)^{n+k}}\rho'(x)\\ &\quad-\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}(n+k)\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\rho'(x)\\ &= \sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\binom{n+k}{k+1}(k+1)\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\rho'(x)\\ &\quad-\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}(n+k)\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\rho'(x).\end{aligned}$$

From the facts that $\binom{n+k}{k+1}(k+1)=\frac{(n+k)!}{(n-1)!k!}$ and $\binom{n+k-1}{k}(n+k)=\frac{(n+k)!}{(n-1)!k!}=n\binom{n+k}{k}$, we can write

$$\begin{aligned}&\frac{dV_n^\rho\left(f;x\right)}{dx}-\frac{V_n^\rho\left(f;x\right)}{dx}\\ &=n\sum_{k=0}^{\infty}\binom{n+k}{k}\rho'(x)\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\left(\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)-\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\right)\\ &-\sum_{k=1}^{\infty}\frac{\rho(x)}{x}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{\left(\rho(x)\right)^{k-1}}{\left(1+\rho(x)\right)^{n+k}}.\end{aligned}$$

Using the equality $\binom{n+k}{k+1}(k+1)=n\binom{n+k}{k}$ and since ρ is $star-shaped$, we obtain

$$\begin{aligned}&\frac{dV_n^\rho\left(f;x\right)}{dx}-\frac{V_n^\rho\left(f;x\right)}{dx}\\ &=n\sum_{k=0}^{\infty}\binom{n+k}{k}\rho'(x)\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\left(\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)-\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\right)\\ &-n\sum_{k=0}^{\infty}\frac{\rho(x)}{x}\frac{1}{k+1}\binom{n+k}{k}\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)\qquad(3.6)\\ &\geq n\sum_{k=0}^{\infty}\frac{\rho(x)}{x}\binom{n+k}{k}\frac{\left(\rho(x)\right)^{k}}{\left(1+\rho(x)\right)^{n+k+1}}\left(\frac{k}{k+1}\left(f\circ\rho^{-1}\right)\left(\frac{k+1}{n}\right)-\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\right).\end{aligned}$$

Since f is $\rho - star - shaped$, we have

$$\frac{k}{k+1}\left(f \circ \rho^{-1}\right)\left(\frac{k+1}{n}\right) \geq \left(f \circ \rho^{-1}\right)\left(\frac{k}{n}\right),$$

also since $\inf_{x\in[0,\infty)} \rho'(x) \geq 1$, we get

$$\frac{\rho(x)}{x} \geq 1. \qquad (3.7)$$

Using inequalities (3.6) and (3.7),

the assertion of the theorem follows.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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